Proc. Univ. of Houston
Lattice Theory Conf..Houston 1973

LAWSON SEMILATTICES DO HAVE A PONTRYAGIN DUALITY

Karl Heinrich Hofmann
and
Michael Mislove

The category $\underline{L}$ of *Lawson semilattices* is the category of all compact topological semilattices S with identity, whose continuous interval morphisms $S \to I$ (where $I = [0,1]$ has the min multiplication) separate the points and whose morphisms are continuous identity preserving semigroup morphisms.

We say that *a category* $\underline{A}$ *has a Pontryagin duality* iff there is a category $\underline{B}$ which is dual to $\underline{A}$ in a way analogous to the fashion in which discrete abelian groups and compact abelian groups are dual. Specifically, there are functors

$$\begin{array}{ccc} \underline{A} & \underset{G}{\overset{F}{\rightleftarrows}} & \underline{B}^{op} \\ U \downarrow & & \downarrow V \\ \mathrm{Set} & \underset{\mathrm{Set}(-,T)}{\overset{\mathrm{Set}(-,T)}{\rightleftarrows}} & \mathrm{Set}^{op} \end{array}$$

Such that the following conditions are satisfied:

(1) There are natural isomorphisms $\eta : I_A \to GF$ and $\varepsilon : I_B \to FG$.

(2) U and V are grounding functors (i.e. faithful functors into the category of sets) and $VF \cong \underline{A}(-,A)$. $UG \cong \underline{B}(-,B)$ with distinguished

objects A and B such that $UA \cong VB \cong T$. (i.e. the functors F and G are given by hom-ing into cogenerating objects which are based on one and the same set).

(3) There are natural monics $\mu_X : FX \to B^{UX}$ and $\nu_Y : GY \to A^{VY}$ such that the diagram

$$\begin{array}{ccccc}
VFX & \xrightarrow{V\mu_X} & V(B^{UX}) & \xrightarrow{\text{nat}} & (VB)^{UX} \\
 & & & & \downarrow \cong \\
\cong \downarrow & & & & T^{UX} \\
 & & & & \| \\
\underline{A}(X,A) & \xrightarrow[U]{} & \underline{Set}(UX,UA) & \xrightarrow{\cong} & \underline{Set}(UX,T)
\end{array}$$

(with nat denoting the natural morphism $V(\Pi Y_j) \to \Pi VY_i$) commutes as well as a similar one for ν.

(This says, intuitively, that FX is defined on the set $\mathrm{Hom}(X,A)$ by inheriting the structure from the product B^{UX} (which itself is based on the function set $\underline{Set}(UX,T)$).

(4) Let $e : S \to \underline{Set}(\underline{Set}(S,T),T)$ be the function defined by evaluation $e(s)(f) = f(s)$. Then the diagram

$$\begin{array}{ccc}
UX & \xrightarrow{U\eta_X} & UGFX \cong \underline{B}(FX,B) \\
 & & \downarrow V \\
e \downarrow & & \underline{Set}(VFX,VB) \\
 & & \downarrow \cong \\
\underline{Set}(\underline{Set}(X,T),T) & \xleftarrow[\underline{Set}(U,T)]{} & \underline{Set}(\underline{A}(X,A),T)
\end{array}$$

commutes. A similar condition holds for ν.

(This condition expresses, in categorical terms, the fact that the isomorphism X identifying an object X in

$\underline{A}$ with its double-dual GFX is obtained by evaluation when everything is viewed in terms of functions and sets.)

Let us remark here that these conditions, which in functorial terms describe the distinguishing features of Pontryagin dualities, are by no means artificial. In fact one has the following

PROPOSITION 1. *Let* $\underline{A} \underset{G}{\overset{F}{\rightleftarrows}} \underline{B}^{op}$ *be a pair of functors between categories with products, such that* F *is left adjoint to* G *and that there are objects* A' *in* $\underline{A}$ *and* B' *in* $\underline{B}$ *such that* $X \longmapsto \underline{A}(A',X)$, $Y \longmapsto \underline{B}(B',Y)$ *are faithful set functors* U, *resp.* V. *Then with front adjunction* η *and back adjunction* ε, *conditions* (2), (3), (4) *are satisfied, with* $A = GB'$ *and* $B = FA'$.
(See Hofmann and Keimel [2], Section 0.)

Note that functors U and V which are of this form are called *representable* and that objects with the properties of A' and B' always exist if the categories $\underline{A}$ and $\underline{B}$ have free functors (i.e. left adjoints to the grounding functors into $\underline{Set}$).

We will also say that a duality such as it is described in (1) through (4) is *a Pontryagin duality*.

The original Pontryagin duality between the category $\underline{A}$ of discrete abelian groups and the category $\underline{B}$ of compact abelian groups satisfies these conditions with FX = character group $\underline{A}(X, \mathbb{R}/\mathbb{Z})$ with the compact topological group structure inherited from the inclusion $\underline{A}(X, \mathbb{R}/\mathbb{Z}) \to (\mathbb{R}/\mathbb{Z})^{UX}$, with GT = character group $\underline{B}(Y, \mathbb{R}/\mathbb{Z})$ with the group structure inherited from the inclusion $\underline{B}(Y, \mathbb{R}/\mathbb{Z}) \to (\mathbb{R}/\mathbb{Z})^{VY}$ (where U and V are the "underlying set"-functors and where we (ambiguously) denote the discrete circle group A, the compact circle group B and the underlying set T with the same symbol $\mathbb{R}/\mathbb{Z}$.

