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REPRESENTATIONS OF LATTICES AS CONGRUENCE LATTICES

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§1. INTRODUCTION

In [1] Birkhoff posed the problem of characterizing the lattice of all congruence relations of an algebra. It is easy to see that this lattice is a complete lattice. In [9] G. Grätzer and E. T. Schmidt showed that every algebraic lattice is isomorphic to the lattice of all congruence relations of some <u>finitary</u> algebra. The converse had been known for some time. Recently, a number of other representation theorems involving the lattice of congruence relations of an algebra have been proved. One such theorem is that every complete lattice is isomorphic to the lattice of all congruence relations of some algebra. In this paper we will survey these results and discuss the basic method used in their proofs. We will also mention some of the open problems. (No originality is claimed for the problems).

§2. TERMS AND NOTATIONS

Let α be an ordinal and A be a set. If $f : A^{\alpha} \to A$, then we say that f is an <u>α-ary operation</u> on A. $\mathfrak{A} = \langle A; F \rangle$ is an <u>algebra</u> iff F is a family of operations on the set A. We say $\mathfrak{A}$ is of characteristic $\mathfrak{m}$ iff $\mathfrak{m}$ is the least regular cardinal such that for any operation f of $\mathfrak{A}$ if f is α-ary then $\alpha < \mathfrak{m}$. $\mathfrak{A}$ is finitary iff $\mathfrak{A}$ is of characteristic $\aleph_0$. $\mathfrak{A}$ is infinitary if $\mathfrak{A}$ is not finitary. If $\underset{\sim}{x} \in A^\alpha$, then the i^{th} component of $\underset{\sim}{x}$ is denoted x_i. If Θ is an equivalence relation on A and if $\underset{\sim}{x}, \underset{\sim}{y} \in A^\alpha$, we write $\underset{\sim}{x} \equiv \underset{\sim}{y}\ (\Theta)$ iff $x_i \equiv y_i\ (\Theta)$ for every $i < \alpha$. Θ is a congruence relation of $\mathfrak{A}$ iff Θ is an equivalence relation on A and for any α and any α-ary operation f and any $\underset{\sim}{x}, \underset{\sim}{y} \in A^\alpha$ $f(\underset{\sim}{x}) \equiv f(\underset{\sim}{y})\ (\Theta)$ whenever $\underset{\sim}{x} \equiv \underset{\sim}{y}\ (\Theta)$. $\mathrm{Con}(\mathfrak{A})$ is the set of all congruence relations of $\mathfrak{A}$. $\underset{\sim}{\mathrm{Con}}(\mathfrak{A}) = \langle \mathrm{Con}(\mathfrak{A}); \subseteq \rangle$ is the congruence lattice of $\mathfrak{A}$. $\mathfrak{A}$ is simple if $\underset{\sim}{\mathrm{Con}}(\mathfrak{A})$ is the two element chain. Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra, and let $B \subseteq A$. B is a subalgebra of $\mathfrak{A}$ iff for every α and for every α-ary operation f of $\mathfrak{A}$ and for every $\underset{\sim}{x} \in B^\alpha$ it holds that $f(\underset{\sim}{x}) \in B$. $\mathrm{Sub}(\mathfrak{A})$ is the set of all subalgebras of $\mathfrak{A}$. By convention $\emptyset \in \mathrm{Sub}(\mathfrak{A})$ iff $\mathfrak{A}$ has no 0-ary operations. $\underset{\sim}{\mathrm{Sub}}(\mathfrak{A}) = \langle \mathrm{Sub}(\mathfrak{A}); \subseteq \rangle$ is the subalgebra lattice of $\mathfrak{A}$. Let $\underset{\sim}{x} \in A^\alpha$ and $\sigma : A \to A$ then $\underset{\sim}{x}\sigma$ is the sequence $\underset{\sim}{y} \in A^\alpha$ with $y_i = x_i\sigma$ for every $i < \alpha$. σ is an endomorphism iff $f(\underset{\sim}{x}\sigma) = f(\underset{\sim}{x})\sigma$ for every operation f and every $\underset{\sim}{x}$. $\mathrm{End}(\mathfrak{A})$ is the set of all endomorphisms of $\mathfrak{A}$, and $\underset{\sim}{\mathrm{End}}(\mathfrak{A}) = \langle \mathrm{End}(\mathfrak{A}); \circ \rangle$ is the endomorphism semigroup of $\mathfrak{A}$. A 1-1 onto endomorphism is an automorphism, and $\underset{\sim}{\mathrm{Aut}}(\mathfrak{A}) = \langle \mathrm{Aut}(\mathfrak{A}); \circ \rangle$ denotes the automorphism group.

