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Disjointness conditions in free products of distributive lattices: An application of Ramsay's theorem.

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1. Introduction. Let L be a lattice. We say that L satisfies the finite disjointness condition if, given any $a \in L$ and any subset $S \subseteq L$ such that $a \notin S$ and such that $x \wedge y = a$ for any distinct $x, y \in S$, it then follows that S is finite. Similarly we say that L satisfies the countable disjointness condition if the above hypotheses imply that S is countable (rather than actually finite). It has long been known that any free Boolean algebra satisfies the countable disjointness condition -- see e.g. R. Sikorski [6], §20, Example L), on page 72, where the countable disjointness condition is called the σ-chain condition. R. Balbes [1] proved that any free distributive lattice satisfies the finite disjointness condition.

In this paper we extend these results to free products in the category $\mathcal{D}$ of distributive lattices and in the category $\mathcal{D}_b$ whose objects are bounded distributive lattices and whose morphisms preserve the bounds. Clearly any free distributive lattice is the free product in $\mathcal{D}$ of a family of one-element lattices, and it is well-known (see [3]) that the

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free Boolean algebra, regarded as a bounded lattice, is the free product in $\mathcal{D}_b$ of a family of four-element lattices. We then generalize the above disjointness conditions by proving the following theorem.

Let $(L_i \mid i \in I)$ *be a family of lattices in* $\mathcal{D}$ (*resp. in* $\mathcal{D}_b$) *and, for each* $i \in I$, *let* L_i *satisfy the finite disjointness condition. Then the free product of the family* $(L_i \mid i \in I)$ *in* $\mathcal{D}$ (*resp. in* $\mathcal{D}_b$) *satisfies the finite disjointness condition (resp. the countable disjointness condition)*.

I should like to thank G. Grätzer and A. Hajnal for many helpful conversations regarding the subject matter of this paper.

2. *The word problem.* To accomplish our aim we shall need a characterization of comparability of elements in the free product in $\mathcal{D}$ and in $\mathcal{D}_b$. Let $(L_i \mid i \in I)$ be a family of lattices in $\mathcal{D}$ or $\mathcal{D}_b$ and let L be the free product of $(L_i \mid i \in I)$ in the appropriate category. We take the point of view that each L_i is a sublattice of L ; it follows that in $\mathcal{D}$ $L_i \cap L_j = \emptyset$ whenever $i \neq j$, and that in $\mathcal{D}_b$ $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$. As usual, 0 denotes the lower bound in $\mathcal{D}_b$ and 1 denotes the upper bound. We denote by P the subset $\bigcup(L_i \mid i \in I)$ of L . Note that, in $\mathcal{D}$, if $x, y \in P$ and $x \leq y$ then there is a unique $i \in I$ such that $x, y \in L_i$ and, clearly, $x \leq y$ in that L_i . Similarly, in $\mathcal{D}_b$, if $x, y \in P$ and $x \leq y$ then either $x = 0$ or $y = 1$ or there is a unique $i \in I$ such that $x, y \in L_i$ (and $x \leq y$ in that L_i) .

Since L is distributive, each $a \in L$ can be expressed in the form $\bigvee(\bigwedge X \mid X \in J)$ where J is finite and nonempty, and each $X \in J$ is a finite nonempty subset of P (2). We can always choose each such X to be <u>reduced</u>, that is, to satisfy $|X \cap L_i| \leq 1$ for all $i \in I$, where $|A|$ denotes the cardinality of the set A. In addition, the term "reduced" will be used only for nonempty sets. Note that in $\mathfrak{D}_b$ if X is reduced and $0 \in X$ then $X = \{0\}$, and similarly for 1.

Any element of L can also be expressed in the dual form $\bigwedge(\bigvee X \mid X \in J)$, J finite and each X reduced.

<u>LEMMA 1.</u> <u>Let</u> X, Y <u>be reduced subsets of</u> P. <u>In either category</u> $\mathfrak{D}$ or $\mathfrak{D}_b$, $\bigwedge X \leq \bigvee Y$ <u>if and only if there are elements</u> $x \in X$ <u>and</u> $y \in Y$ <u>such that</u> $x \leq y$.

