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THE ORDER-SUM IN CLASSES OF PARTIALLY ORDERED ALGEBRAS

by

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The order-sum of partially ordered algebras will be defined as the solution of a universal problem and as a generalization of the coproduct. We show that the order-sum exists without restriction in quasi-primitive classes, and we investigate some of the properties of the order-sum.

1. Order-sum and lexicographic sum

We will consider classes $\mathfrak{K}$ of partially ordered algebras. The algebras under consideration will be *partial algebras* $(A,(f_i)_{i\varepsilon I})$ of arbitrary finitary or infinitary *type* $\Delta = (K_i)_{i\varepsilon I}$. I.e., the *index-sets* K_i may be finite or infinite, and f_i is a mapping of a subset of A^{K_i} into A. If the domain of f_i is all of A^{K_i}, for each $i\varepsilon I$, we may call $(A,(f_i)_{i\varepsilon I})$ a *complete algebra* of type Δ. A *partially ordered algebra* is a triple $(A,(f_i)_{i\varepsilon I}, \leq)$, where $(A,(f_i)_{i\varepsilon I})$ is a partial algebra and $(A, \leq)$ a partially ordered set. The algebraic structure may be empty, $I=\emptyset$. In that case, the partially ordered algebra is nothing but a partially ordered set, and any class of partially ordered sets is an example of a class of partially ordered algebras. Since, on the other hand, the partial order may be total disorder, we can also interpret any class of partial algebras as a class of partially ordered algebras.

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We do not require any kind of compatibility postulates to hold between the algebraic structure and the partial order. But there is no ban on compatibility conditions either.

A *homomorphism* of the partially ordered algebra $(A,(f_i)_{i\varepsilon I}, \leqslant)$ into the partially ordered algebra $(B,(g_i)_{i\varepsilon I}, \leqslant)$ - of the same type Δ - is a mapping $\phi:A \longrightarrow B$ that is *order-preserving:*

(1.1) $$\text{if } x \leqslant y, \text{ then } \phi(x) \leqslant \phi(y) ,$$

for all elements $x,y \varepsilon A$, and at the same time an *algebraic homomorphism:*

(1.2) $$(f_i(a_\kappa | \kappa \varepsilon K_i)) = g_i(\phi(a_\kappa) | \kappa \varepsilon K_i),$$

for each index $i \varepsilon I$, for each sequence $(a_\kappa)_{\kappa \varepsilon K_i}$ in the domain of f_i (making the left side exist - it is understood that the right side will then exist too).

In the sequel, $\mathfrak{K}$ will always be a class of partially ordered algebras of the same type Δ.

Suppose T is a partially ordered set, and assume that a *partially ordered family* of partially ordered algebras $P_t \varepsilon \mathfrak{K}$ is given. A family of homomorphisms $\phi_t:P_t \longrightarrow P$, where P is also supposed to be in $\mathfrak{K}$, is called a *T-family* provided that the following condition holds true for all indices $s,t \varepsilon T$:

(1.3) $$\text{if } s < t \text{ (in T), then } \phi_s(x) \leqslant \phi_t(y) \text{ (in P)},$$

for all elements $x \varepsilon P_s$, $y \varepsilon P_t$. The *order-sum* is now simply a universal T-family. I.e., the T-family $\phi_t:P_t \longrightarrow P$ is an order-sum if, for each algebra $Q \varepsilon \mathfrak{K}$ and each T-family $\psi_t:P_t \longrightarrow Q$, there is a unique homomorphism $\psi:P \longrightarrow Q$ such that $\psi \circ \phi_t = \psi_t$, for all indices $t \varepsilon T$.

In the special case where the index-set T is totally unordered, the order-sum coincides with the coproduct. Clearly, the order-sum (if it exists) will be unique up to unique isomorphism, and in that sense it is justified to talk about "the" order-sum.