$\mathcal{L} = \langle L; \leq \rangle$ is a complete lattice iff $\mathcal{L}$ is a partially ordered set such that any $H \subseteq L$ has a join (sup, $\bigvee H$) and a meet (inf, $\bigwedge H$). Let $\mathfrak{m}$ be a regular cardinal. The element c of the complete lattice $\mathcal{L}$ is <u>m-compact</u> iff whenever $c \leq \bigvee H$ then $c \leq \bigvee H_0$ for some H_0 with $H_0 \subseteq H$ and $|H_0| < \mathfrak{m}$. The complete lattice is <u>m-algebraic</u> iff every element is the join of some set of m-compact elements. $\aleph_0$-algebraic lattices are simply called <u>algebraic lattices</u>. Clearly, any complete lattice $\mathcal{L}$ is $|L|^+$-algebraic.

$\mathcal{L}$ is a <u>partition lattice</u> iff $\mathcal{L}$ is a sublattice of the lattice of all equivalence relations on some set such that equality and the total relation are members of $\mathcal{L}$.

§3. HISTORY AND RESULTS

In [3] G. Birkhoff and O. Frink showed that the congruence lattice of a finitary algebra is an algebraic lattice. The converse appeared in 1963.

<u>Theorem 1</u>. (G. Grätzer and E. T. Schmidt [9]): If $\mathcal{L}$ is any algebraic lattice, then there is a finitary algebra $\mathfrak{A}$ such that $\mathrm{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$.

In [9] Grätzer and Schmidt gave the construction for an algebra $\mathfrak{A}$, all of whose operations were unary, such that $\mathrm{Con}(\mathfrak{A})$ is isomorphic to the specified lattice $\mathcal{L}$. A simpler proof appears in [16]. Other proofs appear in [4], [13], [14] and [21]. The proofs in [14] and [21] are essentially the same. The various proofs differ in detail but all use basically the same construction. The proof in [13] is due to R. N. McKenzie.

Let C be the set of compact elements of $\mathcal{L}$. The algebra in each of the proofs has $|C| \cdot \aleph_0$ elements and $|C| \cdot \aleph_0$ unary operations. A long standing problem is to show that the representation in Theorem 1 can be effected with an algebra having one binary operation (or at least finitely many finitary operations). The known results on this problem are fragmentary.

G. Birkhoff showed in [2] that any group could be isomorphic to the automorphism group of some finitary algebra (in fact a unary algebra). His proof has been extended to show that any semigroup with unit can be the endomorphism semigroup of some finitary algebra. (That such a representation could be effected using only one binary operation or two unary operations was shown in a series of papers which ended with [10]).

The "kernel" of any homomorphism is a congruence relation. This provides a mechanism thru which the endomorphism semigroup of an algebra can affect the congruence lattice. (Very little is known about the connection between $\underset{\sim}{\text{End}}(\mathfrak{A})$ and $\underset{\sim}{\text{Con}}(\mathfrak{A})$. See, for example [5] and [15]). There is no such obvious mechanism through which the automorphism group can affect the congruence lattice.

So it was conjectured some time ago that in general the congruence lattice and the automorphism group are "independent". More precisely, it was conjectured that if $\mathcal{L}$ is any algebraic lattice and $\mathfrak{G}$ is any group then there is a finitary algebra $\mathfrak{A}$ such that $\underset{\sim}{\mathrm{Con}}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$ and $\underset{\sim}{\mathrm{Aut}}(\mathfrak{A})$ is isomorphic to $\mathfrak{G}$. That this conjecture is true follows from Theorem 2. In [20] E. T. Schmidt published an incorrect proof that this conjecture is true. However, the intuitive picture of the construction in Theorem 2 is in some ways similar to E. T. Schmidt's.

