Proc. Univ. of Houston
Lattice Theory Conf..Houston 1973

Structure of Archimedean Lattices

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Abstract An archimedean lattice is a complete algebraic lattice L with the property that for each compact element $c \in L$, the meet of the maximal elements in the interval $[0, c]$ is 0. L is hyper-archimedean if it is archimedean, and for each $x \in L$, $[x, 1]$ is archimedean. The structure of these lattices is analysed from the point of view of their meet irreducible elements. If the lattices are also Brouwer, then the existence of complements for the compact elements characterizes a particular class of hyper-archimedean lattices.

The lattice of ℓ-ideals of an archimedean lattice ordered group is archimedean, and that of a hyper-archimedean lattice ordered group is hyper-archimedean, In the hyper-archimedean case those arising as lattices of ℓ-ideals are fully characterized.

Finally, we examine the role played by these lattices in representations by lattices of open sets of some topological space. We point out a duality between algebraic, Brouwer lattices and certain T_0-spaces with bases of compact open sets.

Notation and terminology Our set theoretic notation is as follows: if A and B are subsets of a set X then $(A \subset B)$ $A \subseteq B$ denotes (proper) containment of A in B; $A \setminus B$ is the complement of B in A.

Our lattice theoretic and topological terminology is standard, except where expressly noted that it is not. The terminology from the theory of lattice ordered groups is for the most part that of Conrad [5].

1. Structure of archimedean and hyper-archimedean lattices We will be dealing exclusively with algebraic lattices: complete lattices generated by compact elements. We call an algebraic lattice archimedean if for each $c \in c(L)$, the semilattice of compact elements, the interval $[0, c]$ has the property that the meet of its maximal elements is 0. The motivation for this notion comes from the theory of ℓ-groups (abbreviation for lattice ordered groups): among the abelian ℓ-groups the archimedean ℓ-groups are characterized precisely by the condition that the lattice of its ℓ-ideals be archimedean as defined above. (Recall: an ℓ-group G is archimedean if for each pair $0 \leq a, b \in G$ $na \not\leq b$, for some natural number n.) This observation concerning the lattice of ℓ-ideals of an archimedean ℓ-group first appeared in [3], and is due to Roger Bleier.

Let us call an algebraic lattice L hyper-archimedean if it is archimedean and for each $x \in L$ $[x, 1]$ is archimedean. Again, here we are motivated by the theory of ℓ-groups: an ℓ-group G is hyper-archimedean if it is archimedean, and each ℓ-homomorphic image of G is archimedean. It is immediate then that G is hyper-archimedean if and only its lattice of ℓ-ideals is hyper-archimedean.

We shall call an element t of a lattice L meet-irreducible if $t = \bigwedge_{\lambda \in \Lambda} x_\lambda$ implies that $t = x_\mu$, for some $\mu \in \Lambda$. The notion of finite meet irreducibility is defined in the obvious manner.

Below, let L be an algebraic lattice; the first three lemmas are well known. See [2] or [7].

1.1 Lemma: If $x < 1$ in L then x is the meet of meet-irreducible elements.

1.2 Lemma: The meet of all the meet-irreducible elements of L is 0.

1.3 Lemma: L is a Brouwer lattice if and only if L is distributive.

(Note: A complete lattice B is Brouwer if and only if the following distributive law holds in B: $a \wedge (\vee_\lambda b_\lambda) = \vee_\lambda (a \wedge b_\lambda)$.)

Now the first structure theorem on archimedean lattices!

1.4 Proposition: Let L be an archimedean lattice, and $0 < c < d \in c(L)$. Then c and d have a value in common. Conversely, if L is a modular algebraic lattice, and any two comparable compact elements have a value in common, then L is archimedean.

(Remark: $p \in L$ is a value of $c \in c(L)$ if p is maximal with respect to not exceeding c. If p is a value of some compact element then p is meet-irreducible, and conversely.

We shall provide a converse to show that we cannot dispense with modularity in the converse of 1.4 .)

Proof: Suppose L is archimedean and $0 < c < d \in c(L)$. There is a maximal element m of [0, d] such that $c \nleq m$. Using Zorn's lemma pick $y \leq m$ so that it is a value of c; one can easily show then that y is a value of d as well.

Conversely, suppose L is modular, and $c, d \in c(L)$ with $0 < c < d$. If p is a value of both c and d, then by modularity $d \wedge p$ is maximal in [0, d] and $c \nleq d \wedge p$. This suffices to show L is archimedean.