Assume now that $\mathfrak{K}$ is the class of all partially ordered algebras of a given type $\Delta = (K_i)_{i\varepsilon I}$, and assume further that the type Δ is without constants, i.e. $K_i \neq \emptyset$ for each $i\varepsilon I$. The algebraic lexicographic sum of partially ordered algebras that we are going to define, is a combination of the partial direct sum of partial algebras (cf. Schmidt [9]) and of the well-known lexicographic sum of partially ordered sets (cf. Birkhoff [2], Schmidt [7],[8]).

For a partially ordered family of partially ordered algebras P_t, we define $\underset{t\varepsilon T}{\llcorner} P_t$ to be the set of all ordered pairs (t,x), where $t\varepsilon T$ and $x\varepsilon P_t$, endowed with the *lexicographic order* :

(1.4) $$(s,x) \leq (t,y) \text{ iff } s < t \quad \text{or} \quad s = t \text{ and } x \leq y$$

The *natural mappings* $i_t : P_t \longrightarrow \llcorner P_t$, defined by $i_t(x) = (t,x)$, are obviously order-preserving, even order- embeddings. On $\llcorner P_t$, there exists now the "weakest" algebraic structure $(f_i)_{i\varepsilon I}$ such that the natural mappings i_t become algebraic homomorphisms, i.e. the final structure for the mappings i_t (cf. Bourbaki[3], Schmidt [9]).

$\llcorner P_t$ with the lexicographic order $\leq$ and this algebraic structure - and with the natural mappings i_t - will be called the *algebraic lexicographic sum* of the partially ordered algebras P_t $(t\varepsilon T)$.

If T and all algebras P_t are totally disordered, then the algebraic lexicographic sum coincides with with the partial direct sum of the algebras P_t (cf. Schmidt [9]). On the other hand, if $I = \emptyset$, our algebraic lexicographic sum is nothing but the ordinary lexicographic sum of partially ordered sets P_t.

<u>Theorem 1.1</u> In the class $\mathfrak{K}$ of all partially ordered algebras of type Δ, the algebraic lexicographic sum $i_t : P_t \longrightarrow L P_t$ is the order-sum.

In the class of partially ordered topological spaces, a topological lexicographic sum can be defined in a similar manner as for partially ordered algebras: It will be a combination of the topological sum of the spaces and the lexicographic sum of the partially ordered sets. An exact analogue of Theorem 1.1 holds true.

Unfortunately, we had to restrict ourselves so far to the case where Δ is a type without constants. This is to a good extend due to

<u>Theorem 1.2</u> Suppose $\psi_t : P_t \longrightarrow P$ is a T-family of order-preserving mappings. Assume that for each $t \in T$, there is an $a_t \in P_t$ such that $\psi_t(a_t) = a$, where a is independent of t. Suppose $s < t$ in T. Then $\max \psi_s(P_s) = \min \psi_t(P_t) = a$.

<u>Corollary 1.</u> $\max \psi_s(P_s) = a$ if s is not maximal in T, $\min \psi_s(P_s) = a$ if s is not minimal in T. $\psi_s(P_s)$ collapses into $\{a\}$ if s is not extremal in T (neither maximal nor minimal).

<u>Corollary 2.</u> If s is not maximal in T, and $\min P_s = a_s$, then again $\psi_s(P_s)$ collapses into $\{a\}$.

Let us show which damage Theorem 1.2 does to the order-sum in the presence of constants: Let $\mathfrak{K}$ be the class of partially ordered algebras with least and greatest elements, the latter explicitly listed among the constants. I.e., the homomorphisms in $\mathfrak{K}$ are supposed to preserve both least and greatest elements. We now assume that T contains a pair of comparable elements $s < t$. Consider a T-family $\psi_t : P_t \longrightarrow P$. Since s is not maximal, $\psi_s(P_s)$ consists of the least element of P only, according to Corollary 2. On the other hand, it contains the greatest element of P. So the latter has to coincide with the least element, thus squeezing P down to one element. In such a class, the old coproduct will be the only meaningful order-sum. If we give up insisting on the preservation of extrema, however, other order-sums become highly meaningful.

