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REPRESENTATIONS OF FINITE LATTICES AS PARTITION LATTICES ON FINITE SETS

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§ 0. A lattice is a set with two associative commutative and idempotent binary operations $\vee$ (meet) and $\wedge$ (join) satisfying

$$x \wedge (x \vee y) = x \vee (x \wedge y) = x .$$

We put $x \leq y$ if $x \vee y = y$ and $x < y$ if $x \leq y$ and $x \neq y$. We consider here only lattices L with a least element 0_L and a greatest element 1_L. A sublattice of a lattice L is a subset X of L such that $a \in X$ and $b \in X$ imply that $a \wedge b \in X$ and $a \vee b \in X$. If 0_L and $1_L \in X$, X is called a normal sublattice.

For any set S we denote by $\Pi(S)$ the lattice of partitions on S, that is, the lattice of all equivalence relations on S with $\leq$ defined as set inclusion, relations being treated as sets of ordered pairs. Thus $1_{\Pi(S)} = S \times S$, $0_{\Pi(S)} = \{(x,x): x \in S\}$ and $a \wedge b = a \cap b$ for all $a,b \in \Pi(S)$.

A representation of a lattice L as a lattice of partitions is an isomorphism $\varphi: L \to \Pi(S)$. Then we call φ a representation of L on S. The representation φ is called normal if $\varphi(L)$ is a normal sublattice of $\Pi(S)$. For each lattice L, let $\mu(L)$ be the least cardinal μ such that L has a representation on S, where $|S| = \mu$. Whitman has shown [10] that $\mu(L) \leq \aleph_0 + |L|$. A well-known and still unsolved problem of Birkhoff [2, p. 97] is whether $\mu(L)$ is finite whenever L is finite.

§ 1. For any $x \in \Pi(S)$ and $a,b \in S$ we write $a(x)b$ for $(a,b) \in x$. Let A and B be sets such that $A \cap B = \{v\}$. Let L and M be normal sublattices of $\Pi(A)$ and $\Pi(B)$, respectively. For $x \in L$ and $y \in M$, let $x \circ y$ denote the partition of $A \cup B$ defined by $a(x \circ y)b$ if and only if $a(x)b$ or $a(y)b$ or both $a(x)v$ and $b(y)v$.

<u>Theorem 1</u>. The set N of all partitions of the form $x \circ y$ with $x \in L$ and $y \in M$ is a normal sublattice of $\Pi(A \cup B)$ and this lattice is isomorphic to $L \times M$.

<u>Proof</u>. Clearly the map $\varphi\colon L \times M \to N$ given by $\varphi(x,y) = x \circ y$ is a bijection. We need only establish for all $x,u \in L$ and $y,v \in M$ the equations

(i) $1_{\Pi(A)} \circ 1_{\Pi(B)} = 1_{\Pi(A \cup B)}$,

(ii) $0_{\Pi(A)} \circ 0_{\Pi(B)} = 0_{\Pi(A \cup B)}$,

(iii) $(x \circ y) \vee (u \circ v) = (x \vee u) \circ (y \vee v)$,

(iv) $(x \circ y) \wedge (u \circ v) = (x \wedge u) \circ (y \wedge v)$.

These equations can be proved by examining all possible special cases. In place of (iii) and (iv) it is sufficient to prove the cases

(v) $x \circ y = (x \circ 0_M) \vee (0_L \circ y) = (x \circ 1_M) \wedge (1_L \circ y)$,

$$
\text{(vi)} \quad \begin{cases} (x \circ 0_M) \vee (u \circ 0_M) = (x \vee u) \circ 0_M , \\ (0_L \circ y) \vee (0_L \circ v) = 0_L \circ (y \vee v) , \\ (x \circ 1_M) \wedge (u \circ 1_M) = (x \wedge u) \circ 1_M , \\ (1_L \circ y) \wedge (1_L \circ v) = 1_L \circ (y \wedge v) \end{cases}
$$

which are obvious. We prove (iii) from (v) and (vi) as follows:

$$
\begin{aligned}
(x \circ y) \vee (u \circ v) &= (x \circ 0_M) \vee (0_L \circ y) \vee (u \circ 0_M) \vee (0_L \circ v) \\
&= (x \circ 0_M) \vee (u \circ 0_M) \vee (0_L \circ y) \vee (0_L \circ v) \\
&= ((x \vee u) \circ 0_M) \vee (0_L \circ (y \vee v)) \\
&= (x \vee u) \circ (y \vee v) .
\end{aligned}
$$

The remaining facts are established in a similar way.