<u>Proof.</u> Assume that for each $\langle x, y\rangle \in X \times Y$, $x \nleq y$. Observe first that $0 \notin X$, $1 \notin Y$ if we are in $\mathfrak{D}_b$. In the remainder of the proof it is irrelevant whether we are in $\mathfrak{D}$ or in $\mathfrak{D}_b$. Let

$$I_1 = \{i \in I \mid |X \cap L_i| = 1,\ |Y \cap L_i| = 0\}$$

$$I_2 = \{i \in I \mid |X \cap L_i| = 0,\ |Y \cap L_i| = 1\}$$

$$I_3 = \{i \in I \mid |X \cap L_i| = |Y \cap L_i| = 1\}$$

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(2) This notation is preferable for our purpose to the equivalent double index notation
$a = (x_1^1 \wedge \cdots \wedge x_1^{n_1}) \vee (x_2^1 \wedge \cdots \wedge x_2^{n_2}) \vee \cdots \vee (x_k^1 \wedge \cdots \wedge x_k^{n_k})$, $x_i^j \in P$.

Let $\underset{\sim}{2}$ be the two-element lattice $\{0, 1\}$ with $0 < 1$. For each $i \in I$ we define a homomorphism $\varphi_i : L_i \to \underset{\sim}{2}$ using the Prime Ideal Theorem:

If $i \in I - (I_1 \cup I_2 \cup I_3)$ φ_i is arbitrary.

If $i \in I_1$, let $x\varphi_i = 1$ where $X \cap L_i = \{x\}$. (This is clearly possible in $\mathfrak{D}$ by taking the constant $L_i \to \underset{\sim}{2}$. In $\mathfrak{D}_b$ we note that $x \neq 0$ and so by the Prime Ideal Theorem we can take $0\varphi_i = 0$, $x\varphi_i = 1$, and, perforce, $1\varphi_i = 1$.)

Similarly, if $i \in I_2$, let $y\varphi_i = 0$ where $Y \cap L_i = \{y\}$.

If $i \in I_3$, let $X \cap L_i = \{x\}$, $Y \cap L_i = \{y\}$. Since $x \nleq y$, we can define φ_i so that $x\varphi_i = 1$, $y\varphi_i = 0$.

The family of homomorphisms $(\varphi_i \mid i \in I)$ then extends to a homomorphism $\varphi: L \to \underset{\sim}{2}$ such that $x\varphi = 1$ for all $x \in X$ and $y\varphi = 0$ for all $y \in Y$. Thus $(\bigvee Y)\varphi = 0 < 1 = (\bigwedge X)\varphi$, showing that $\bigwedge X \nleq \bigvee Y$, and proving the lemma.

A more complete treatment of the word problem can be found in Grätzer and Lakser [3].

3. <u>The finite disjointness condition in</u> $\mathfrak{D}$. If Γ is any set we denote the diagonal $\{\langle\gamma, \gamma\rangle \in \Gamma \times \Gamma\}$ by ω_Γ. We first recall the classic result of Ramsay in the following form:

<u>LEMMA 2</u> (Ramsay's Theorem). <u>Let</u> Γ <u>be an infinite set and let</u> $R_1, \cdots, R_n$ <u>be binary symmetric relations on</u> Γ <u>such that</u> $\omega_\Gamma \cup R_1 \cup \cdots \cup R_n = \Gamma \times \Gamma$.

Then there is a subset $\Gamma' \subseteq \Gamma$ *and an* $i \leq n$ *such that*

(i) *for any distinct* $\alpha, \beta \in \Gamma'$, $\langle \alpha, \beta \rangle \in R_i$;

and

(ii) Γ' *is infinite*.

For our purposes the following alternative characterization of the finite and countable disjointness conditions is preferable.