G. Birkhoff and O. Frink proved in [3] that any algebraic lattice was isomorphic to $\underset{\sim}{\mathrm{Sub}}(\mathfrak{A})$ for some finitary $\mathfrak{A}$. E. T. Schmidt gave a very nice proof in [19] that $\underset{\sim}{\mathrm{Sub}}(\mathfrak{A})$ and $\underset{\sim}{\mathrm{Aut}}(\mathfrak{A})$ are independent. This result is also a Corollary to Theorem 2. There is obviously a third corollary to Theorem 2 which gives a representation for any pair of algebraic lattices.

Theorem 2. (W. A. Lampe [18]): If $\mathfrak{G}$ is any group and $\mathcal{L}_0$ and $\mathcal{L}_1$ are any two algebraic lattices each having two or more elements, then there is a finitary algebra $\mathfrak{A}$ such that:

(i) $\underset{\sim}{\mathrm{Con}}(\mathfrak{A})$ is isomorphic to $\mathcal{L}_0$;

(ii) $\underset{\sim}{\mathrm{Sub}}(\mathfrak{A})$ is isomorphic to $\mathcal{L}_1$;

(iii) $\underset{\sim}{\mathrm{Aut}}(\mathfrak{A})$ is isomorphic to $\mathfrak{G}$.

The $\mathfrak{A}$ in the proof of Theorem 2 actually has n-ary operations for every $n > 0$. Binary operations would have done as well, but the proof would have been a little bit longer. If C_i represents the set of compact elements of $\mathcal{L}_i$, then $\mathfrak{A}$ has $[|C_0| \cdot |C_1| \cdot \aleph_0]$ elements and operations.

In what ways can one "improve" this representation? If $\mathfrak{A}$ is a finitary algebra having at most countably many operations, then each finitely generated subalgebra is countable, and so each finitely generated subalgebra has at most countably many finitely generated subalgebras. Thus, in $\mathrm{Sub}(\mathfrak{A})$ each compact element has at most countably many compact elements below it. (The converse was first proved by W. Hanf. It appeared in [13] and [22]). It is clear then that in general one cannot put a bound on the number of operations that the $\mathfrak{A}$ in Theorem 2 has. But if one omits conclusion (ii), then it seems likely that one could produce a representation using only finitely many finitary operations.

One must use at least one binary operation in the $\mathfrak{A}$ of Theorem 2 for two reasons. First, among other things, G. Grätzer showed in [5] that the automorphism group of a simple algebra having only unary or nullary operations was a group of order p where $p = 1$ or p is a prime. (A corollary of the main result of [5] is that any group is the automorphism group of some simple algebra having one binary and many unary operations. The unary operations have been eliminated by J. Ježek in a recent paper appearing in Comm. Math. Univ. Carolinae). Secondly, if $\mathfrak{A}$ is unary then the join in $\mathrm{Sub}(\mathfrak{A})$ is just set union, and so $\mathrm{Sub}(\mathfrak{A})$

is then a "completely" distributive lattice.

Let Θ and Φ be equivalence relations on some set, and let $\Theta \cdot \Phi$ represent the "composition" of Θ and Φ. Let $\Psi_0 = \Theta$, $\Psi_1 = \Theta \cdot \Phi$, $\Psi_2 = \Theta \cdot \Phi \cdot \Theta$, $\Psi_3 = \Theta \cdot \Phi \cdot \Theta \cdot \Phi$, etc. In the lattice of all equivalence relations on the set, $\Theta \vee \Phi = \bigcup(\Psi_i \mid i = 0, 1, \ldots)$. We say the join in a partition lattice is of <u>type-n</u> if for any Θ, Φ, $\Theta \vee \Phi = \Psi_n$. B. Jónsson showed in [12] that a lattice $\mathcal{L}$ is modular iff $\mathcal{L}$ is isomorphic to a partition lattice in which the join is of type-2. $\mathbf{Con}(\mathfrak{A})$ is a partition lattice but it is a special kind of partition lattice. So a natural and non-trivial question arises which is answered by Theorem 3.