Corollary 2. If L is a sublattice of the product of the lattices L_i $(i = 1,\dots,k)$, then

$$
\mu(L) \leq \sum_{i=1}^{k} \mu(L_i) - k + 1 .
$$

Proof. The proof follows directly from Theorem 1 by induction.

Theorem 3. If L is a subdirect product of M and P, if $\mu(M)$ and $\mu(P)$ are finite and if $(0_M, 1_P) \in L$, e.g., $L = M \times P$, then

$$
\mu(L) = \mu(M) + \mu(P) - 1 .
$$

Proof. For each $x \in M$ there exists a $y_x \in P$ such that $(x, y_x) \in L$. Similarly, for each $y \in P$ there exists an $x_y \in M$ such that $(x_y, y) \in L$. Thus for each $x \in M$ and $y \in P$ we have

$(0_M,y) = (0_M,1_P) \wedge (x_y,y) \in L$ and $(x,1_P) = (0_M,1_P) \vee (x,y_x) \in L$.
By Corollary 2, we know that $\mu(L) \le \mu(M) + \mu(P) - 1$. Suppose that φ is a representation of L on a set T with $\mu(L)$ elements. Suppose that $\varphi(0_M,1_P)$ has k equivalence classes $A_1,A_2,\ldots,A_k$ of cardinalities $n_1,n_2,\ldots n_k$. Let P_{A_i} be the lattice of partitions of A_i formed by restricting the elements $\varphi(0_M,y)$ with $y \in P$ to A_i , that is, $P_{A_i} = \{\varphi(0_M,y)|_{A_i} : y \in P\}$. Let $\varphi(y) = (\varphi(0_M,y)|_{A_1}, \varphi(0_M,y)|_{A_2},\ldots,\varphi(0_M,y)_{A_k})$. Then φ is an isomorphism of P into $P_{A_1} \times \ldots \times P_{A_k}$ and thus Corollary 2 yields

$$\mu(P) \le \sum_{i=1}^{k} n_i - k + 1 = \mu(L) - k + 1 .$$

On the other hand, M is isomorphic to $\{(x,1) | x \in M\} \subseteq L$. Thus M can be represented on $T/(\varphi(0_M,1_P))$ (T factored by the equivalence relation $\varphi(0_M,1_P)$), so $k \ge \mu(M)$. Hence

$$\mu(L) \ge \mu(P) + k - 1 \ge \mu(P) + \mu(M) - 1 .$$

<u>Corollary 4</u>. If $\mu(L)$ is finite and L is a sublattice of $\Pi(S)$, where $|S| = \mu(L)$, then L is a normal sublattice. Thus a minimum finite representation is a normal representation.

<u>Proof</u>. Since L can be represented on $S/0_L$, the fact that $\mu(L)$ is minimum implies that $0_L = 0_{\Pi(S)}$. If 1_L has equivalence classes $A_1,A_2,\ldots,A_k$, then L is isomorphic to a sublattice of the product of the L_{A_i} . Corollary 2 gives $\mu(L) \le \sum_{i=1}^{k} |A_i| - k + 1 = \mu(L) - k + 1$, a contradiction unless $k = 1$. Thus $1_L = 1_{\Pi(S)}$.

Remark 1. By Theorem 3, the problem of finding $\mu(L)$ for all finite lattices L reduces to the determination of $\mu(L)$ for all finite directly indecomposable L's. This reduces this problem for various special classes of lattices: Dilworth [3] has shown that every finite relatively complemented lattice is a product of simple lattices. This applies also to finite geometric lattices since they can be characterized as finite relatively complemented semi-modular lattices [2; p. 89]. Birkhoff has shown that every modular geometric lattice is a product of a Boolean algebra and projective geometries [2; § 7]. Dilworth (see [2; p. 97]) has shown that every finite lattice is isomorphic to some sublattice of a finite semi-modular lattice. Hartmanis [5] has shown both that every finite lattice is isomorphic to some sublattice of the lattice of subspaces of a geometry on a finite set and that every finite lattice is isomorphic to the lattice of geometries of a finite set. Jónsson [7] has shown that every finite lattice is isomorphic to a sublattice of a finite subdirectly irreducible lattice.

Remark 2. The assumption $(0_M, 1_P) \in L$ in Theorem 3 is essential. In fact, if C_n is the n-element chain and if $L = C_3 \times C_2$, then Theorem 3 gives $\mu(L) = 4$; however, by Figure 1, L is also isomorphic to a subdirect product of $\Pi(2) \times \Pi(2)$ and $\Pi(2) \times \Pi(2)$, which would lead to $\mu(L) = 5$ if Theorem 3 applied.

