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REPRESENTATIONS OF LATTICE-ORDERED RINGS

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In this paper we present two typical representation theorems for archimedean lattice-ordered rings with identity, a classical one by means of continuous extended real valued functions and a less classical one by means of continuous sections in sheaves.

0. Introduction.

The oldest question in the theory of lattice-ordered rings, groups, and vector spaces probably is the question of representations by real valued functions. In the forties F. MAEDA and T.OGASAWARA [17], H. NAKANO [19], T. OGASAWARA [2o] and K. YOSIDA [23] and probably others established such representation theorems by continuous functions for vector lattices, M.H. STONE [22] and H. NAKANO [18] for lattice-ordered real algebras. (See also R.V. KADISON [13].) In the sixties, this question has been taken up in a more general and modern presentation e.g. by S.J. BERNAU [1], M. HENRIKSEN and D.G. JOHNSON [9], D.G. JOHNSON [11], D.G. JOHNSON and J. KIST [12], J. KIST [15].

Our first theorem has been proved in various ways and various generality in almost all of the papers listed above. Our proof might contain some new aspects: It is a self-contained proof not using any ideal theory, based on a notion of characters like GELFAND's representation theorem for commutative C -algebras. In the case of lattice-ordered groups this idea is implicitely used by D.A. CHAMBLESS [4], in the case of Banach lattices it is explicitely used by H.H. SCHAEFER [24].

Our second representation theorem as well as its proof is inspired by GROTHENDIECK's construction of the affine scheme of a commutative ring with the one exception that to some extent the lattice operations are used instead of the ring operations. The sheaf associated with a lattice-ordered ring also reminds the sheaf of germs of continuous functions, although this second theorem applies to a much bigger class of lattice-ordered rings than that representable by extended real valued functions. As references for theorem 2 we give [7], [14], [15].

1. Representation by continuous extended real valued functions.

In this paper, rings are always supposed to have an identity e ; but commutativity is not required (although archimedean f-rings turn our to be commutative).

DEFINITION 1. A lattice-ordered ring is a ring A endowed with a lattice order $\leq$ in such a way that $a+b \geq 0$ and $ab \geq 0$ for all elements $a \geq 0$ and $b \geq 0$ in A . We denote by $A_+ = \{a \in A \mid a \geq 0\}$ the positive cone of A , and by $\vee$ and $\wedge$ the lattice operations.

If A and A' are lattice-ordered rings, a function $f:A \to A'$ is called an ℓ-homomorphism, if f is a ring and a lattice homomorphism (preserving the identity).

Unfortunately, only few things can be said about lattice-ordered rings in general. Usually one considers a more special class of lattice-ordered rings:

DEFINITION 2. A lattice-ordered ring A is called an abstract function ring (shortly f-ring) if A is a subdirect product of totally ordered rings.

BIRKHOFF and PIERCE [3] have shown that a lattice-ordered ring A is an f-ring if and only if one has:

$a \wedge b = 0$ implies $a \wedge bc = 0 = a \wedge cb$ for all $c \in A_+$.

In a fist approach we call concrete function ring every ℓ-subring (i.e. subring and sublattice) of the f-ring $C(X)$ of all continuous real valued functions on some topological space X. The answer to the question, whether every abstract function ring is isomorphic to a concrete function ring is obviously negative; for a non-archimedean field cannot be represented in this way.

DEFINITION 3. A lattice-ordered ring A is called <u>archimedean</u>, if for every pair of elements a,b in A with $a \neq 0$ there is an integer n such that $na \nleq b$.

BIRKHOFF and PIERCE [3] have shown that an archimedean lattice-ordered ring is an f-ring if and only if the identity e is a weak order unit, i.e. $e \wedge x > 0$ for every $x > 0$.

Every archimedean abstract function ring can be represented as a concrete function ring, if one generalises slightly the notion of concreteness: Let X be a topological space. Denote by $E(X)$ the set of all continuous functions $f:U_f \to \mathbb{R}$, where U_f is any open dense subset of X. We identify two such functions $f:U_f \to \mathbb{R}$, $g:U_g \to \mathbb{R}$, if f and g agree on $U_f \cap U_g$. (Note that the intersection of two open dense subsets is open and dense.) Then $E(X)$ is an f-ring.