The following further dualities are Pontryagin dualities:

(a) Stone duality between Boolean algebras and compact zero dimensional spaces.

(b) Gelfand duality between commutative C*-algebras with identity and compact spaces (although this is not entirely obvious).

(c) The duality between discrete and compact semilattices.

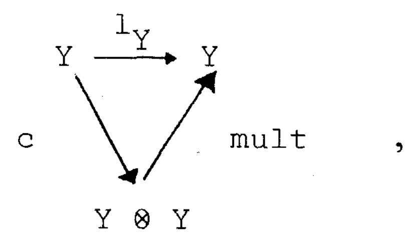
(d) The duality between complete lattices in which every element is a meet of primes and arbitrary sup-preserving morphisms on one hand and the category of spectral spaces on the other (Hofmann and Keimel).

(e) Tannaka duality for compact groups(this again is not obvious).

The duality between the category of all compact topological semigroups and the category of commutative C*-bigebras co-semigroups in the symmetric monoidal category of commutative C*-algebras with identity introduced by Hofmann is not a Pontryagin duality, since neither category has a cogenerator.

The purpose of this note is to point out that the category of Lawson semilattices, contrary to the suspicions of people who have worked in the area, does have a Pontryagin duality. This duality may not be as useful and as applicable as some of the dualities listed above because the dual category of $\underline{L}$ is rather involved; our observation may still serve a useful purpose since it will avoid futile efforts to prove the contrary. In view of the Hofmann duality mentioned above it is known that Lawson semilattices indeed do have a dual category. Let us record this fact in the following

PROPOSITION 2. *The category* E^* *of commutative and co-commutative* C*-*bialgebras* Y *with identity, idempotent comultiplication* $c : Y \to Y \otimes^* Y$ *and co-identity* $u : A \to C$ (where idempotency of the comultiplication is expressed in the commutativity of the diagram



,

together with all C-bialgebra morphisms is dual to the category of all compact topological semilattices with identity. If C(I) denotes the dual of the interval semilattice in this category, then the full subcategory E_I^* of E* generated by C(I) is dual to the category L of Lawson semilattices. The duality is given by the functors* $C : \underline{L} \to E^*$ *and* $\mathrm{Spec} : E^* \to \underline{L}$.
(For more details of the Hofmann duality see Hofmann [1], notably pp. 136-139.)

The rest of the note is now concerned with a discussion of

THEOREM 3. *The duality between* E_I^* *and* $\underline{L}$ *is a Pontryagin duality.*

Since $\underline{E}_I^*$ does not appear to have a free functor the routine proof which works in many of the examples does not apply. However, due to the special property of having a generator, the category $\underline{E}_I^*$ has a representable grounding functor. The category $\underline{L}$ even has a free functor hence a representable grounding functor. In view of Proposition 3, the hypotheses of Proposition 1 are satisfied; thus Theorem 3 is proved.

Let us see some of the details. Let $2 \in \mathrm{ob}\, \underline{L}$ be the two element semilattice. Then $S \longmapsto \underline{L}(2,S)$ clearly is (equivalent to) the "underlying set" functor U. The grounding functor V of $\underline{E}_I^*$ is given by $VY = \underline{E}^*(C(I),Y)$. Since C(I) generates $\underline{E}_I^*$, then this functor is faithful on E_I^*, We note $VC(2) = E^*(C(I),C(2)) \cong \underline{L}(2,I) \cong [0,1] \cong$ $\cong UI \cong U\,\mathrm{Spec}\; C(I)$.
Thus, as is to be expected for a Pontryagin duality the objects which serve as the fixed domains for the hom set representation of the duality functors are modelled on one

and the same set, namely the unit interval $[0,1]$. The one in $\underline{L}$ is I, the unit interval semilattice, the one in E^*_I is C(2). Note that the underlying C*-algebra of C(2) is $C \times C$. The monic $\mu_S : C(S) \to C(2)^{L(2,S)}$ of condition (3) is given by $\mu_S(f)(\phi) = f \circ \phi$; similarly $\nu_Y : \text{Spec } Y \to I^{E^*(C(I),Y)}$ is given by $f(\nu_Y(\psi)(\alpha)) = (\psi \circ \alpha)(f)$, $f \in C(I)$, $\psi : Y \to C$ an element of Spec Y, $\alpha : C(I) \to Y$. The verification of the commuting of the diagrams in (3) and (4) (which we know on the basis of the general Proposition 2) is left as an exercise.

The deviation from the more customary Pontryagin duality theories is due to the somewhat unfamiliar grounding functor for the category E^*_I, which is not the underlying set functor, nor the unit ball functor (which is the natural underlying set functor for C*-algebras in the context of Gelfand duality). This accounts for the somewhat curious appearance of the unit interval in the guise of C(2) in E^*_I.

Let us conclude with the remark, that a similar theory holds for semilattices without identity. In that case the generator for the category of Lawson semilattices without identity is 1, the one element semilattice, and the cogenerator for the dual category is $C(1) \cong C$.

REFERENCES

1. Hofmann, K. H., The Duality of Compact Semigroups and and C*-Bigebras, Lecture Notes in Mathematics 129 Springer-Verlag, Heidelberg 1970.

2. Hofmann, K. H., and K. Keimel, A General Character Theory for Partially Ordered Sets and Lattices Memo. Amer. Math. Soc. 122 (1972).