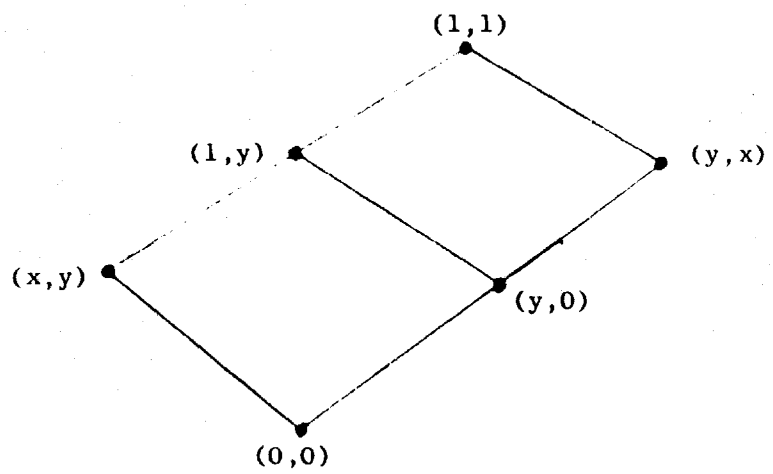

Figure 1.

Remark 3. Let $L \lhd \Pi(n)$ mean that L has a normal representation on n. Theorem 1 shows that $\Pi(\ell) \times \Pi(\ell) \lhd \Pi(2\ell - 1)$. Since $\Pi(\ell) \lhd \Pi(\ell) \times \Pi(\ell)$, this suggests the question: For what ℓ and m is $\Pi(\ell) \lhd \Pi(m)$? If $\Pi(\ell) \lhd \Pi(\ell_1)$ and $\Pi(\ell) \lhd \Pi(\ell_2)$, then $\Pi(\ell) \lhd \Pi(\ell_1 + \ell_2 - 1)$. Since $\Pi(3) \lhd \Pi(4)$, we have $\Pi(3) \lhd \Pi(m)$ for all $m \geq 3$. Ralph McKenzie has proved (private communication) that $\Pi(\ell) \lhd \Pi(\ell + 1)$ does not hold for $\ell \geq 4$.

§ 2. We now examine μ for some special lattices. We recall that by a complement of x in a lattice L is meant an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$.

Lemma 5. If $P_1, P_2, \ldots P_k$ and Q are partitions of a set S with n elements and $P_1 \vee \ldots \vee P_k = Q$, then $\sum_{i=1}^{k} |S/P_i| \leq n(k-1) + |S/Q|$ in addition, $P_i \vee P_j = Q$ for all $i \neq j$, then $\sum_{i=1}^{k} |S/P_i| \leq \frac{k}{2}(n + |S/Q|)$.

Proof. For every $A \in S/P_i$ $(i = 1,2,\ldots,k)$ form a path through all points of A. Thus S obtains a graph structure and by $P_1 \vee \cdots \vee P_k = Q$, this graph has $\ell = |S/Q|$ connected components containing, in some order, $n_1, n_2, \ldots, n_\ell$ points. Since a connected graph with m points has at least $m-1$ edges,

$$\sum_{i=1}^{k} \sum_{A \in S/P_i} (|A| - 1) \geq \sum_{j=1}^{\ell} (n_j - 1) \ ;$$

$$\sum_{i=1}^{k} \left(\sum_{A \in S/P_i} |A| - |S/P_i| \right) \geq n - \ell \ ;$$

$$\sum_{i=1}^{k} (|S| - |S/P_i|) \geq n - |S/Q| \ ;$$

$$kn - \sum_{i=1}^{k} |S/P_i| \geq n - |S/Q| \ ;$$

$$\sum_{i=1}^{k} |S/P_i| \leq n(k-1) + |S/Q| \ .$$

Now suppose $P_i \vee P_j = Q$ for all $i \neq j$. Then by the last equation with $k = 2$, for all $i \neq j$, $|S/P_i| + |S/P_j| \leq n + |S/Q|$. Hence we have

$$(k-1) \sum_{i=1}^{k} |S/P_i| = \sum_{i \neq j} (|S/P_i| + |S/P_j|)$$

$$= \binom{k}{2} (n + |S/Q|) \ .$$

The lemma follows.