LEMMA 3. *A distributive lattice* L *satisfies the finite (resp. countable) disjointness condition if and only if the following condition holds*.

Given any $a \in L$ *and any subset* $S \subseteq L$ *such that* $x \nleq a$ *for all* $x \in S$ *and such that* $x \wedge y \leq a$ *for distinct* $x, y \in S$, *it then follows that* S *is finite (resp. countable)*.

Proof. The proof follows immediately by observing that if S satisfies the condition of the lemma then

(i) $x \vee a > a$ for all $x \in S$;

(ii) If $x, y \in S$ are distinct then

$(x \vee a) \wedge (y \vee a) = (x \wedge y) \vee a = a$ (and so the correspondence $x \to x \vee a$ from S to $\{x \vee a \mid x \in S\}$ is one-to-one).

THEOREM 1. *Let* $(L_i \mid i \in I)$ *be a family of lattices in* $\mathcal{D}$ *satisfying the finite disjointness condition*. *Then* L, *the free product in* $\mathcal{D}$, *also satisfies the finite disjointness condition*.

Proof. Let $a \in L$ and let $(s_\gamma \mid \gamma \in \Gamma)$ be any family of elements of L such that

(A) for each $\gamma \in \Gamma$, $s_\gamma \nleq a$;

and

(B) if $\alpha, \beta \in \Gamma$ are distinct then $s_\alpha \wedge s_\beta \leq a$.

We show that Γ must be finite by proving a sequence of statements involving successively weaker hypotheses about the form of the s_γ and of a .

Statement 1. If $a \in P$ and $s_\gamma \in P$ for all $\gamma \in \Gamma$ then Γ is finite.

Let $a \in L_i$ for some $i \in I$ and let α, β be distinct elements of γ . Then, since $s_\alpha \wedge s_\beta \leq a$, it follows that $s_\alpha, s_\beta \in L_i$ by Lemma 1 and condition (A). Thus $\{s_\gamma \mid \gamma \in \Gamma\} \subseteq L_i$ also and perforce Γ is finite since L_i satisfies the finite disjointness condition.

Statement 2. If $a \in P$ and $s_\gamma = \bigwedge X_\gamma$ for each $\gamma \in \Gamma$ where X_γ is a reduced subset of P then Γ is finite.

For each $\gamma \in \Gamma$ and each $x \in X_\gamma$, $x \nleq a$ by Lemma 1 and (A). Let $a \in L_i$. By (B) if $\alpha, \beta \in \Gamma$ are distinct $\bigwedge X_\alpha \wedge \bigwedge X_\beta \leq a$. There are thus $x \in X_\alpha \cap L_i$, $y \in X_\beta \cap L_i$ such that $x \wedge y \leq a$. But $|X_\gamma \cap L_i| \leq 1$ for all $\gamma \in \Gamma$. Thus we have a family $(x_\gamma \mid \gamma \in \Gamma)$ such that $x_\gamma \in L_i$ for all $\gamma \in \Gamma$, such that $x_\gamma \nleq a$ for all $\gamma \in \Gamma$ and such that $x_\alpha \wedge x_\beta \leq a$ for distinct α, β . Thus, by Statement 1, Γ is finite.

Statement 3. If $s_\gamma = \bigwedge X_\gamma$, X_γ reduced, for each γ , and if $a = \bigvee Y$, Y reduced, then Γ is finite.

Let $Y = \{y_1, \cdots, y_p\}$. Then for each $j \le p$ and each $\gamma \in \Gamma$ $\bigwedge X_\gamma \nleq y_j$, by (A). Define binary relations $R_1, \cdots, R_p$ on Γ by setting $\langle \alpha, \beta \rangle \in R_j$ if and only if $\bigwedge X_\alpha \wedge \bigwedge X_\beta \le y_j$. Since, for any distinct $\alpha, \beta \in \Gamma$, $\bigwedge X_\alpha \wedge \bigwedge X_\beta \le \bigvee Y$ it follows, by Lemma 1, that $\omega_\Gamma \cup R_1 \cup \cdots \cup R_p = \Gamma \times \Gamma$. Now let $j \le p$ and let Γ' be a subset of Γ such that $\langle \alpha, \beta \rangle \in R_j$ for any two distinct $\alpha, \beta \in \Gamma'$. Then, by Statement 2, Γ' is finite. Thus, by Ramsay's Theorem, Γ is finite.

