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Distributive Topological Lattices

(Dedicated to L. W. Anderson)

by

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It is our purpose to discuss some recent developments in the theory of distributive topological lattices. As is usual in such discussions the topics most interesting to the author are those in which he has made some contributions. We shall carry forward that hallowed tradition. L. W. Anderson, to whom this paper is dedicated, gave a survey of the theory of topological lattices in 1961 [1]. We shall take that survey as a foundation for our subsequent remarks.

We begin with some results about lattices of (semilattice) ideals of compact semilattices. Aside from the intrinsic value of such lattices we begin our discussion here because such lattices provide us with examples and counter-examples needed in later sections. We then construct representations for compact, distributive lattices of finite breadth. This topic leads naturally to questions involving compactification of lattices which will be discussed in section 3. We conclude with some remarks about the congruence extension property for compact lattices.

Let $\mathcal{P}$ be the category of Hausdorff topological spaces equipped with closed partial orders. The morphisms of $\mathcal{P}$ will be continuous order-preserving maps. For $(S,\leq)$ in $\mathcal{P}$ and $x,y \in S$, $x \vee y = \text{l.u.b. } \{x,y\}$ and $x \wedge y = \text{g.l.b. } \{x,y\}$ where they exist. $\mathcal{S}$ is the subcategory of $\mathcal{P}$ consisting of those objects S for which $x \wedge y$ exists for all $x,y \in S$ and the map $\wedge : S \times S \longrightarrow S$ is continuous. The morphisms of $\mathcal{S}$ will be those $\mathcal{P}$-morphisms which in addition preserve $\wedge$. $\mathcal{L}$ will be that subcategory of $\mathcal{S}$ consisting of those objects L for which $x \vee y$ exists for all $x,y \in L$ and the map $\vee : L \times L \longrightarrow L$ is continuous. The morphisms of $\mathcal{L}$ will be those $\mathcal{S}$-morphisms which also preserve $\vee$. $\mathcal{DL}$ will be the full subcategory of distributive lattices in $\mathcal{L}$. By $\mathcal{CS}$ we shall mean the full subcategory of compact semilattices, $\mathcal{CL}$ and $\mathcal{CDL}$ are defined accordingly. For a lattice L, $J(L)$ will be the set of join-irreducible elements of L and $M(L)$ will be the set of meet-irreducible elements of L.

1. The lattice of ideals of a compact semilattice.

Suppose that S is an object of $\mathcal{CS}$. We define $\mathcal{M}(S)$ to be the set of all closed (semilattice) ideals of S i.e. closed subsets A of S such that if $a \in A$, $s \in S$ and $s \leq a$ then $s \in A$. When $\mathcal{M}(S)$ is ordered by set-theoretic inclusion and endowed with the Vietoris topology it becomes a compact, distributive topological lattice. There is a natural imbedding $\rho_S : S \longrightarrow \mathcal{M}(S)$ defined by $\rho_S(s) = s \wedge S$. (cf. [7]). For $f : S \longrightarrow T$ a morphism in $\mathcal{CS}$ we define $\mathcal{M}(f) \, \mathcal{M}(S) \longrightarrow \mathcal{M}(T)$ by $\mathcal{M}(f)(A) = f(A) \wedge T$. $\mathcal{M}$ is a covariant functor of $\mathcal{CS}$ to $\mathcal{CDL}$. The following results appear in [13] and [10] or can be derived from results therein.

(1) $\rho_s(S)$ is the set of join-irreducible elements of S.

(2) The set of meet-irreducible elements of $\mathcal{M}(S)$ is the set of closed prime ideals of S.

(3) If I is the closed real interval from 0 to 1 with its natural order then $\mathcal{M}((I,\wedge))$ is isomorphic with $(I,\wedge,\vee)$.

(4) The following are equivalent

(a) $\mathcal{S}(S,I)$ separates points.

(b) $\mathcal{L}\mathcal{M}(S),I)$ separates points.

(c) Every point of $\mathcal{M}(S)$ is a meet of members of $M(\mathcal{M}(S))$.

(d) [6] S is a Lawson semilattice (i.e. the topology of S has a neighborhood base of subsemilattices of S).

(e) The topology of $\mathcal{M}(S)$ has a neighborhood base of lattices.

We define $\mathcal{K}(\mathcal{K}')$ to be the full subcategory of objects K of $\mathcal{CDL}$ such that the topology of K has a neighborhood base of $\vee$-semilattices (lattices).

(5) If $L \in \mathcal{CDL}$ then L is the lattice of ideals of an object of $\mathcal{CS}$ if and only if (a) $L \in \mathcal{K}$ (b) $J(L) \in \mathcal{CS}$ and (c) every element of L is a join of a subset of $J(L)$. In this case L is isomorphic with $\mathcal{M}(J(L))$.