Theorem 6. Consider the lattice $L(\ell,m)$ consisting of 0 and 1 and of two chains $P_1 > \dots > P_\ell$ of length ℓ and the other $Q_1 > \dots > Q_m$ of length m, such that P_i and Q_j are complementary for all i and j (see Figure 2). If $\ell > 1$, then

$$\mu(L(\ell,m)) = \ell + m - 1 + \{2\sqrt{\ell + m - 2}\} .$$

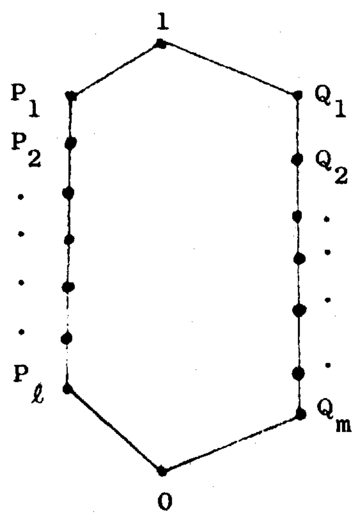

Figure 2.

Proof. Here, the symbol $\{x\}$ denotes the least integer not less than x. We suppose that $k = |P_1| \le |Q_1|$. Then $|P_\ell| \ge |P_1| + \ell - 1$ and $|Q_m| \ge |Q_1| + m - 1$. By Lemma 5, if $\mu(L) = n$, then

$$n + 1 \ge |P_\ell| + |Q_m| \ge |P_1| + \ell - 1 + |Q_1| + m - 1 .$$

Letting $x = \ell + m$, we have

$$k \le |Q_1| \le n + 3 - k - x .$$

Since $P_1 \wedge Q_1 = 0$, no class of Q_1 can have more than k elements. Thus

$$n \le k|Q_1| \le k(n + 3 - k - x) .$$

Since the maximum of the right hand side of this equation occurs when $k = \frac{1}{2}(n+3-x)$,

$$n \le \left(\frac{n+3-x}{2}\right)^2 .$$

Solving this equation, we find that

$$n \ge x - 1 + 2\sqrt{x-2} .$$

We first demonstrate a representation of $L(\ell,1)$. Let k be the first integer such that $k^2 \ge \ell + 2\sqrt{\ell-1}$ $(k = 1 + \{\sqrt{\ell-1}\})$. Let n be the initial segment of length $\ell + \{2\sqrt{\ell-1}\}$ in the lexicographic ordering on $Z_k \times Z_k$. The partition P_1 on n is defined by $((x,y),(u,v)) \in P_1$ if and only if $x = u$. The partition Q_1 on n is defined by $((x,y),(u,v)) \in Q_1$ if and only if $y = v$. (Note that $\ell \ge 2$ implies that $k \ge 4$ and thus $P_1 \ne Q_1$.) The partition P_ℓ is defined by $((s,y),(u,v)) \in P_\ell$ if and only if either $x = 0 = u$ or $(x,y) = (u,v)$. The partitions P_i with $1 < i < \ell$ are formed by interpolation between P_1 and P_ℓ (separating off each of the singletons in P_ℓ one at a time from P_1). We must verify that a sufficient number of partitions can be formed in this way. Since $|P_\ell| = n - k + 1$ and $|P_1| = \{\frac{n}{k}\}$, if all possible interpolations were made, the length of the chain from P_1 to P_ℓ would be

$$p = n - k + 1 - \{\tfrac{n}{k}\} + 1 .$$

If $\{\frac{n}{k}\} \le k - 1$, we have

$$p \ge \ell + 1 + \{2\sqrt{\ell-1}\} - 2\{\sqrt{\ell-1}\} \ge \ell .$$

If $\{\frac{n}{k}\} = k$, we have

$$p = \ell + \{2\sqrt{\ell-1}\} - 2\{\sqrt{\ell-1}\} .$$

Suppose $s < \sqrt{\ell - 1} \leq s + \frac{1}{2}$ for some integer s. Then $n = \ell + 2s + 1$, $k = s + 2$ and $\ell \leq s^2 + s + \frac{5}{4}$. Since ℓ is an integer, $\ell \leq s^2 + s + 1$ and thus

$$n \leq s^2 + 3s + 2 = k(k-1) .$$

This gives $\{\frac{n}{k}\} = k - 1$, a contradiction. Thus $\{2\sqrt{\ell - 1}\} = 2\{\sqrt{\ell - 1}\}$ and hence $p = \ell$.