A more formal construction of $E(X)$ goes as follows: Let $\mathcal{U}$ be the collection of all open dense subsets of X.

For each $U \in \mathcal{U}$ consider $C(U)$, the f-ring of all continuous real valued functions defined on U. If $U,V \in \mathcal{U}$ and $V \subset U$, define the ℓ-homomorphism $\rho_U^V : C(U) \to C(V)$ to be the restriction map $f \mapsto f|V$. Then

$$E(X) = \varinjlim_{U \in \mathcal{U}} C(U) .$$

With the exception of some rather special classes of spaces X, the f-ring $E(X)$ cannot be represented in any $C(Y)$, as one may conclude from some results of CHAMBLESS [5].

If we call <u>concrete function ring</u> every ℓ-subring of some $E(X)$, we can state:

THEOREM 1. <u>Every archimedean f-ring with identity can be represented as a concrete function ring</u>.

One can prove something more precise by using the extended real line

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty , +\infty \},$$

endowed with the usual order and topology; we also use the usual conventions for addition and multiplication with $\pm\infty$, as far as reasonable.

A continuous function $f: X \to \overline{\mathbb{R}}$ is called an almost finite extended real valued function, if the open set $U_f = \{x \in X \mid f(x) \neq \pm\infty \}$ is dense in X. The set $D(X)$ of all these functions can be naturally embedded in $E(X)$ by the assignment $f \mapsto f|U_f$. This allows us to consider $D(X)$ as a subset of $E(X)$. $D(X)$ always is a sublattice of $E(X)$, but it need not be a subring. Every ℓ-subring of $E(X)$ contained in $D(X)$ will be called an f-ring of continuous extended almost finite real valued functions. Now we state:

Theorem 1'. Every archimedean f-ring (with identity e) is isomorphic to a lattice-ordered ring of continuous extended almost finite real valued functions defined on some compact Hausdorff space.

The proof is carried out in several steps. In a sense, the whole proof is based on the following result credited to PICKERT [22] by FUCHS [6], but probably known for quite some time:

(a) THEOREM (αρχιμεδησ ([1]) ?). If A is an archimedean totally ordered ring with identity, then there is a unique order preserving isomorphism from A onto some subring of $\mathbb{R}$.

(b) Let A be any f-ring with identity e . A function $\omega : A \to \overline{\mathbb{R}}$ is called a character of A , if it satisfies:

(1) $\omega(e) = 1$;

(2) $\omega(a \vee b) = \omega(a) \vee \omega(b)$, $\omega(a \wedge b) = \omega(a) \wedge \omega(b)$

(3) $\omega(a+b) = \omega(a) + \omega(b)$, $\omega(ab) = \omega(a)\omega(b)$, whenever the right hand side is defined in $\overline{\mathbb{R}}$.

Let X denote the set of all characters of A . Note that X is a subset of $\overline{\mathbb{R}}^A$. Endow $\overline{\mathbb{R}}^A$ with the product topolo-

([1]) Archimedes, Greek mathematician (287? to 212 b.c.)

gy which is compact Hausdorff. It is easily checked that X is a closed subset of $\overline{\mathbb{R}}^A$. Consequently, X is a compact Hausdorff space, called the <u>character space</u> of A.

(c) For every a in A define a function $\hat{a}:X \to \overline{\mathbb{R}}$ by $\hat{a}(\omega) = \omega(a)$ for all $\omega \in X$. As $\hat{a}$ is the a-th projection $\overline{\mathbb{R}}^A \to \overline{\mathbb{R}}$ restricted to X, it is a continuous function.

(d) For all a,b in A we have:

$$(a \vee b)\hat{} = \hat{a} \vee \hat{b} \quad \text{and} \quad (a \wedge b)\hat{} = \hat{a} \wedge \hat{b}.$$

For all $\omega \in X$ on has indeed $(\hat{a} \vee \hat{b})(\omega) = \hat{a}(\omega) \vee \hat{b}(\omega) =$ $= \omega(a) \vee \omega(b) = \omega(a \vee b) = (a \vee b)\hat{}(\omega)$, and likewise for $\hat{a} \wedge \hat{b}$. In the same way one shows that

$$(a + b)\hat{}(\omega) = (\hat{a} + \hat{b})(\omega) \quad \text{and} \quad (ab)\hat{}(\omega) = \hat{a}(\omega)\hat{b}(\omega)$$

whenever $\hat{a}(\omega) + \hat{b}(\omega)$ and $\hat{a}(\omega)\hat{b}(\omega)$, respectively, are well defined in $\overline{\mathbb{R}}$.

