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ON THE DUALITY OF SEMILATTICES AND ITS APPLICATIONS TO LATTICE THEORY

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This article reports on a monograph in which the authors discuss the duality between the category $\underline{S}$ of semilattices with identity and identity preserving morphisms on one hand and the category $\underline{Z}$ of compact zero dimensional topological semilattices with identity and identity preserving continuous morphisms.

In itself, this duality theory is not new. Various authors discovered the duality on objects some time ago and the full duality theory itself together with various ramifications was described in the context of other duality theories by Hofmann. The duality theory for discrete and compact abelian groups was introduced by Pontryagin with the express purpose of immediate applications to algebraic topology. It was soon applied in group theory, topology and analysis. Thus it became fruitful by producing results in either of two directions: from the discrete theory to the topological one and indeed also vice versa. By contrast, the duality of semilattices has not been noticed as a vehicle for applications at all. We hope to demonstrate that it, too can have useful applications to discrete and topological lattice theory and to the theory of compact semilattices as a part of compact abelian semigroup theory.

I. As a first step we set apart a chapter describing basic functorial properties of the categories $\underline{S}$ and $\underline{Z}$ such as their completeness, cocompleteness, their having biproducts, and the existence of free functors (i.e. left adjoints for the obvious grounding functors into the category Set of sets). We then give a proof of the duality theory which is based on a fairly general, yet useful functorial device which e.g. has been applied recently by Roeder to give a new proof of the self duality of locally compact abelian groups. This proof is based on some generalities on functorial density and continuous (i.e. limit or colimit preserving functors) which we describe in a preliminary chapter, preceding Chapter I, which in itself does not refer to semilattices. The proof of the duality theorem then proceeds as follows: We show that the category F of finite semilattices is co-dense in S and dense in Z. It is very elementary to show that F is naturally dual to itself. Then we push the button and the functorial machinery yields the desired duality. The advantage is that this method allows generalizations beyond the application we have in mind. Alternative proofs of the duality are available in the literature.

II. In the second chapter we view the duality theory as an instance of a character theory, thereby exhibiting its closeness to Pontryagin duality theory for abelian groups. This requires that we first give a description of the category Z from the view point of compact topological semigroups. We record a characterization theorem for zero dimensional compact semilattices known to semigroupers for some time, in which the existence of small semilattices, the existence of sufficiently many ultra-pseudometrics, and the separation of points by characters (and some other properties) are used to characterize the objects of Z. We introduce the concept of a local minimum $m \in S$ and give different semigroup theoretical characterizations: Indeed m is a local minimum iff $\{m\}$ is isolated in Sm iff $\uparrow m$ (the set $\{s \in S \mid sm = m\}$) is open. Further $m \in S$ is called a strong local maximum iff there is a local minimum $n \in S$ such that m is maximal in the ideal $S \setminus \uparrow n$. We observe that the set of local minima is dense in S and in fact even in every principal ideal Ss, and that the set of strong local maxima is dense.

In the second part of the chapter we correlate the concepts of characters and filters; *a character* of S is a morphism $S \to 2$ (in $\underline{S}$, respectively $\underline{Z}$), a filter $F \subseteq S$ is a subsemilattice such that $s \in F$ implies $\uparrow s \in F$. Since a function $f : S \to 2$ for a discrete S is a character iff $f^{-1}(1)$ is a filter, we have the following.

PROPOSITION. *The character semilattice* $\hat{S}$ *of a discrete semilattice* S *is naturally isomorphic to the filter semilattice* $\mathcal{F}(S)$ *under intersection as operation.*

The search for a concrete realization of the character semilattice of a $T \in \text{ob}\ \underline{Z}$ is a bit more involved. Firstly we observe that the underlying semilattice of T is in fact a complete lattice. We then prove the following

PROPOSITION. *Let* $k \in T$, *where* T *is a compact zero dimensional semilattice. Then the following statements are equivalent:*

(1) k *is a local minimum of the topological semilattice* T.

(2) k *is a compact element of the underlying complete lattice.*

We denote the sup-semilattice of all compact elements of a semilattice T by $K(T)$; recall that an element k of a semilattice T is *compact* iff $k \leq \sup X$ for some $X \subseteq T$ implies the existence of a finite subset $F \subseteq X$ with $k \leq \sup F$. For each $k \in K(T)$ there is precisely one $\underline{T}$-character $f : T \to 2$ with $k = \min f^{-1}(1)$, and each $\underline{T}$-character is so defined.