To complete the proof, we show that $L(\ell - 1, m+1)$ can be represented on the same set as $L(\ell, m)$. Suppose $\ell \geq 2$ and $P_\ell \in L(\ell, m)$ has classes C_i, $1 \leq i \leq n$. Since $P_{\ell-1} \geq P_\ell$, we may assume that $P_{\ell-1}$ has a class containing $C_1 \cup C_2$. Since $P_{\ell-1} \wedge Q_m = 0$, for every $x \in C_1$ and $y \in C_2$, $(x,y) \notin Q_m$. Consider a shortest $P_\ell - Q_m$ path $x_1 x_2 \ldots x_n$ $(n \geq 3)$ from C_1 to C_2. Then $x_1 \in C_1$ and $x_n \in C_2$ but $x_i \notin C_1 \cup C_2$, $2 \leq i < n$. Thus $(x_1, x_2) \in Q_m$. Let $Q_{m+1} \leq Q_m$ be the partition defined by: for all $x, y \neq x_1$, $(x,y) \in Q_{m+1}$ if and only if $(x,y) \in Q_m$; for all x, $(x, x_1) \in Q_{m+1}$ if and only if $x = x_1$. To show $P_{\ell-1} \vee Q_{m+1} = 1$, we need only show $(x_1, x_2) \in P_{\ell-1} \vee Q_{m+1}$ for then $P_{\ell-1} \vee Q_{m+1} \geq P_{\ell-1} \vee Q_m = 1$. Since the $P_\ell - Q_m$ path $x_2 \ldots x_n$ does not contain x_1, it is a $P_\ell - Q_{m+1}$ path. Since $(x_n, x_1) \in P_{\ell-1}$, $x_2 \ldots x_n x_1$ is a $P_{\ell-1} - Q_{m+1}$ path from x_2 to x_1.

We now consider the lattice L_n of subspaces of the geometry G_n with n points and 1 line. L_n consists of n mutually complementary elements and 0 and 1 (see Figure 3). Hartmanis [6] has shown that $\mu(L_n) \leq 2p$ where p is the first prime larger than n. We shall prove $\mu(L_n) \leq p$, where p is the first prime not less than n (see Theorems 7, 8 and 9 below).

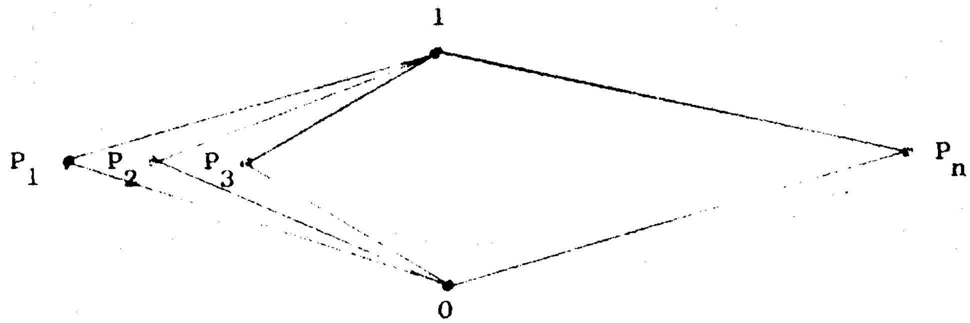

Figure 3.

Theorem 7.

$$\mu(L_n) \geq \begin{cases} n+1 \ ; & n \text{ even} \\ n \ ; & n \text{ odd} . \end{cases}$$

Proof. Suppose L_n can be represented as a sublattice L of the lattice of partitions of m. Each non-trivial $P \in L$ defines a set of edges $L_P = \{\{a,b\}: (a,b) \in P, \ a \neq b\}$. Since $P \wedge Q = 0$ and $P \vee Q = 1$ when $P \neq Q$, we have that $L_P \cup L_Q$ is a connected graph. Thus

(i) $$|L_P| + |L_Q| = |L_P \cup L_Q| \geq m-1 ,$$

(ii) $$\sum_{P \in L} |L_P| \leq \frac{1}{2} m(m-1) .$$

From (i) we get

$$(n-1) \sum_{P \in L} |L_P| = \sum_{P \neq Q} (|L_P| + |L_Q|) \geq \frac{n(n-1)}{2} (m-1) .$$

Hence from (ii),

$$\frac{1}{2}\, m(m-1) \geq \sum_{P \in L} |L_P| \geq \frac{1}{2}\, n(m-1)$$

which yields $m \geq n$. Equality can occur only if $|L_P| + |L_Q| = m-1$ for all non-trivial $P \neq Q \in L$, which implies that $m-1$ is even whenever $m = n = 3$. Small cases are handled by inspection.