(e) PROPOSITION. Let B be the ℓ-subring of all bounded elements of A, i.e. B is the set of all $a \in A$ such that $-ne \leq a \leq ne$ for some $n \in \mathbb{N}$. Then the assignment $a \mapsto \hat{a}$ gives an ℓ-homomorphism from B into $C(X)$ the kernel of which is the set of all a such that $na \leq e$ for all integers n. In particular, if A is archimedean, this ℓ-homomorphism is injective.

Indeed, if $a \in B$, then $\hat{a}(\omega) = \omega(a) \in \mathbb{R}$ for every character ω. By (c) and (d), $a \mapsto \hat{a}$ is then an ℓ-homomorphism from B into $C(X)$. The assertion about the kernel follows from the following lemma:

(f) LEMMA. If a is an element of A such that $na \nleq e$ for some integer n, then there is a character ω of A such that $\omega(a) \neq 0$.

Proof. Let $na \nleq e$. As A is a subdirect product of totally ordered rings, there is an ℓ-homomorphism α from A onto some totally ordered ring $\bar{A}$ such that $\alpha(na) > \alpha(e)$. Denote $\bar{x} = \alpha(x)$ for all x. Now let $\bar{B}$ be the ring of all bounded elements of $\bar{A}$ and $\bar{I}$ the set of all $\bar{x}$ with $n\bar{x} < \bar{e}$ for all integers n. Then $\bar{I}$ is a convex ideal of $\bar{B}$ and $\bar{B}/\bar{I}$ is an archimedean totally ordered ring with identity. Using (a) we can find an order preserving homomorphism $\bar{\omega}:\bar{B} \to R$ such that $\bar{\omega}(\bar{e}) = 1$, whence $\bar{\omega}(a) \neq 0$. By defining

$$\bar{\omega}(\bar{x}) = \begin{cases} +\infty & \text{if } \bar{x} < n\bar{e} \text{ for all } n > 0, \\ -\infty & \text{if } \bar{x} > n\bar{e} \text{ for all } n > 0, \end{cases}$$

we have extended $\bar{\omega}$ to a character of $\bar{A}$. Then $\omega = \bar{\omega}\circ\alpha$ is a character of A such that $\omega(a) \neq 0$.

In order to achieve the proof of theorem 1' we need two more lemmas. As in the preceding lemmas, we are working an an f-ring with identity, not necessarily archimedean.

(g) LEMMA. The sets of the form

$$V(f) = \{\omega \in X \mid \hat{f}(\omega) = \omega(f) > 0\}, \quad 0 \leq f \leq e, \ f \in A,$$

constitute a basis of the topology on X.

Proof. We first note that, by the definition of the product topology on R^A, the sets $\overline{V}(f,q) = \{\omega \in X \mid \omega(f) > q\}$ and $\underline{V}(f,q) = \{\omega \in X \mid \omega(f) < q\}$ with $f \in A$ and $q = \frac{n}{m} \in \mathbb{Q}$ form a subbasis of the topology on X. As $\omega(f) > \frac{n}{m}$ iff $\omega(mf) > n = \omega(ne)$ iff $\omega(mf - n) > 0$, we conclude that $\overline{V}(f,q) = \overline{V}(mf-ne,0) = V(mf-ne)$; likewise $\underline{V}(f,q) = V(ne-mf)$. Thus, the $V(f)$, $f \in A$, already form a subbasis. They even form a basis, as $V(f) \cap V(g) = V(f \wedge g)$. As $V(f) = V((f \vee 0) \wedge e)$, we may restrict our attention to elements f with $0 \leq f \leq e$.

(h) LEMMA. If A is archimedean, one has $a = \bigvee_{n \in \mathbb{N}} (a \wedge ne)$ for all $a \in A_+$.