Theorem 1.3 Let the class $\mathfrak{K}$ be closed under taking subalgebras. Let $\phi_t : P_t \longrightarrow P$ be an order-sum in $\mathfrak{K}$. Then the union $\bigcup \mathrm{im}\ \phi_t$ generates P.

2. The algebraic lexicographic sum with constants

We want to extend the notion of the algebraic lexicographic sum to the general case where the type Δ may now contain some constants, $K_i = \emptyset$ for some $i \in I$. This should be done in such a way that Theorem 1.1 remains true. The construction is similar to the construction of the partial direct sum of algebras (cf. Schmidt [9]), but somewhat more involved in the presence of partial orders.

Throwing out the indices $i \in I$ standing for constants, we arrive at the reduced index-domain $I^* = \{ i \mid K_i \neq \emptyset \}$ and the corresponding *reduced type* Δ^*, without constants. The partially oredered algebras P_t are turned into partially ordered algebras $P_t{}^*$ of type Δ^*. We can consider the algebraic lexicographic sum of the latter, $\mathrm{L} P_t^*$. In order to arrive at an appropriate factorization, we consider quasi-orders ρ of $\mathsf{L}P_t^*$ which are admissible in the sense that the following three conditions hold:

(i) $\rho \wedge \rho^{-1}$ is a congruence relation of the algebra $\mathsf{L}P_t^*$;

(ii) ρ contains the lexicographic order of $\mathsf{L}P_t^*$;

(iii) ρ takes care of the constants insofar as $(s,f_{si})\ \rho\ (t,f_{ti})$, for each $s,t\in T$ and for each $i\in I\setminus I^*$.

It is easy to see that there is a least admissible quasi-order, say σ. The contraction $\mathsf{L}P_t^*/\sigma\wedge\sigma^{-1}$ is then a partially ordered algebra of type Δ^*, and the natural projection $p:\mathsf{L}P_t^*\longrightarrow\mathsf{L}P_t^*/\sigma\wedge\sigma^{-1}$ is a homomorphism between them. One makes $\mathsf{L}P_t^*/\sigma\wedge\sigma^{-1}$ an algebra of type Δ by introducing the constants $g_i = p(t,f_{ti})$, for each $i\in I$, this definition is independent of t. The partially ordered algebra $\mathsf{L}P_t^*/\sigma\wedge\sigma^{-1}$ so enriched may be called the *algebraic lexicographic sum* of the partially ordered algebras P_t $(t\in T)$ and again be denoted by $\mathsf{L}P_t$. Clearly, in the case without constants, $I^* = I$, $\Delta^* = \Delta$, $P_t^* = P_t$, nothing has happened at all. We introduce the mappings $j_t = p\circ i_t:P_t\longrightarrow\mathsf{L}P_t$, which are homomorphisms by construction (in particular, they preserve the constants).

<u>Theorem 2.1</u> In the class of all partially ordered algebras of type Δ, the algebraic lexicographic sum $j_t:P_t\longrightarrow\mathsf{L}P_t$ is the order-sum.

Note that the homomorphisms j_t need no longer be one-one since p can not be expected to be one-one. Indeed, σ and $\sigma\wedge\sigma^{-1}$ may become the universal relation in P_t^*, forcing $\mathsf{L}P_t$ to collapse into one element.

3. A general existence theorem

The order-sum in a class $\mathfrak{K}$ of partially ordered algebras can be build up in two steps. The first one of these has been described in sections 1 and 2. In general, of course, the algebraic lexicographic sum of algebras P_t will not be in $\mathfrak{K}$. So the second step will consist in associating with the latter a universal object in $\mathfrak{K}$.

<u>Theorem 3.1</u> Consider a T-family of homomorphisms $\phi_t : P_t \longrightarrow P$ in $\mathfrak{K}$ and the associated homomorphism $\phi : \llcorner P_t \longrightarrow P$ (which exists according to Theorem 2.1). Then the following two conditions are equivalent:

(i) $\phi_t : P_t \longrightarrow P$ is the order-sum in $\mathfrak{K}$;

(ii) $\phi : \llcorner P_t \longrightarrow P$ is the universal homomorphism of $\llcorner P_t$ into a $\mathfrak{K}$-algebra.