Theorem 8. The following four statements are equivalent:

(i) $\mu(L_{2n-1}) = 2n-1$;

(ii) The complete graph on $2n-1$ points, K_{2n-1}, can be edge-colored with $2n-1$ colors so that the union of any two color classes is a spanning path;

(iii) K_{2n} can be edge-colored with $2n-1$ colors so that the union of any two color classes is a spanning cycle;

(iv) The symmetric group on $2n$ elements, S_{2n}, contains a set $\{I_i : i = 1,2,\ldots,2n-1\}$ of involutions such that the group generated by I_i and I_j is transitive whenever $i \neq j$.

Proof. (i) $\leftrightarrow$ (ii). If we assume (ii), each color class is a partition, so (i) follows easily. Suppose (i) holds. As we have seen above $|L_P \cup L_Q| = 2n-2$ for all $P \neq Q$. Since $L_P \cup L_Q$ is connected, it must be a tree. Thus $|L_P| = n-1$ and L_P contains no cycles, that is, P is a maximum matching of the points of K_{2n-1}. (ii) now follows.

(ii) $\leftrightarrow$ (iii). Suppose K_{2n} has been $(2n-1)$ edge-colored so that the union of any two color classes is a spanning cyle. Clearly $K_{2n} \setminus \{v\}$ satisfies (ii). On the other hand, if K_{2n-1} has been $(2n-1)$ edge-colored so that the union of two color classes is a spanning path, each point misses one color and, by counting, each color misses one point. $K_{2n} = K_{2n-1} \cup \{\{v,a\} : a \in K_{2n-1}\}$ is $2n-1$ edge-colored by coloring $\{v,a\}$, $a \in K_{2n-1}$, with the color missing at a. It is easy to show that this coloring satisfies (iii).

(iii) $\leftrightarrow$ (iv). Each 1-factor of K_{2n} defines an involution on $2n$ and vice versa. Since the elements of the group generated by the involutions I and J have the form $\ldots IJIJ \ldots$, the union of two 1-factors spans K_{2n} if and only if the group generated by the corresponding involutions is transitive.

Theorem 9. The statement 8 (i) holds if n (see [1] and [8]) or $2n-1$ (see [1] and [9]) is a prime.

Remark 4. B. A. Anderson (private communication) has also shown that 8 (i) holds for $n = 8$ and $n = 14$. Thus the first unknown case is $n = 18$. We would like to know a similar result to Theorem 6 about a lattice $L(\ell_1, \ell_2, \ldots, \ell_w)$ consisting of 0 and 1 and of w chains $P_{i1} > \ldots > P_{i\ell_i}$, $1 \le i \le w$, such that P_{ij} and $P_{i'j'}$ are complementary when $i \ne i'$. However, the method of proof used in Theorem 6 gives only $\mu(L(\ell_1, \ldots, \ell_w)) \ge f(\bar{\ell}, w)$ where $\bar{\ell} = w^{-1} \sum_{i=1}^{w} \ell_i$ and

$$f(\bar{\ell}, w) = 2\bar{\ell} - 3 + 8\,\frac{w-1}{w^2} + 4\,\frac{\sqrt{w-1}}{w^2}\,\sqrt{4 + w^2(2\ell - 3)}\ .$$

Although this reduces to Theorem 6 when $w = 2$, for large values of w it is a very bad estimate since $\lim_{w \to \infty} f(\bar{\ell}, w) = 2\bar{\ell} - 3$, an absurdity. Actually, proofs of this type seem to indicate that the best results for these lattices are obtained by partitions with nearly equal classes. For this reason, we mention the following theorem.

Theorem 10. L_{k+2} has a normal representation $\varphi: L_{k+2} \to \Pi(S)$, where $|S| = n^2$ such that $|S/\varphi(a)| = n$ and $|A| = n$ for each $A \in S/\varphi(a)$ whenever $a \in L_{k+2}$, $a \neq 0_{L_{k+2}}, 1_{L_{k+2}}$, if and only if there are k mutually orthogonal Latin squares of order n.