Proof. By the way of contradiction, we suppose that there is an element b in A such that $a \wedge ne \leq b < a$ for all $n \in \mathbb{N}$. As $0 < a-b$ and as e is a weak order unit, $e \wedge (a-b) > 0$. The element $d = e \wedge (a-b)$ satisfies $0 < d \leq e$ and $d \leq a$. Under the hypothesis that $(n-1)d \leq a$, we can conclude that $(n-1)d \leq (n-1)e \wedge a \leq b$, which together with $d \leq a-b$ implies $nd \leq a$. Thus, we have shown by induction that $nd \leq a$ for all $n \in \mathbb{N}$ which is incompatible with the archimedean hypothesis.

(j) Now we are ready to achieve the proof of theorem 1': We first show that $\hat{a} = \hat{b}$ implies $a = b$. As $a = (a \vee 0) - (-a \vee 0)$, it suffices to consider the case where $a,b \geq 0$. If $\hat{a} = \hat{b}$, then $\hat{a} \wedge n\cdot 1 = \hat{b} \wedge n\cdot 1$ for all $n \in \mathbb{N}$, whence $(a \wedge ne)\hat{} = (b \wedge ne)\hat{}$ for all $n \in \mathbb{N}$ by (d). As $a \wedge ne$ and $b \wedge ne$ are bounded, we conclude that $a \wedge ne = b \wedge ne$ for all $n \in \mathbb{N}$ by (e). Hence, $a = b$ by (h). Now we prove that $\hat{a} \in D(X)$: If U is an open subset of X such that, for exemple, $\hat{a}(\omega) = +\infty$ for all $\omega \in U$, then by (g) we may suppose that $U = V(f)$ for some f in A with $0 \leq f \leq e$, and we conclude that $\hat{a} = (a+f)\hat{}$. Consequently, $f = 0$ by the preceding, i.e. $U = V(f) = \emptyset$. Finally, (d) shows that $a \mapsto \hat{a}$ is an ℓ-homomorphism.

REMARKS. 1. Using property (g), one can show easily that $(\bigvee_{i \in I} a_i)\hat{} = \bigvee_{i \in I} \hat{a}_i$, whenever $\bigvee_{i \in I} a_i$ exists in A. The same holds for arbitrary meets.

2. Every archimedean f-ring without nilpotent elements can be embedded in an f-ring with identity which is archimedean, too. Consequently, all archimedean f-rings with identity have representations as concrete function rings.

3. Let $\psi : Y \to X$ be a continuous map of topological spaces such that $\psi^{-1}(U)$ is dense in Y for every dense open subset U of X. For every $f \in E(X)$ the function $f \circ \psi$ belongs to $E(X)$. Thus, we obtain an ℓ-homomorphism $E(\psi) : E(X) \to E(Y)$; moreover, $D(X)$ is mapped into $D(Y)$. If, in addition, the image $\psi(Y)$ is dense in X, then $E(\psi)$ is injective. This gives the idea, how to obtain

representations of A on other spaces Y from the above representation on the character space X . We list two cases:

Let $\pi : P \to X$ be the projective cover of the character space X of the archimedean f-ring A (cf.GLEASON [8]). Then π is surjective and has the property required above. Moreover, P is extremally disconnected, compact and Hausdorff. Thus, we obtain a representation of A in E(P) for some extremally disconnected compact Hausdorff space P. One can show that this representation of A is just the representation of BERNAU [1].

In a similar way one can obtain JOHNSON's [10] and KIST's [15] representation theorems from theorem 1'; for the character space X is homeomorphic with the "space of maximal ℓ-ideals";further there is a continuous map from the space of all "prime ℓ-ideals" of A onto X which has all the required properties.

2. Representation by continuous sections in sheaves.

This section is not as self-contained as the first. But the proofs are complete. We refer to [14] and [15] for further information.

Let A be an arbitrary f-ring (with identity e). A subset I of A is called an ℓ-ideal, if I is a ring ideal and a convex sublattice. For an ℓ-ideal I , the

the quotient ring A/I becomes an f-ring by defining $a+I \leq b+I$ if there is an $x \in I$ with $a \leq b+x$. For every subset C of A , we define $C^{\perp} = \{x \in A \mid |x| \wedge |c| = 0 \ \forall c \in C\}$. Then $C^{\perp}$ is an ℓ-ideal, called _polar_ ℓ-_ideal_.