1.5 Theorem: Suppose L is a hyper-archimedean lattice; then the subset of meet-irreducibles is trivially ordered. Conversely, if L is modular and the set of meet-irreducibles is trivially ordered, then L is hyper-archimedean.

Proof: Suppose first that L is hyper-archimedean. A meet-irreducible element t (in any complete lattice) always has a <u>cover</u> $\bar{t}$: namely, the meet of all the eleeents that exceed t properly. Here we show that $\bar{t} = 1$ for each meet-irreducible element t. [t, 1] is an archimedean lattice in which $\bar{t}$ is the unique atom; if $\bar{t} < 1$, one can show that a compact element d of [t, 1] exceeds $\bar{t}$. This contradicts the fact that [t, 1] is archimedean.

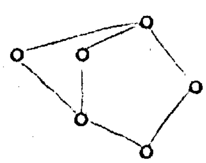
Conversely, suppose L is modular and the set $\{ t_\lambda \mid \lambda \in \Lambda \}$ is the trivially ordered set of meet-irreducibles. Then each one is maximal and their meet is 0 by lemma 1.2, so if $c \in c(L)$ and $c > 0$ then some t_μ fails to exceed c. By modularity $c \wedge t_\mu$ is maximal in [0, c] for each such t_μ, and the intersection of all these $c \wedge t_\mu$ is 0. This proves L is archimedean.

If one observes that for each $x < 1$ $\{ t_\lambda \mid t_\lambda \geqq x \}$ is the complete set of meet-irreducibles of [x, 1] the argument of the preceding paragraph shows [x, 1] is archimedean, and hence that L is hyper-archimedean.

1.6 <u>Examples</u>: a) If E is any vector space, the lattice $\underset{\sim}{V}(E)$ of subspaces of E is a hyper-archimedean, modular lattice. In fact, if R is any semisimple, Artinian ring and M is a left R-module then the lattice of submodules of M is hyper-archimedean. The author will explore this matter further elsewhere.

b) Examples can be found of non-modular archimedean and hyper-archimedean lattices; see [9].

c) Below we exhibit a lattice satisfying the condition of proposition 1.4 which is not modular and not archimedean.



Notice also that this lattice satisfies the condition of theorem 1.6, but is not hyper-archimedean.

We now direct our attention to archimedean, Brouwer lattice. Recall that in a complete Brouwer lattice it is true that for each pair of elements x and y the set $\{ z \mid x \wedge z \leq y \}$ has a unique largest element. In particular if $y = 0$, there is a largest element x' such that $x \wedge x' = 0$. It is well known that this "complementation" is an auto-Galois connection on the Brouwer lattice. The set of all elements with the property that $x = x''$ form a Boolean algebra in which the meet operation agrees with that of the underlying lattice. We shall refer to it as the <u>Boolean algebra of polars</u> and to its elements as <u>polars</u>.

1.7 <u>Proposition</u>: Let L be an algebraic, Brouwer lattice. Then L is archimedean if and only if $c' = \wedge \{$ all values of c $\}$, for each $c \in c(L)$.

Proof: Suppose L is archimedean, and $0 < c \in c(L)$ and let $\{ p_\lambda \mid \lambda \in \Lambda \}$ be the set of values of c; since $c \wedge c' = 0$ and p_λ is prime, $p_\lambda \geq c'$, for each $\lambda \in \Lambda$. If $c' < \wedge p_\lambda$ there is a compact element $d \leq \wedge p_\lambda$ so that $d \not\leq c'$, ie. $d \wedge c > 0$. Since L is archimedean there is an m maximal below c such that $d \wedge c \not\leq m$. Let y be the largest element of L such that $y \wedge c = m$; then y is a value of c, and so $y = p_\mu$, for some $\mu \in \Lambda$. But then $d \leq y$ and hence $d \wedge c \leq y \wedge c = m$, a contradiction. Thus $c' = \wedge p_\lambda$.

Using the same notation of the preceding paragraph, let us assume the indicated condition holds. It is not hard to see that the elements $c \wedge p_\lambda$ are precisely the maximal elements of $[0, c]$. Now $\wedge_\lambda (c \wedge p_\lambda) = c \wedge (\wedge_\lambda p_\lambda) = c \wedge c' = 0$, and so L is archimedean.