<u>Theorem 3</u>. (G. Grätzer and W. A. Lampe [7]): If $\mathcal{L}$ is a modular algebraic lattice, then there is a finitary algebra $\mathfrak{A}$ such that $\mathbf{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$ and the join in $\mathbf{Con}(\mathfrak{A})$ is of type-2.

Incorrect proofs for the above theorem appeared in [9] and [21].

The algebra $\mathfrak{A}$ in the proof is unary and has $|C| \cdot \aleph_0$ elements and operations where C is the set of compact elements of $\mathcal{L}$. One can ask the familiar questions about the number and kind of operations required for this representation.

The new techniques of [16] were essential to the proof of Theorem 3. Incidentally, the join in $\mathbf{Con}(\mathfrak{A})$ is "automatically" of type-3 for the particular algebra $\mathfrak{A}$ in the proof of Theorem 1 given in [16]. The same is probably true for the other proofs.

By generalizing the technique in the proof of Theorem 3 we can make the algebra $\mathfrak{A}$ in Theorem 2 be such that the join in $\underset{\sim}{Con}(\mathfrak{A})$ is of type-n and not type n-1 for any $n \geq 3$. We can also make the join in $\underset{\sim}{Con}(\mathfrak{A})$ be of "type ω" - i.e. not of type n for any n. If $\mathfrak{G}$ is the one-element group and $\mathcal{L}_0$ is modular, we can construct an $\mathfrak{A}$ for Theorem 2 such that the join in $\underset{\sim}{Con}(\mathfrak{A})$ is of type-2. Another problem is: what are the automorphism groups of algebras having modular congruence lattices in which the join is of type-2?

As mentioned in the introduction, we also know that

Theorem 4: If $\mathcal{L}$ is a complete lattice, then there is an algebra $\mathfrak{A}$ such that $\underset{\sim}{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$.

More generally, we know

Theorem 5. (G. Grätzer and W. A. Lampe [8]): If $\mathcal{L}$ is an $\mathfrak{m}$-algebraic lattice, then there is an algebra $\mathfrak{A}$ of characteristic $\mathfrak{m}$ such that $\underset{\sim}{Con}(\mathfrak{A})$ is isomorphic to $\mathcal{L}$.

In general, the congruence lattice of an infinitary algebra is not a partition lattice. However, we can build the $\mathfrak{A}$ for the proof of Theorem 5 in such a way that $\underset{\sim}{Con}(\mathfrak{A})$ is a partition lattice in which the join is of type-3.

Such a result is not automatic for Theorem 5 as it was for Theorem 1. In fact, one uses a generalization of the technique for Theorem 3.

Once again the algebra has very many operations, and it's not clear one needs so many.

Consider Theorems 2 and 3 and all their previously mentioned extensions. A natural question is, "Are all the straightforward generalizations of all these theorems to $\mathfrak{m}$-algebraic lattices and algebras of characteristic $\mathfrak{m}$ true?" The answer is yes. But the proofs are not exactly straightforward generalizations of the corresponding finitary case proofs. There is also a corresponding array of open problems.

A "master" construction from which all these theorems follow will appear in [8].

§4. THE BASIC METHOD

All the above mentioned theorems are proved using constructions that have their roots in the original construction by Grätzer and Schmidt for Theorem 1. In this section we will make some remarks about this method.

To some extent, the method is derived from the proof of the Birkhoff-Frink Theorem on $\mathrm{Sub}(\mathfrak{A})$. So we will start the discussion there. But first we need to define some more terms.

Let C be some family of subsets of the set A. C is a <u>closure system</u> iff given any family $(D_i \mid i \in I)$

with $D_i \in C$ for every $i \in I$ it also holds that $\cap(D_i \mid i \in I) \in C$. For $B \subseteq A$ we define the C-closure (or simply, closure) of B by $[B]_C = \cap(D \mid D \in C, B \subseteq D)$. Since $A \in C$, $B \subseteq [B]_C \in C$. B is <u>closed</u> iff $B = [B]_C \in C$. The closure system C is an <u>algebraic closure system</u> iff C is also closed under directed unions; i.e., if the family $(D_i \mid i \in I)$ is a directed partially ordered set (under set inclusion) and each $D_i \in C$, then $\cup(D_i \mid i \in I) \in C$. In an algebraic closure system a set is closed iff it contains the closure of each of its finite subsets. For a regular cardinal $\mathfrak{m}$ one can define an $\mathfrak{m}$-algebraic closure system to be a closure system in which a set is closed iff it contains the closure of each of its subsets having less than $\mathfrak{m}$ elements.