DEFINITION 4. The f-ring A is called _quasi-local_, if A has a unique maximal ℓ-ideal.

DEFINITION 5. A _sheaf of_ [_quasi-local_] f-_rings_ is a triple $\mathcal{F} = (E, \eta, X)$, where E and X are topological spaces and $\eta : E \to X$ is a local homeomorphism; moreover, every _stalk_ $E_x = \eta^{-1}(x)$, $x \in X$, has to bear the structure of a [quasi-local] f-ring in such a way that the functions

$$(x,y) \mapsto x+y \ , \ (x,y) \mapsto xy \ , \ (x,y) \mapsto x \wedge y$$

from $\bigcup_{x \in X} (E_x \times E_x)$ into E are continuous, where $\bigcup_{x \in X} (E_x \times E_x) \subset E \times E$ is endowed with the topology induced from the product space $E \times E$.

DEFINITION 6. Let $\mathcal{F} = (E, \eta, X)$ be a sheaf of [quasi-local] f-rings. Call _section_ of $\mathcal{F}$ every continuous function $\sigma : X \to E$ such that $\sigma(x) \in E_x$ for all $x \in X$. Denote by $\Gamma\mathcal{F}$ the set of all sections of $\mathcal{F}$. By defining on $\Gamma\mathcal{F}$ addition, multiplication and order pointwise, $\Gamma\mathcal{F}$ becomes an f-ring, in fact , an ℓ-subring of the direct product of the stalks.

Now we are ready to state:

THEOREM 2. <u>For every f-ring</u> A <u>(with identity</u> e) <u>there is a sheaf</u> $\mathcal{F} = (E,\eta,X)$ <u>of quasi-local f-rings over a compact Hausdorff space</u> X <u>such that</u> A <u>is isomorphic to the f-ring</u> $\Gamma\mathcal{F}$ <u>of all</u> (continuous global) <u>sections of</u> $\mathcal{F}$.

The proof is carried out in several steps. Let B be the f-ring of all bounded elements of A . We use the character space X of A and the representation $a \mapsto \hat{a}: B \to C(X)$ established in Proposition (e) of section 1.

(a) For every $\omega \in X$, let I_ω be the union of all the polars $a^\perp$, where a runs through all elements of A such that $\omega(a) > 0$. Then I_ω is an ℓ-ideal. Let $A_\omega = A/I_\omega$.

(b) CONSTRUCTION. Let E be the disjoint union of the quotient rings A_ω , $\omega \in X$. For every $a \in A$, define

$$\tilde{a}: X \to E \quad \text{by} \quad a(\omega) = a + I_\omega \in A_\omega \; .$$

It is easily shown that the sets of the form $\tilde{a}(U)$ with $a \in A$ and $U \subset X$ open, form a basis of a topology on E such that the triple $\mathcal{F} = (E,\eta,X)$ is a sheaf of f-rings, where $\eta: E \to X$ is the obvious projection which maps A_ω onto ω . The stalks of $\mathcal{F}$ are the f-rings A_ω . Moreover, every $\tilde{a}$ is a section of $\mathcal{F}$ and the assignment $a \mapsto \tilde{a}: A \to \Gamma\mathcal{F}$ is an ℓ-homomorphism.

(c) LEMMA. Let U be an open neighborhood of $\omega_o \in X$. There is an element p in A_+ such that $\tilde{p}(\omega_o) = \tilde{e}(\omega_o)$ and $\tilde{p}(\omega) = 0$ for all $\omega \notin U$.

Proof. By lemma (g) in section 1, there is an element f in A_+ such that $\omega_o \in V(f) \subset U$. Then $\omega_o(f) > 0$ and $\omega(f) = 0$ for all $\omega \notin U$. After replacing f by $nf \wedge e$ for a suitably large n, we may suppose that $\omega_o(f) = 1$. Now let $g = 3f-e$ and $h = 2f-e$. We use the notation $x_+ = x \vee 0$ and $x_- = -x \vee 0$ and note that $x_+ \wedge x_- = 0$. Let