Statement 4. If $a = \bigvee Y_1 \wedge \cdots \wedge \bigvee Y_r$ where each Y_j is a reduced subset of P and if, for each $\gamma \in \Gamma$, $s_\gamma = \bigvee(\bigwedge X \mid X \in J_\gamma)$ for some finite nonempty set J_γ of reduced subsets of P , then Γ is finite.

Since for each $\gamma \in \Gamma$ $s_\gamma \nleq a$ then for each $\gamma \in \Gamma$ there is an $X_\gamma \in J_\gamma$ and a $j(\gamma) \le r$ such that $\bigwedge X_\gamma \nleq \bigvee Y_{j(\gamma)}$. For each $j \le r$ let let $\Gamma_j = \{\gamma \in \Gamma \mid j(\gamma) = j\}$. Then if α, β are distinct elements of Γ_j , $\bigwedge X_\alpha \wedge \bigwedge X_\beta \le s_\alpha \wedge s_\beta \le a \le \bigvee Y_j$. But, by definition of Γ_j , $\bigwedge X_\gamma \nleq \bigvee Y_j$ if $\gamma \in \Gamma_j$. Thus, by Statement 3, Γ_j is finite. It thus follows that $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_r$ is finite, proving Statement 4.

Since each element of L can be expressed in both forms $\bigvee(\bigwedge X \mid X \in J)$ and $\bigwedge(\bigvee Y \mid Y \in K)$, Statement 4 is the statement of the theorem.

4. *The countable disjointness condition in* $\mathcal{D}_b$. The situations in $\mathcal{D}$ and in $\mathcal{D}_b$ differ essentially because of the following fact. In $\mathcal{D}$, if $x, y \in L_i$, if $z \in L_j$, and if $x \wedge y \leq z$ then $i = j$. In $\mathcal{D}_b$, however, it is possible that $i \neq j$; if $z \neq 1$ then $x \wedge y \leq z$ if and only if $x \wedge y = 0$. It is precisely this difference which yields the countable disjointness condition only, rather than finite disjointness. We will also need a more delicate analysis since the argument establishing Statement 2 of Theorem 1 does not apply in $\mathcal{D}_b$ precisely because of this difference.

THEOREM 2. Let $(L_i \mid i \in I)$ *be a family of lattices in* $\mathcal{D}_b$ *satisfying the finite disjointness condition. Then* L, *the free product in* $\mathcal{D}_b$, *satisfies the countable disjointness condition.*

Proof. Let $a \in L$ and let $(s_\gamma \mid \gamma \in \Gamma)$ be any family of elements of L such that

(A) for each $\gamma \in \Gamma$, $s_\gamma \not\leq a$;

and

(B) if $\alpha, \beta \in \Gamma$ are distinct then $s_\alpha \wedge s_\beta \leq a$.

We show that Γ is countable by proving a sequence of statements involving successively weaker hypotheses about the form of the s_γ and of a.

Statement 1. If $a \in P$ and $s_\gamma \in P$ for all $\gamma \in \Gamma$ then Γ is finite.

Let $a \in L_i$. Since, for each $\gamma \in \Gamma$, $s_\gamma \nleq a$ and if $\alpha \neq \beta$ then $s_\alpha \wedge s_\beta \leq a$, it follows that there is a $j \in I$ such that $s_\gamma \in L_j$ for all $\gamma \in \Gamma$. If $i = j$ the finiteness of Γ follows as in Statement 1 of Theorem 1. If $i \neq j$ then $s_\alpha \wedge s_\beta = 0$ for distinct α, β. Since $s_\gamma \nleq a$ implies $s_\gamma \nleq 0$, the finiteness of Γ follows in this case from the fact that L_j satisfies the finite disjointness property.