PROPOSITION. *The character semilattice* T *of a compact zero dimensional semilattice* T *is naturally isomorphic with the* (sup) *semilattice* $K(T)$ *of compact elements of the underlying lattice of* T.

By our earlier observation we know that for $T \in \text{ob}\ \underline{Z}$ the set of local minima, hence $K(T)$ is dense in every principal ideal Tt of T. This rather directly implies that the underlying lattice of T is *algebraic*, i.e. is a complete lattice in which every element is a sup of the elements in $K(T)$ which it dominates. We prove, conversely, that every algebraic lattice has a unique compact zero dimensional semilattice topology relative to which $K(T)$ is the set of local minima. Since it is not hard to see that a semilattice morphism $T \to T'$ between algebraic lattices is continuous relative to these topologies iff it is an order continuous lattice morphism, i.e. iff arbitrary infs and sups of upward directed sets are preserved, we have the following

THEOREM. The category $\underline{Z}$ of compact zero dimensional semilattices and continuous identity preserving semilattice morphisms is isomorphic to the category of algebraic lattices and order continuous semilattice morphisms (and this latter category is then dual to the category $\underline{S}$ of discrete semilattices and identity preserving semilattice morphisms).

If we call a lattice T arithmetic if it is algebraic and if in addition $K(T)$ is a sublattice, we have the

COROLLARY. The category of lattices with identity and identity preserving semilattice morphisms is dual to the category of arithmetic lattices and order continuous semilattice morphisms.

III. The third chapter contains various applications of the duality theory to lattice theory. We begin with a preliminary section in which we record simple consequences of the duality, such as e.g. the following: If $f \in \underline{S} \cup \underline{Z}$, then $\hat{f}$ is injective iff f is surjective. A family $S \to S_j$ of morphisms is a product diagram in one of the two categories iff the family $\hat{S}_j \to \hat{S}$ is a coproduct diagram in the other. (In fact this holds for arbitrarily limits and colimits). Quotients are dual concepts for subobjects.

We proceed to discuss concepts which are of key importance in lattice theory.

A fundamental role is played by the prime elements in a semilattice. We say that $p \in S$ is prime iff $ab \leq p$ implies $a \leq p$ or $b \leq p$, and we call the set of primes Prime S. We say that S is primally generated iff Prime S generates S (in either $\underline{S}$ or $\underline{Z}$; note that $T \in \mathrm{ob}\, \underline{Z}$ is generated by $A \subseteq T$ iff T is the smallest closed subsemilattice of T containing A). A morphism $f : S \to T$ between semilattices will not automatically preserve primes; if indeed we have $f(\text{Prime } S) \subseteq \text{Prime } T$, then we call f <u>a prime-morphism</u>. A prime-morphism into 2 is a <u>prime-character</u>. A <u>prime filter</u> is a prime element in the filter semilattice.

In a semilattice finite sups need not exist. Nevertheless, various concepts of distributivity are possible. We say that a semilattice is distributive iff $\uparrow a(\uparrow b \cap \uparrow c) = \uparrow ab \cap \uparrow ac$ for all a, b, c. We say that a morphism $f: S \to T$ is a <u>sup-morphism</u> iff $f^{-1}(Q)$ is a prime filter in S for every prime filter Q of T. These morphisms do preserve existing sups if the prime filters of T separate the points. Thus all <u>sup-characters</u> of S (i.e. sup-morphisms $S \to 2$) preserve existing sups. The duality theory sheds light on the mutual relation of these concepts:

<u>THEOREM</u>. <u>A morphism $f \in \underline{S} \cup \underline{Z}$ is a prime morphism iff its dual $\hat{f}$ is a sup-morphism. If $S \in \text{ob } \underline{S}$ and $T \in \text{ob } \underline{Z}$ is its dual then the following statements equivalent</u>:

(1) <u>S is a distributive semilattice</u>. (2) <u>The sup-characters of S separate the points</u>. (3) <u>S is a subsemilattice of a distributive lattice (such that the inclusion preserves sups)</u>. (4) <u>T is primally generated</u>. (5) <u>T is a distributive lattice</u>. (6) <u>T is a Brouwerian lattice</u>.

<u>Further, the following statements are equivalent</u>:

(i) <u>S is primally generated</u>. (ii) <u>T is a topological distributive lattice</u>. (iii) <u>The lattice characters of T separate the points</u>. (iv) <u>$K(T)$ is primally generated</u>.

<u>Finally, the following are equivalent</u>:

(I) <u>S is a distributive lattice</u>. (II) <u>T is an arithmetic distributive lattice</u>.