As in universal algebra without partial order, a *quasi-primitive class* of partially ordered algebras will be a class closed under taking cartesian products, subalgebras, and isomorphic images. After reinterpretation of the partial orders as partial operations, such a class will become a quasi-primitive class in the ordinary sense of universal algebra.

<u>Theorem 3.2 (Existence of Order-sums)</u>
In a quasi-primitive class all order-sums exist.

4. When is the order-sum an extension of the lexicographic sum?

Suppose that $\mathfrak{K}$ is a class of partially ordered algebras. Suppose $\phi_t : P_t \longrightarrow P$ to be an order-sum in $\mathfrak{K}$ and $i_t : P_t \longrightarrow \llcorner P_t$ the lexicographic sum. Let $\phi : \llcorner P_t \longrightarrow P$ be the universal homomorphism of Theorem 3.1.

In order to avoid the difficulties connected with the constants, we shall assume from now on that the type Δ be *without constants* ($K_i \neq \emptyset$ for each $i \in I$).

Theorem 4.1 Equivalent are:

(i) $\phi : \coprod P_t \longrightarrow P$ is one-one;

(ii) the homomorphisms $\phi_t : P_t \longrightarrow P$ are one-one, and their images are pairwise disjoint;

(iii) there is a $\mathfrak{K}$-algebra Q and a one-one homomorphism $\psi : \coprod P_t \longrightarrow Q$.

Theorem 4.2 Equivalent are:

(i) $\phi : \coprod P_t \longrightarrow P$ is an order-embedding;

(ii) the homomorphisms $\phi_t : P_t \longrightarrow P$ are order-embeddings, and the indexed family of their images is not only pairwise disjoint, but a "lexicographic decomposition" of the partially ordered set $\bigcup_{t \in T} \operatorname{im} \phi_t$ (= $\operatorname{im}\phi$);

(iii) there is a $\mathfrak{K}$-algebra Q and an order-embedding (and algebraic homomorphism) $\psi : \coprod P_t \longrightarrow Q$.

Theorem 4.3 Equivalent are:

(i) the homomorphisms $\phi_t : P_t \longrightarrow P$ are one-one;

(ii) for all indices $s \in T$ and all elements $x, y \in P_s$ such that $x \neq y$, there is a $\mathfrak{K}$-algebra Q and a T-family $\psi_t : P_t \longrightarrow Q$ separating x and y, $\psi_s(x) \neq \psi_s(y)$.

Theorem 4.4 Equivalent are:

(i) the homomorphisms $\phi_t : P_t \longrightarrow P$ are order-embeddings;

(ii) for all indices $s \in T$ and all elements $x, y \in P_s$ such that $x \nleq y$, there is a $\mathfrak{K}$-algebra Q and a T-family $\psi_t : P_t \longrightarrow Q$ such that $\psi_s(x) \nleq \psi_s(y)$.

Whenever the universal homomorphism ϕ is an order-embedding, we can replace it by the inclusion mapping of the lexicographic sum into an isomorphic copy of P, due to the well-known Zermelo - van der Waerden replacement procedure.I.e., the order-sum can be considered as an extension of the lexicographic sum. $\bigsqcup P_t$ becomes a subset of P, the lexicographic order of $\bigsqcup P_t$ is the restriction of the partial order of P. However, the inclusion of the partial algebra $\bigsqcup P_t$ into the algebra P will only be a homomorphism, not necessarily an embedding. I.e., $\bigsqcup P_t$ may only be a *weak* relative algebra of P. We will refer to this situation by saying that P is an *order-extension* of $\bigsqcup P_t$ (algebraically, it may only be a *weak extension*). If all $\mathfrak{K}$-algebras are complete, at least the inclusions of the pieces $i_t(P_t)(= \phi_t(P_t))$ into P are strong, the pieces are then genuine subalgebras of the complete algebra P.