$$P = g_+^{\perp} \text{ and } Q = h_+^{\perp} .$$

We have $\omega_o(h_+) = (2\omega_o(f) - \omega_o(e)) \vee 0 = 1$, whence $Q = h_+^{\perp} \subset I_{\omega_o}$. For every $\omega \notin U$, one has $\omega(g_-) = \omega(e-3f) \vee 0 = (\omega(e) - 3\omega(f)) \vee 0 = 1$, Hence, $P^{\perp} \subset g_-^{\perp} \subset I_{\omega}$. The ℓ-ideal $P^{\perp}+Q$ contains $g_+ + h_- = (3f-e) \vee 0 + (e-2f) \vee 0$, and this element is not contained in any proper ℓ-ideal of A, as its image in every non zero totally ordered ring is easily seen to be strictly positive. Consequently, $P^{\perp}+Q = A$. Thus, there are positive elements $p \in P^{\perp}$ and $q \in Q$ such that $p+q = e$. This means that $p+Q = e+Q$ and consequently $p+I_{\omega_o} = e+I_{\omega_o}$ and $p \in I_{\omega}$ for all $\omega \notin U$; thus, p has the required properties.

(d) LEMMA. A_{ω} is a quasi-local f-ring for every $\omega \in X$.

Proof. We first note that I_{ω} is contained in $\ker \omega$. From (c) it follows that $I_{\omega} \not\subset \ker \omega'$ for every $\omega' \neq \omega$. Let M_{ω} be the greatest ℓ-ideal of A contained in $\ker \omega$, i.e. M_{ω} is the sum of all ℓ-ideals contained in $\ker \omega$. Then M_{ω} is a maximal ℓ-ideal of A. It is the unique maximal ℓ-ideal containing I_{ω}; indeed, every maximal

ℓ-ideal is easily seen to be contained in the kernel of some character.

(e) LEMMA. $\bigcap_{\omega \in X} I_\omega = \{0\}$.

Proof. Suppose that $b \in I_\omega$ for all $\omega \in X$. Then $b \in a_\omega^\perp$ for some element a_ω satisfying $\omega(a_\omega) > 0$. After replacing a_ω by na_ω for a suitable n , we may suppose that $\omega(a_\omega) > 1$. The sets $W(a_\omega) = \{\omega' \mid \omega'(a_\omega) > 1\}$ are open in X and cover X . Hence, there is a finite subset F in X such that $X = \bigcup_{\omega \in F} W(a_\omega)$. Let $a = \bigvee_{\omega \in F} a_\omega$. Then $\omega(a) > 1$ for all $\omega \in X$,whence $a > e$; further $|b| \wedge a = 0$ as $b \in a_\omega^\perp$ for all ω . As e and consequently a is a weak order unit, this implies $b = 0$.

(f) The proof of theorem2 will be achieved, if we show that the assignment $a \mapsto \tilde{a} : A \to \Gamma F$ is bijective. The injectivity is a straightforward consequence of lemma (e). For the surjectivity let σ be an arbitrary section of F . We want to find an element a in A such that $\tilde{a} = \sigma$. As $\sigma = (\sigma \vee 0) + (\sigma \wedge 0)$, we may restrict ourselves to the case $\sigma \geq 0$. By the construction of the sheaf F , for every $\omega \in X$ there is an element $a_\omega \in A_+$ such that $\tilde{a}_\omega(\omega) = \sigma(\omega)$. If two sections of a sheaf coincide in a point, they agree in a whole neighborhood; hence, there is a neighborhood U_ω of ω such that $\sigma | U_\omega = \tilde{a}_\omega | U_\omega$. By lemma. By lemma (c), there is an element $p_\omega \in A_+$ such that $\tilde{p}_\omega(\omega) = \tilde{e}(\omega)$ and $\tilde{p}_\omega(\omega') = 0$

for all $\omega'\notin U_\omega$. One may suppose $p_\omega \leq e$. Let $b_\omega = a_\omega p_\omega$; then $\tilde{b}_\omega(\omega) = \sigma(\omega)$ and $\tilde{b}_\omega \leq \sigma$. Let V_ω be an open neighborhood of ω such that $\tilde{b}_\omega | V_\omega = \sigma | V_\omega$. The V_ω, $\omega \varepsilon X$, form an open covering of X . As X is compact, we may find a finite subset $F \subset X$ such that the V_ω with $\omega \varepsilon F$ already form a covering of X . Let $a = \bigvee_{\omega \varepsilon F} b_\omega$. Then $\tilde{a} = \sigma$.

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