At this point we can easily tie in results of other duality theories which are exemplified by recent results of Keimel and Hofmann (Memoir of the Amer. Math. Soc. 122 (1972)). We exemplify the amalgamation of these two theories by the following

THEOREM. The subcategory in Z of distributive lattices and lattice morphisms is dual to the category of continuous maps between topological spaces X having the following properties:

(a) X is a T_o-space in which every irreducible set is a singleton closure. (A set Z is irreducible in X if it is closed and not contained in the union of two closed subsets unless at least one of the two contains Z.)

(b) X has a basis of quasicompact open sets (i.e. every open set is the union of the quasi-compact open subsets which it contains).

Thus the category of these spaces is equivalent to the category of distributive semilattices and prime morphisms.

Remark. The spaces described in (a) and (b) have been called spectral spaces since they occur, e.g., as the spectra of commutative rings.

In a subsequent section we proceed to discuss Boolean lattices in S and in Z (a Boolean lattice in Z is a Boolean topological lattice and as such is equivalent to a compact topological Boolean algebra). Recall that a semilattice in S is free (over Set) iff it is the $\cup$-semilattice of all finite subsets of some set X. We denote such a semilattice by ${}^{X}2$ (since indeed it is the coproduct of X copies of 2). The category Z has a free functor from the category ZComp of compact zero dimensional spaces (which is left adjoint to the forgetful functor). It associates with a space $X \in$ ZComp the $\cup$-semilattice C(X) of all closed subsets of X with the Hausdorff topology. We say that such a semilattice is free over ZComp. We have the

THEOREM. Let $S \in$ ob S and $T \in$ ob Z its dual. Then

(a) S is a Boolean lattice iff T is free over ZComp

(b) S is free over Set iff T is a Boolean topological lattice.

In particular, the compact topological Boolean lattices are precisely the 2^X for some set X.

A morphism $f \in S \cup Z$ between Boolean objects in either category is a Boolean morphism (i.e. preserves complements) iff its dual $\hat{f}$ is co-atomic (i.e. maps all co-atoms in its domain into the set of co-atoms of the co-domain.) (A co-atom a is an element which is maximal relative to the property $a < 1$.).

In a further section we complement the work of Kimura and Horn about the injectives and projectives in S.

The results are as follows:

THEOREM. Let $S \in$ ob S and $T \in$ ob Z be its dual. Then the following are equivalent.
(1) S is projective in S (2) S is a retract of some X_2 (3) S is a distributive lattice with $\uparrow s$ finite for all $s \in S$ (4) S is primally generated and $\uparrow p$ is finite for all $p \in$ Prime S (5) T is injective in Z (6) T is a retract of some 2^X (7) T is a distributive arithmetic lattice such that Tk is finite for all $k \in K(T)$ (8) T is a distributive arithmetic and topological lattice such that Tk is finite for all $k \in K(T)$.
Furthermore, the following conditions are equivalent:
(i) S is injective in S. (ii) S is a retract of a complete Boolean lattice. (iii) S is a complete Brouwerian lattice. (iv) T is projective in Z. (v) T is a retract of some free object (over Set). (vi) T is a retract of some C(E) with an extremally disconnected space E.

IV. In a final chapter we discuss application of duality theory to the theory of compact semilattices. A portion of this is presented in another contribution (K. H. Hofmann and M. Mislove, Stability in compact zero dimensional semilattices). As an example of material not presented at this conference which will be discussed in detail in the monograph let us mention the following results. If X is a topological space we may associate with it two cardinals, its weight $w(X) = \min\{a \mid$ there is a basis for the topology of X of cardinality $a\}$ and its density character $d(X) = \min\{a \mid$ there is a dense subset of X of cardinality $a\}$. These cardinals in a sense describe the size of the space X. We then have the following

THEOREM. *Let* T *be a compact zero dimensional semilattice and* S *its dual semilattice. Then* $w(T) = \text{card } S \le 2^{d(T)}$. *In fact if, for a cardinal* a *we let* $\log a$ *denote the smallest cardinal* b *with* $a \le 2^b$, *then* $d(T) = \log \text{card } S$.

We also use duality to characterize extremally disconnected objects in $\underline{Z}$:

THEOREM. *Let* T *be a zero dimensional compact semilattice. Then the following are equivalent statements.*

1) T *is extremally disconnected.*
2) *Every converging sequence is finally constant.*
3) T *satisfies the ascending chain condition and for each* t *the set of minimal elements in* $\uparrow t \setminus \{t\}$ *is finite.*
4) T *is finite.*

An account of the history of the subject and detailed references are to follow in the complete presentation of the material indicated in this report.