If C is an algebraic closure system, then $\langle C; \subseteq \rangle$ is an algebraic lattice. Conversely, any algebraic lattice is isomorphic to some $\langle C; \subseteq \rangle$ where C is an algebraic closure system. Similar statements hold for $\mathfrak{m}$-algebraic lattices and $\mathfrak{m}$-algebraic closure systems.

Let C be an algebraic closure system on the set A. It is easy to describe a family F of finitary operations on A such that $C = \mathrm{Sub}(\langle A; F \rangle)$. In particular, for each finite sequence $a_0, \ldots, a_n$ of elements of A such that $a_n \in [a_0, \ldots, a_{n-1}]_C$ define an n-ary operation $f_{a_0, \ldots, a_n}$

by $f_{a_0, \ldots, a_n}(a_0, \ldots, a_{n-1}) = a_n$ and $f_{a_0, \ldots, a_n}(x_0, \ldots, x_{n-1}) = x_0$ otherwise. One takes F to be the family of all such operations.

Suppose now you have some algebraic lattice $\mathcal{L}$ that you want to represent as $\mathrm{Sub}(\mathfrak{A} \times \mathfrak{A})$. A first step is to find some algebraic closure system C on a set of the form $B \times B$ where $\mathcal{L}$ is isomorphic to $\langle C; \subseteq \rangle$. Obviously, one then should try the approach from the preceding paragraph. So for each $\langle a_0, b_o \rangle, \ldots, \langle a_n, b_n \rangle$ with $\langle a_n, b_n \rangle \in [\langle a_0, b_o \rangle, \ldots, \langle a_{j-1}, b_{n-1} \rangle]_C$ one defines an operation f on B with $f(a_0, \ldots, a_{n-1}) = a_n$ and $f(b_0, \ldots, b_{n-1}) = b_n$ and $f(x_0, \ldots, x_{n-1}) = x_0$ otherwise. Unfortunately, this doesn't work. Such an f has some unwanted side effects. In particular $f(\langle a_0, c_0 \rangle, \ldots, \langle a_{n-1}, c_{n-1} \rangle) = \langle a_n, c_0 \rangle$ and it may happen of course that $\langle a_n, c_0 \rangle \notin [\langle a_0, c_0 \rangle, \ldots, \langle a_{n-1}, c_{n-1} \rangle]_C$. So one drops the statement "$f(x_0, \ldots, x_{n-1}) = x_0$ otherwise" and leaves f undefined otherwise. One can take B together with these partly defined operations and form a "partial algebra" $\mathfrak{B}$. One can extend $\mathfrak{B}$ to the "algebra freely generated by $\mathfrak{B}$" ($\underset{\sim}{F}(\mathfrak{B})$) by filling in the "tables" for the operations as freely as possible. The subalgebras generated by subsets of $B \times B$ in $\underset{\sim}{F}(\mathfrak{B}) \times \underset{\sim}{F}(\mathfrak{B})$ are "right". But there are many new subsets that don't generate the "right" subalgebras. So add some new partial operations to take care of this. Freely generate. Repeat ad infinitum. Take the direct limit, and call it $\mathfrak{A}$. $\underset{\sim}{\text{Sub}}(\mathfrak{A} \times \mathfrak{A})$ is isomorphic to $\mathcal{L}$. (Actually one must choose the initial C so that the "diagonal" is the smallest member.) (That this works is shown in [6], essentially. See [11] also.)