(6) Let $L \in \mathcal{CDL}$. Define $\Phi : \mathcal{M}((L,\wedge)) \longrightarrow (L,\wedge,\vee)$ by $\Phi(A) = \vee A$ (i.e. the sup of A when A is considered as a subset of $(L,\wedge,\vee)$). Then $\Phi : \mathcal{M}(L) \longrightarrow L$ is an $\mathcal{L}$-morphism if and only if $L \in \mathcal{K}$.

It was pointed out to the author by J.D. Lawson and J.W. Stepp that from (6) and [6] we have

(7) $(\mathcal{K}, \mathcal{CS}, U, \mathcal{M})$ and $(\mathcal{K}', \mathcal{CLS}, U, \mathcal{M})$ are adjunctions where U in each case is the suitable restriction of the functor $U : \mathcal{CDL} \longrightarrow \mathcal{CS}$ which forgets the $\vee$-operation and $\mathcal{CLS}$ is the category of compact Lawson semilattices.

2. Compact distributive lattices of finite breadth.

A lattice L has breadth n (n a positive integer) if (1) given any finite subset A of L there is $B \subseteq A$ such that card $B \leq n$ and $\wedge B = \wedge A$ and (2) there is $A \subseteq L$ with card $A = n$ and for $B \subseteq A$ with $B \neq A, \wedge B \neq \wedge A$. The main result of this section is a representation theorem obtained by the author and Kirby Baker in [2]. The steps of this theorem are of some independent interest.

(2.1) If L is a complete lattice of finite breadth n and the operations of L are continuous with respect to order convergence then every element of L is the meet of a subset A of $M(L)$ with card $A \leq n$. Also each element of L is a join of a subset B of $J(L)$ with card $B \leq n$.

(2.2) If L satisfies the hypotheses of (2.1) and is distributive then by applying the Dilworth coding theorem [4] and Zorn's lemma $J(L) = K_1 \cup \ldots \cup K_n$ where each K_i is a maximal chain in $J(L)$. When endowed with the interval topology each K_i becomes a compact chain.

(2.3) If L satisfies the hypotheses of (2.2) the maps $\sigma_i : L \longrightarrow K_i$ defined by $\sigma_i(x) = \vee\{k \in K_i ; k \leq x\}$ are continuous lattice homomorphisms.

(2.4) If L satisfies the hypotheses of (2.2) then L is a member of $\mathcal{CDL}$ with either the interval on order topology (which coincide). Moreover the map $\sigma_i \times \ldots \times \sigma_n : L \longrightarrow K_1 \times \ldots \times K_n$ is an imbedding of L into a product of n compact chains.

A. C. Dempster by a very different method had obtained the representation theorem for lattices of breadth two at about the same time [3].

From (2.4) the class of those objects L of $\mathcal{CDL}$ such that $B_n(L) \leq n$ and the class of all closed sublattices of products of n compact chains coincide.

A result similar to (2.3) was obtained by the author and E. D. Shirley in [9].

(2.5) Let L and M be locally compact, connected topological lattices of finite breadth and let M be distributive. If $\varphi : L \longrightarrow M$ is a $\wedge$ and $\vee$ preserving map of L onto M then φ is continuous.

We will now give some examples to show what happens to these results when some of the hypotheses are dropped.

(2.6) J. D. Lawson in [7] gave an example of a metric, connected, one-dimensional object of $\mathcal{CS}$, which we denote by Law, with the property that $\mathcal{S}(\text{Law}, I)$ is trivial. Then $\mathcal{L}(\mathcal{M}(\text{Law}), I)$ is also trivial. Hence not every element of $\mathcal{M}(\text{Law})$ is the meet of meet-irreducibles. In fact, if $\mathcal{M}(\text{Law})$ does not already have this property, it is possible to create an object $L \in \mathcal{CDL}$ with $B_r(L) = \infty$ such that $M(L) = \{1\}$. Since $\mathcal{M}(\text{Law})$ is a lattice of ideals every element of $\mathcal{M}(\text{Law})$ is a join of join-irreducibles. $\mathcal{M}(\text{Law})^{op}$ ($\mathcal{M}(\text{Law})$ with order reversed) has the opposite properties.