Statement 2. Let $n \geq 1$ be an integer, let $a \in P$, and let $s_\gamma = \bigwedge X_\gamma$ for each $\gamma \in \Gamma$, where X_γ is a reduced subset of P with $|X_\gamma| = n$. Then Γ is finite.

The case $n = 1$ is Statement 1. We prove Statement 2 by induction on n. Let $n > 1$. First fix $\gamma_0 \in \Gamma$ and let $X_{\gamma_0} = \{x_1, \cdots, x_n\}$. Then there are distinct $i(1), \cdots, i(n)$ in I such that $x_k \in L_{i(k)}$ for each $k \leq n$. For each $k \leq n$ let $\Gamma_k = \{\gamma \in \Gamma \mid X_\gamma \cap L_{i(k)} \neq \emptyset\}$. Now $\Gamma_1 \cup \cdots \cup \Gamma_n = \Gamma$; since $\bigwedge X_{\gamma_0} \nleq a$, $\bigwedge X_\gamma \nleq a$ if $\gamma \neq \gamma_0$, and $\bigwedge X_{\gamma_0} \wedge \bigwedge X_\gamma \leq a$ it follows that, for each γ, $X_\gamma \cap L_{i(k)} \neq \emptyset$ for some k. It suffices thus to prove that each Γ_k is finite. For each $\gamma \in \Gamma_k$ let x_γ be defined by setting $X_\gamma \cap L_{i(k)} = \{x_\gamma\}$ and let $X'_\gamma = X_\gamma - L_{i(k)}$. Then $|X'_\gamma| = n-1$ and $X_\gamma = X'_\gamma \cup \{x_\gamma\}$. We define two symmetric binary relations R and S on Γ_k. We set $\langle\alpha, \beta\rangle \in R$ if and only if $x_\alpha \wedge x_\beta \leq a$ and we set $\langle\alpha, \beta\rangle \in S$ if and only if $\alpha \neq \beta$ and $\langle\alpha, \beta\rangle \notin R$. Then $\langle\alpha, \beta\rangle \in S$ only if $\bigwedge X'_\alpha \wedge \bigwedge X'_\beta \leq a$. Since $n > 1$ and $|X'_\gamma| = n-1$ if $\gamma \in \Gamma_k$ we conclude by Ramsay's Theorem and the induction hypothesis that Γ_k is finite for each k. Thus Γ is finite.

Statement 3. Let $n \geq 1$. For each $\gamma \in \Gamma$ let $s_\gamma = \bigwedge X_\gamma$ where X_γ is reduced and $|X_\gamma| = n$. Let $a = \bigvee Y$, Y reduced. Then Γ is finite.

The proof of this statement is a word-for-word duplicate of the proof of Statement 3 of Theorem 1.

Statement 4. Let $a = \bigvee Y_1 \wedge \cdots \wedge \bigvee Y_r$ where each Y_j is a reduced subset of P. For each $\gamma \in \Gamma$ let J_γ be a finite nonempty set of reduced subsets of P such that $s_\gamma = \bigvee(\bigwedge X \mid X \in J_\gamma)$. Then Γ is countable.

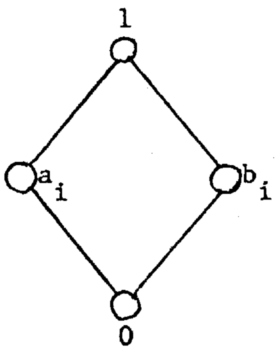
For each $\gamma \in \Gamma$ there is an $X_\gamma \in J_\gamma$ and a $j(\gamma) \leq r$ such that $\bigwedge X_\gamma \nleq \bigvee Y_{j(\gamma)}$. For each $j \leq r$ and $n \geq 1$ let

$$\Gamma_{jn} = \{\gamma \in \Gamma \mid j(\gamma) = j \text{ and } |X_\gamma| = n\} .$$