We now find convenient sufficient conditions on the class $\mathfrak{K}$ to garantee that our mapping $\phi: \bigsqcup P_t \longrightarrow P$ will be an order-embedding.

(I) $\mathfrak{K}$ is *non-trivial*, i.e. it contains a *non-trivial algebra* Q in the sense that Q contains a pair of distinct comparable elements.

(II) All constant mappings between $\mathfrak{K}$-algebras are homomorphisms.

(III) For every $\mathfrak{K}$-algebra P and all elements $x,y \in P$ such that $x \nleq y$, there is a $\mathfrak{K}$-algebra Q and a homomorphism $\alpha: P \longrightarrow Q$ such that $\alpha(y) = \min \alpha(P) < \alpha(x) = \max \alpha(P)$ ("separability").

Condition (III) may be replaced, for our purposes, by the following:

(III') Every $\mathfrak{K}$-algebra is embeddable into a non-trivial $\mathfrak{K}$-algebra with least and greatest element.

Note that in the class $\mathfrak{K}$ of all distributive lattices, all four conditions hold. In the class of modular lattices, at least (I), (II), (III') hold.

Theorem 4.5 Let $\phi_t : P_t \longrightarrow P$ be an order-sum in $\mathfrak{K}$. Suppose that $\mathfrak{K}$ fulfills the conditions (I), (II), and (III) or, alternatively (III'). Then the universal homomorphism $\phi : \sqcup P_t \longrightarrow P$ is an order-embedding.

Corollary Suppose is a class of complete algebras fulfilling conditions (I), (II), and one of (III) or (III'). Then the order-sum P (provided it exists) is an order-extension (and weak algebraic extension) of the lexicographic sum $\sqcup P_t$, and the pieces $i_t(P_t)$ are subalgebras of P.

Recall that the order-sum exists, if $\mathfrak{K}$ is quasi-primitive.

5. The order-sum extends the lexicographic sum in some nice classes

Theorem 5.1 In the class $\mathfrak{K}$ of semilattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

Theorem 5.2 In the class $\mathfrak{K}$ of lattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

Theorem 5.3 In the class $\mathfrak{K}$ of distributive (modular) lattices, the order-sum exists without restriction and is an order-extension of the lexicographic sum.

Theorem 5.4 In the class $\mathfrak{K}$ of semilattices (lattices, distributive, modular lattices), the order-sum over a chain T coincides with the lexicographic sum of the partially ordered sets.

REFERENCES

[1] R. Balbes and A. Horn, Order-sums of distributive lattices, Pacific J. Math. 21 (1967), 421-435.

[2] G. Birkhoff, Lattice theory, 2nd ed., New York 1948; 3rd ed. Providence 1967.

[3] N. Bourbaki, Theorie des ensembles, Chap.4: Structures, Paris 1957.

[4] G. Grätzer, Lattice theory, San Francisco 1971.

[5] M. Höft, The order-sum in classes of partially ordered algebras, Dissertation, University of Houston 1973.

[6] H. Lakser, Free lattices generated by partially ordered sets, Dissertation, University of Manitoba 1968.

[7] J. Schmidt, Die Theorie der halbgeordneten Mengen, Dissertaion, Berlin 1952.

[8] J. Schmidt, Zusammensetzungen und Zerlegungen halbgeordneter Mengen, J. Ber. DMV 56 (1952), 19-20.

[9] J. Schmidt, Allgemeine Algebra, Mimeographed Lecture Notes, Bonn 1966.

[10] J. Schmidt, A general existence theorem on partial algebras and its special cases, Coll. Math. 14 (1966), 73-87.

[11] J. Schmidt, Universelle Halbgruppe, Kategorien, freies Product, Math. Nachr. 37 (1968), 345-358.

[12] J. Schmidt, Direct sums of partial algebras and final algebraic structures, Canad. J. Math. 20 (1968), 872-887.

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