Now suppose you want an $\mathfrak{A}$ so that $\underset{\sim}{\text{Con}}(\mathfrak{A})$ is isomorphic to the algebraic lattice $\mathcal{L}$. It is easy to check that $\text{Con}(\mathfrak{A})$ is always an algebraic closure system on $A \times A$. So one might look for a set B and some algebraic closure system C on $B \times B$ such that each member of C is an equivalence relation on B and such that $\langle C; \subseteq \rangle$ is isomorphic to $\mathcal{L}$. One could then hope to proceed as in the preceding paragraph. Unfortunately, transitivity rears its ugly head, and that idea doesn't work either. The following modification does work. Given $\langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle$ with $\langle a_n, b_n \rangle \in [\langle a_0, b_0 \rangle, \ldots, \langle a_{n-1}, b_{n-1} \rangle]_C$ one defines three partial operations, say f, g, h, with $f(a_0, \ldots, a_{n-1}) = a_n$, $f(b_0, \ldots, b_{n-1}) = g(b_0, \ldots, b_{n-1})$, $g(a_0, \ldots, a_{n-1}) = h(a_0, \ldots, a_{n-1})$ and $h(b_0, \ldots, b_{n-1}) = b_n$. Now when Θ is a congruence relation with $a_i \equiv b_i\ (\Theta)$ for $0 \leq i \leq n-1$ then under Θ we have $a_n = f(a_0, \ldots, a_{n-1})$ $\equiv f(b_0, \ldots, b_{n-1}) = g(b_0, \ldots, b_{n-1}) \equiv g(a_0, \ldots, a_{n-1})$ $= h(a_0, \ldots, a_{n-1}) \equiv h(b_0, \ldots, b_{n-1}) = b_n$. Transitivity gives us the desired result, $a_n \equiv b_n\ (\Theta)$. Now if one replaces each partial operation of the proceeding paragraph by three partial operations (as in this paragraph), and if one otherwise proceeds as in the preceeding paragraph, one then obtains an algebra $\mathfrak{A}$ with $\underset{\sim}{\text{Con}}(\mathfrak{A})$ isomorphic to $\mathcal{L}$.

Now let us go back to the $\mathfrak{B}$ and the $\underset{\sim}{F}(\mathfrak{B})$ above. Each congruence relation Θ of $\mathfrak{B}$ has an extension $\underset{\sim}{F}(\Theta)$ to a congruence of $\underset{\sim}{F}(\mathfrak{B})$. It is fairly obvious that if the ideas are going to work then one must have $\underset{\sim}{F}(\Theta \cap \Phi) = \underset{\sim}{F}(\Theta) \cap \underset{\sim}{F}(\Phi)$. Unfortunately, this fails in general. This is the technical problem that is cured by using a triple of operations in place of each "natural" operation. This problem is caused by transitivity.

So it becomes important to discover lemmas giving sufficient conditions on a partial algebra $\mathfrak{B}$ so that $\underset{\sim}{F}(\cap(\Theta_i \mid i \in I)) = \cap(\underset{\sim}{F}(\Theta_i) \mid i \in I)$. Such a lemma was implicit in [9]. It was made explicit in both [14] and [21]. But this lemma was true only if $\mathfrak{B}$ was a <u>unary</u> partial algebra. A lemma of this sort for arbitrary finitary partial algebras appears in [17]. This made Theorems 2 and 3 possible. (There are some other innovations required also.)

One would hope that the construction outlined above (when appropriately generalized) would work for proving Theorem 5. It does, but a new proof is required. One of the main new ingredients is a new, mildly complicated lemma giving sufficient conditions on an <u>infinitary</u> partial algebra $\mathfrak{B}$ so that $\underset{\sim}{F}(\cap \Theta_i \mid i \in I) = \cap(\underset{\sim}{F}(\Theta_i \mid i \in F)$ always holds.

The proofs of all the theorems use variations on the above construction.

The reader has probably noticed that the construction outlined above for Theorem 1 gives an algebra $\mathfrak{A}$ having n-ary operations for every $n > 0$. Yet it was stated in §3 that the algebra $\mathfrak{A}$ used in the proof had only unary operations. One can do this by starting with a C such that an equivalence relation Θ is closed iff it contains the closure of its one element subsets. If $\mathcal{L}$ is algebraic, such a C exists. As previously noted, Grätzer and Schmidt were forced to do this because their techniques were valid only for unary partial algebras.

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