If α, β are distinct elements of Γ_{jn} then $\bigwedge X_\alpha \wedge \bigwedge X_\beta \leq s_\alpha \wedge s_\beta \leq a \leq \bigvee Y_j$. By definition of Γ_{jn}, $|X_\gamma| = n$ if $\gamma \in \Gamma_{jn}$ and $\bigwedge X_\gamma \nleq \bigvee Y_j$. Thus Γ_{jn} is finite by Statement 3. But $\Gamma = \bigcup(\Gamma_{jn} \mid n \geq 1, 1 \leq j \leq r)$; thus Γ is countable, proving Statement 4.

Statement 4 is the statement of the Theorem.

To complete this section we present an example of a countable family of finite lattices whose free product in $\mathcal{D}_b$ does not satisfy the finite disjointness condition. Let the index set I be the set of positive integers and, for each $i \in I$, let the lattice L_i be the four-element lattice in the diagram.



L_i

Let L be the free product in $\mathcal{D}_b$ of the L_i, $i \in I$. Let $s_1 = b_1$ and for each $n > 1$ let $s_n = a_1 \wedge a_2 \wedge \cdots \wedge a_{n-1} \wedge b_n$. Let $S = \{s_n\}$. Then S is infinite, $0 < s_n$ for each n, and if $m \neq n$, say $m < n$, then $s_m \wedge s_n = 0$, since $s_m \leq b_m$ and $s_n \leq a_m$.

Thus L does not satisfy the finite disjointness condition. Of course, L is just the underlying lattice of the free Boolean algebra generated by a countable set, and this example shows that it need not satisfy the finite disjointness condition.

5. Epilogue. For any infinite cardinal $\mathfrak{m}$ one can of course define the $\mathfrak{m}$-disjointness condition: a lattice L is said to satisfy the $\mathfrak{m}$-disjointness condition if, given any $a \in L$ and any $S \subseteq L$ such that $a \notin S$ and $x \wedge y = a$ for distinct $x, y \in S$, it then follows that $|S| < \mathfrak{m}$. An obvious question is the following:

In either category $\mathcal{D}$ or $\mathcal{D}_b$ is the $\mathfrak{m}$-disjointness condition preserved under free products for $\mathfrak{m} > \aleph_0$?

The methods presented in sections 3 and 4 cannot be applied to answer this question in the affirmative because, as first observed by

Sierpiński [5], the obvious extension of Ramsay's Theorem to infinite cardinals does not hold.

There are Ramsay-type theorems for infinite cardinals; see Erdös, Hajnal, Rado [2] for a rather complete survey. Of particular interest to our problem is the following result of Kurepa [4], under the assumption of the generalized continuum hypothesis:

Let α be any ordinal. Let Γ be a set such that $|\Gamma| \geq \aleph_{\alpha+2}$, and let $R_1, \cdots, R_n$ be binary symmetric relations on Γ such that $\omega_\Gamma \cup R_1 \cup \cdots \cup R_n = \Gamma \times \Gamma$. Then there is a subset $\Gamma' \subseteq \Gamma$ and an $i \leq n$ such that $|\Gamma'| \geq \aleph_{\alpha+1}$ and for any distinct $\alpha, \beta \in \Gamma'$ $\langle \alpha, \beta \rangle \in R_i$.

Using this result in place of Ramsay's Theorem the methods of sections 3 and 4 carry over to prove:

Let $(L_i \mid i \in I)$ be a family of lattices in $\mathfrak{D}$ or $\mathfrak{D}_b$ satisfying the $\aleph_{\alpha+1}$-disjointness condition, $\alpha \geq 0$. Then the free product in $\mathfrak{D}$ or $\mathfrak{D}_b$ satisfies the $\aleph_{\alpha+2}$-disjointness condition.

Unfortunately I have been unable to construct an example to show that $\aleph_{\alpha+2}$ cannot be replaced by $\aleph_{\alpha+1}$. This is thus to date an open problem.

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