(2.7) Let $T = \{1 - \frac{1}{n}\ ;\ n = 1,2,\ldots\} \cup \{1\} \leq I$. With the inherited order from I T is a compact chain. Form $S = T \times I/_{T \times \{0\}}$. (The Rees quotient of $T \times I$ by $T \times \{0\}$ i.e. $T \times \{0\}$ is shrunk to a point). Then $T \in \mathcal{CS}\backslash\mathcal{CL}$. Let I_o be the unique chain from 0_S to 1_S. Define $\varphi : S \longrightarrow I_o$ by $\varphi(s) = \vee\{u \in I_o;\ u \leq s\}$. Then φ is a $\wedge$-preserving map of S onto I but is not continuous because $\varphi(S\backslash I_o) = 0$. φ induces a lattice homomorphism $\Phi : \mathcal{M}(S) \longrightarrow (I_o) = I_o$ defined by $\Phi(A) = \vee\{\rho_S(u) \in \rho_S(I_o); \rho_S(u) \leq A\}$. $\rho_S(I_o)$ is a maximal chain in $J\mathcal{M}(S))$ and the map $\Phi : \mathcal{M}(S) \longrightarrow I_o$ satisfies all of the hypotheses of (2.5) except for finite breadth. However Φ cannot be continuous because Φ restricted to $\rho_S(S)$ is the same as $\varphi : S \longrightarrow I_o$.

Thus (2.3) and (2.5) do not hold without finite breadth. This example is found in [9].

3. Locally convex lattice

A subset A of a lattice L is called convex if whenever $x,y \in A$ and $x < y$ then $[x,y] \subseteq A$. A topological lattice is called locally convex if its topology has a neighborhood base of convex sets. To see that many lattice are locally convex we have:

(3.1) The following classes of lattices are locally convex

(a) Compact lattices (Nachbin [8])

(b) Locally compact and connected lattices (L.W. Anderson [1]).

(c) Discrete lattices

(d) Sublattices of locally convex lattices.

To see that some lattices are not locally convex we have:

(3.2) In the plane let $L = \{(1,\frac{1}{2n});\ n = 1,2,\ldots\} \cup \{(0,\ \frac{1}{2n+1});n = 1,2,\ldots\} \cup \{(1,0)\}$. An order $\leq$ is defined on L by setting $(x_1,y_1) \leq (x_2 y_2)$ if and only if $y_1 \leq y_2$. With the topology L inherits from the plane it becomes a topological lattice, in fact a chain. However L is not locally convex.

From [11] we have

(3.3) Let L be an object of $\mathcal{DL}$ which is locally convex and is of finite breadth n. Then L can be imbedded in a member of $\mathcal{CDL}$. Hence from (2.4) L can be imbedded in a product of n compact chains.

(3.3) characterizes all sublattices of finite products of compact chains in the same way that (2.3) characterizes all closed sublattices of finite products of compact chains.

For infinite breadth we have more difficulty. First we note that $\mathcal{M}(\mathrm{Law})$ is an object of $\mathcal{CDL}$ which cannot be imbedded in any product of compact chains.

(3.4) Let L be a locally compact and connected distributive topological lattice then L can be imbedded in a compact lattice [14].

(3.5) Let L be the product of countably many copies of the two point lattice. When L is given the discrete topology it cannot be imbedded (as a topological lattice) in any compact lattice [14].

4. Congruence extension property for $\mathcal{CDL}$.

It is well-known that the congruence extension property characterizes distributive lattices (cf. [5]). We make the obvious modification of this property to $\mathcal{L}$ as follows: $L \in \mathcal{L}$ has the congruence extension property (c.e.p.) if given A a closed sublattice of L and $\varphi : A \longrightarrow B$ an $\mathcal{L}$-morphism on A there is an $\mathcal{L}$-morphism $\Phi : L \longrightarrow M$ such that the following diagram commutes

$$\begin{tikzcd} L \arrow[r, "\Phi"] & M \\ A \arrow[u, "i"] \arrow[r, "\varphi"] & B \arrow[u, "j"'] \end{tikzcd}$$

where i the inclusion map of A into L and j is an imbedding of B into M.

For $L \in \mathcal{CL}$ let $\mathcal{C}(L)$ be the lattice of closed congruence on L. From [12] we have the following results:

(4.1) If $L \in \mathcal{CDL}$ and $Br(L) < \infty$ then $\mathcal{C}(L)$ is a distributive lattice.

(4.2) If $L \in \mathcal{CDL}$ and $Br(L) < \infty$ then L has c.l.p. It seems likely that the following conjecture should hold

(C) If $L \in \mathcal{CDL}$ and $\dim L = 0$ then $\mathcal{C}(L)$ is a distributive lattice.

However the general question remains

(Q) If $L \in \mathcal{L}(\mathcal{CL})$ is $\mathcal{C}(L)$ distributive?

(4.3) Let X be a countable product of copies of the two point lattice endowed with the Cartesian product topology. (as such $X \in \mathcal{CL}^{T}\mathcal{L}$). Then X can have no dimension-raising, continuous $\wedge$-preserving maps.

Then because the usual chain lattice C in the Cantor set can be imbedded in X and C has dimension-raising lattice homomorphisms it follows that

(4.4) X does not have c.e.p.

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