Proof. Suppose L exists. Let the partitions be $C_i = \{C_{i1}, \ldots, C_{in}\}$, $1 \leq i \leq k$, $A = \{A_1, \ldots, A_n\}$, and $B = \{B_1, \ldots, B_n\}$. We form the Latin square $L^i_{\ell m}$ as follows: let $L^i_{\ell m} = j$ if $C_{ij} \cap A_\ell \cap B_m \neq \phi$. The definition is possible since $A_\ell \cap B_m = \{x_{\ell m}\}$ for all ℓ and m, and given i, some C_{ij} must contain $x_{\ell m}$. Suppose $L^i_{\ell m} = L^i_{\ell' m} = j$. Then $C_{ij} \cap A_\ell \cap B_m \neq \phi$ and $C_{ij} \cap A_{\ell'} \cap B_m \neq \phi$, contradicting $A_\ell \cap A_{\ell'} = \phi$ unless $\ell = \ell'$. Similarly $L^i_{\ell m} = L^i_{\ell m'}$ if and only if $m = m'$. Thus $L^i_{\ell m}$ is a Latin square. Suppose $L^i_{\ell m} = L^i_{rs} = p$ and $L^j_{\ell m} = L^j_{rs} = q$ with $i \neq j$. Then

$$\begin{cases} C_{ip} \cap A_\ell \cap B_m = \{x_{\ell m}\} \\ C_{ip} \cap A_r \cap B_s = \{x_{rs}\} \\ C_{jq} \cap A_\ell \cap B_m = \{x_{\ell m}\} \\ C_{jq} \cap A_r \cap B_m = \{x_{rs}\} \ . \end{cases}$$

Thus $C_{jq} \cap C_{ip} = \{x_{\ell m}\} = \{x_{rs}\}$, so $\ell = r$ and $m = s$. Hence the $L^i_{\ell m}$ are mutually orthogonal Latin squares.

Conversely, suppose $\{L^i_{\ell m}\}_{i=1}^k$ is a set of mutually orthogonal Latin squares. We consider the n^2 elements in $\mathbb{Z}_n \times \mathbb{Z}_n$. We let $A_i = \{i\} \times \mathbb{Z}_n$ and $B_j = \mathbb{Z}_n \times \{j\}$. We put $(\ell, m) \in C_{ij}$ if and only if

$L^i_{\ell m} = j$. It is easily verified that the partitions $C_i = \{C_{i1},\ldots,C_{in}\}$, $1 \le i \le k$, $A = \{A_1,\ldots,A_n\}$, and $B = \{B_1,\ldots,B_n\}$ generate the desired lattice.

<u>Corollary 11</u>. (See [4; p. 177]). The following statements are equivalent:

(i) The edges of the complete graph K_{n^2} on n^2 points can be decomposed into $n+1$ sets so that each set consists of n components isomorphic to K_n and so that the union of any two sets is a connected graph.

(ii) There exists a projective plane P_n of order n .

(iii) There are $n-1$ mutually orthogonal Latin squares of order n .

(iv) There is a partition lattice L on n^2 elements consisting of $n+1$ mutually complementary elements plus 0 and 1 such that each non-trivial partition has n classes of n elements.

<u>Proof</u>. We shall sketch the proof. The equivalence of (i) and (iv) follows from the method used in the proof of Theorem 7. That is, to each partition $P \neq 0,1$ in L there corresponds a set of edges $L_P = \{\{a,b\}: (a,b) \in P\}$. (Note that each of these partitions turns out to be nothing more than a parallel class of lines in an affine geometry.) The equivalence of (iii) and (iv) follows from the theorem. The proof of the equivalence of (i) and (ii) follows standard lines: Suppose (i) holds. To form P_n add to the points of K_{n^2} the points $c_1,\ldots,c_{n+1}$, corresponding to the $n+1$ sets $C_1,\ldots,C_{n+1}$. We suppose the components of C_i are $C_{i1},\ldots,C_{in}$. The lines of P_n are then the sets $C_{ij} \cup \{c_i\}$, $i = 1,\ldots,n+1$, and the set $\{c_1,\ldots,c_{n+1}\}$. Conversely, if (ii) holds, let $\{c_1,\ldots,c_{n+1}\}$ be a

line in P_n . The points of K_{n^2} are then the points of $P_n \setminus \{c_1, \ldots, c_{n+1}\}$. The edge $\{x,y\}$ of K_{n^2} is in the set c_i if x , y and c_i are colinear in P_n .

§ 3 . By Whitman's Theorem (see § 0), every lattice is a sublattice of the lattice of all partitions of some set. If φ is a representation of a lattice L as a lattice of partitions of A , and B is a subset of A , then for every $x \in L$ let $\varphi_B(x)$ be the restriction of the partition $\varphi(x)$ to B . Of course, $\varphi_B(L)$ does not necessarily have to be a sublattice of L . Even if $\varphi_B(L)$ is a sublattice, φ_B does not have to be an isomorphism. If $\varphi_B(L)$ is a sublattice and φ_B is an isomorphism, then the subset B is called faithful.