If $\underset{\sim}{L}(G)$ is the lattice of ℓ-ideals of a hyper-archimedean ℓ-group G then the set of prime elements of $\underset{\sim}{L}(G)$ is trivially ordered; see [6]. As we shall see this is not true of any hyper-archimedean, Brouwer lattice. Also $\underset{\sim}{L}(G)$ (for any abelian ℓ-group G) has the property that the meet of two compact elements is compact; once again this is not true in general in the abstract lattice setting. The above considerations may serve to motivate the following definitions. If L is an algebraic, Brouwer lattice we say it has the <u>finite intersection property (FIP)</u> if the meet of any two compact elements is compact. L has the <u>compact splitting property (CSP)</u> if each compact element of L is complemented, ie. if $c \vee c' = 1$, for each $c \in c(L)$.

Our next theorem ties things together properly.

1.8 <u>Theorem</u>: Let L be an algebraic, Brouwer lattice; the following are equivalent:

(a) L has the CSP.

(b) L has the FIP, and the set of primes of L is trivially ordered.

In particular, with either of these conditions L is hyper-archimedean.

Proof: (a) → (b) Suppose $c, d \in c(L)$ and $c \wedge d = \vee_{i \in I} x_i$, where the x_i are upward directed. $1 = d \vee d'$, so $c = (c \wedge d) \vee (c \wedge d')$, and hence $c = \vee_{i \in I}(x_i \vee (c \wedge d'))$. But then $c = x_{i_o} \vee (c \wedge d')$ for a suitable index i_o; this implies that $c \wedge d = x_{i_o}$. This suffices to show $c \wedge d$ is compact.

If $p < q$ are both prime, there is a $c \in c(L)$ with $c \leq q$ yet $c \nleq p$. Since $c \wedge c' = 0$, $c' \leq p$, and so $1 = c \vee c' = q \vee p = q$, a contradiction.

The converse of theorem 1.8 requires a technical lemma which we shall not prove; its proof may be found in [9].

1.9 Lemma: Suppose L has the FIP; there is a one to one correspondence between minimal primes of L and ultrafilters of $c(L)$.† This correspondence is given as follows: if p is a minimal prime, let $N(p) = \{ c \in c(L) \mid c \not\leq p \}$; its inverse assigns to an ultrafilter M of $c(L)$ the element $\vee \{ c' \mid c \in M \}$.

(† Filter here means proper filter; an ultrafilter is a maximal filter.)

1.9.1 Corollary: If L has the FIP, then $p \in L$ is a minimal prime if and only if $p = \vee \{ c' \mid c \not\leq p, c \in c(L) \}$. If p is a minimal prime and $p \geq d \in c(L)$, then $p \not\geq d'$.

Now let us prove that (b) implies (a) in theorem 1.8: suppose $c \in c(L)$ yet $c \vee c' < 1$. Let p be a meet irreducible so that $p \geq c \vee c'$; by assumption p is a minimal prime, and so by 1.9.1 $p \geq c \rightarrow p \not\geq c'$, a contradiction. This completes the proof of theorem 1.8 .

We should check that the pair of conditions contained in (b) of 1.8 are irredundant. So consider an infinite set X with the finite complement topology, and let $L = \underset{\sim}{O}(X)$, the lattice of open sets of X; this is a hyper-archimedean, Brouwer lattice (interpreting infinite meets as interiors of intersections of open sets.) However, L has the FIP (each $x \in L$ is compact) while 0 is prime.

On the other hand let $X = \{ x_1, x_2, \ldots, y, z \}$, and $X' = X \setminus \{y, z\}$. Any subset of X' shall be open , and the open neighbourhoods of y (resp. z) are the sets with a finite complement in X'. Again let $L = \underset{\sim}{O}(X)$; L is a hyper-archimedean, Brouwer lattice in which every prime is maximal, yet if $U = X \setminus \{y\}$ and $V = X \setminus \{z\}$, then U and V are compact whereas $X' = U \cap V$ is not. The author owes this example to Jed Keesling.

We close this section with a rather striking analogue of a well known result about archimedean ℓ-groups. For its proof we refer the reader to [9].

1.10 Theorem: Suppose L is an archimedean, Brouwer lattice and $x \in L$ is a polar. Then $[x, 1]$ is archimedean.

2. Realizations of hyper-archimedean, Brouwer lattices as lattices of ℓ-ideals We were motivated to study this concept of an archimedean lattice in order to discover which lattices arise as the lattice $L(G)$ of ℓ-ideals of an archimedean ℓ-group G. Although some necessary conditions become obvious rather early in the game, (such as: the lattice must be an archimedean, Brouwer lattice with the FIP plus a good deal more), the problem is in general quite hard. In the case of hyper-archimedean ℓ-groups the matter as a lot simpler; we can fully characterize those lattices arising as the lattice of ℓ-ideals of a hyper-archimedean ℓ-group.