Remark 5. Every representation of the lattice L_2 has a finite faithful subset. The simplest example of a finite lattice which has a representation without finite faithful subsets is L_3 . The representation is constructed as follows: the points of the set are the vertices of the regular triangular lattice on the plane. Three points form an equivalence class with respect to a given color if they are the vertices of a triangle which has this color (see Figure 4). It is clear that if we take any

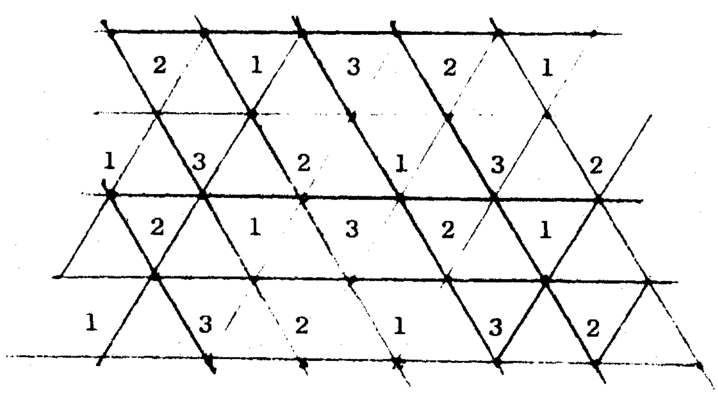

Figure 4.

finite subset S of this triangulation, there will be at least one vertex which appears in only one colored triangle, say color 1. Thus this vertex is not $2 \vee 3$ equivalent to any other, so S cannot be a faithful subset. We can also show that the lattice of Figure 5 has a representation without finite faithful subsets.

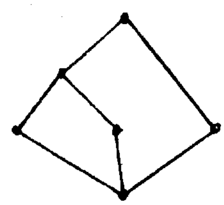

Figure 5.

There exists a finite distributive lattice with a representation without finite faithful subsets. The lattice generated by the partitions induced by the colors 1, 2 and 3 in Figure 6 is isomorphic to $\{0,1\}^3$.

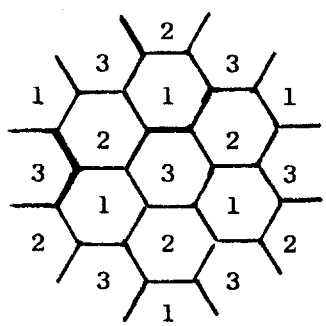

Figure 6.

The lattice L in Figure 7 is a finite lattice with an infinite representation without proper faithful subsets. Partitions A, B, A_1 and B_1 of $\mathbb{Z}$ are formed as follows: A has classes $\{2n, 2n+1\}$ for all $n \in \mathbb{Z}$, B has classes $\{2n-1, 2n\}$ for all n, A_1 has classes $\{2n-1, 2n+4\}$ for all n, and B_1 has classes $\{2n+2, 2n-1\}$

for all n. It is clear that these partitions generate a lattice isomorphic to L. For any proper subset of $\mathbb{Z}$, one of the relations $A \vee B = 1$, $A_1 \vee B_1 = 1$ would fail, so this representation of L has no proper faithful subsets.

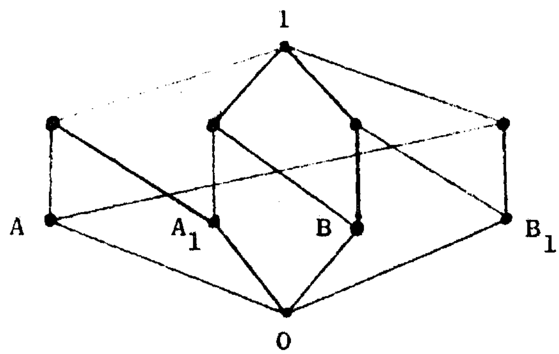

Figure 7.

Problems.

1. Suppose $P \subsetneq Q$ are lattices and P has a representation without finite faithful subsets. Does Q have such a representation? Can a given representation φ of P without finite faithful subsets be extended to a representation $\bar{\varphi}$ of Q such that $\bar{\varphi}$ also does not have finite faithful subsets?

2. Characterize the class of lattices which can be generated by colorings of tesselations of the plane.

3. (See Remark 3.) For what ℓ and m is $\Pi(\ell) \lhd \Pi(m)$?

4. (See Theorems 7, 8 and 9 and [1], [8] and [9].) Find $\mu(L_n)$ for all n.

5. (See Remark 4.) Find $\mu(L(\ell_1, \ell_2, \ldots, \ell_w))$ for all w-tuples of positive integers $(\ell_1, \ell_2, \ldots, \ell_w)$.

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