2.1 Theorem: A hyper-archimedean, Brouwer lattice L arises as the lattice of ℓ-ideals of an ℓ-group if and only if L has the CSP.

Proof: The necessity is well known (see [6]), so we pass to a sketch of the proof of the sufficiency; further details may be found in [9]. Let $\{ p_\lambda \mid \lambda \in \Lambda \}$ be the family of primes of L, and G^* be the ℓ-group of integer-valued functions on Λ with finite range; alternatively, the ℓ-group of integral step functions on Λ. We define a mapping σ: $c(L) \to G^*$ by: $c\sigma_\lambda = 1$, if $c \not\leq p_\lambda$, and 0 if $c \leq p_\lambda$. It is easy to verify that σ is a lattice embedding.

Let G be the ℓ-subgroup of G^* generated by $c(L)\sigma$, and $P(G)$ denote its lattice of principal ℓ-ideals; these are the compact elements of $L(G)$. Define a mapping

$\tau: c(L) \to \mathbf{P}(G)$ by letting $c\tau = G(c\sigma) \equiv$ the ℓ-ideal generated by $c\sigma$ in G. Once again it is easily verified that τ is a lattice embedding, so one is only left with proving that τ is onto. Once this is done $c(L)$ and $\mathbf{P}(G)$ are isomorphic lattices, and hence so are L and $\mathbf{L}(G)$. It is here that one uses the full force of the CSP, in the following way: if $0 \neq g \in G$ expressible by $g = m_1(c_1\sigma) + \ldots + m_k(c_k\sigma)$, then this expression can be rewritten so that the compact elements of L that appear are pairwise disjoint.

2.1.1 Corollary: If G is a hyper-archimedean ℓ-group then one cannot tell from the lattice of ℓ-ideals whether G is embeddable as an ℓ-subgroup of a group of real valued step functions.

3. Topological realizations of algebraic, Brouwer lattices and dualities

For further amplification on the material in this section the reader is urged to consult Bruns [4], Hofmann & Keimel [8], Martinez [10] and Schmidt [12], plus probably many, many others.

If L is an algebraic, Brouwer lattice, let $\mathbf{I}(L)$ denote the set of meet-irreducibles, and $\mathbf{P}(L)$ denote the set of primes of L. Topologize $\mathbf{P}(L)$ by taking for its open sets the sets $P(x) = \{ p \in \mathbf{P}(L) \mid p \ngeq x \}$, for all $x \in L$; topologize $\mathbf{I}(L)$ with the subspace topology. Then $\mathbf{P}(L)$ is a T_0-space with a base of compact, open sets, (it is spectral in the terminology of [8],) and $\mathbf{I}(L)$ also has a base of compact, open sets and is t_1: every point is isolated in its closure; Bruns [4] first dealt with this separation axiom and called it $T_{1/2}$. Moreover, L is isomorphic with the lattice of open sets of both $\mathbf{P}(L)$ and $\mathbf{I}(L)$.

Let us say that a topological space X coordinatizes L if $L \simeq \mathbf{O}(X)$, the lattices of open sets of X. Bruns [4] showed that if X is a T_o-coordinatization of L then X is homeomorphic to a set $\underset{\sim}{B}$, with $\underset{\sim}{I}(L) \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{P}(L)$, having the subspace topology of $\underset{\sim}{P}(L)$. It can easily be shown that $\underset{\sim}{I}(L)$ is (up to homeomorphism) the only t_1-coordinatization, and likewise $\underset{\sim}{P}(L)$ the only spectral one. The author will take up coordinatizations of non-algebraic lattices elsewhere.

Coordinatizations by $\underset{\sim}{P}(L)$ gives rise to a duality between the category of algebraic, Brouwer lattices and lattice homomorphisms preserving all joins, and the category of spectral spaces with bases of compact, open sets, together with all continuous mappings, see [8]. Coordinatization by $\underset{\sim}{I}(L)$ also gives rise to a duality; qua objects, a one to one correspondence between algebraic, Brouwer lattices and t_1-spaces with bases of compact, open sets. The morphism-classes pertinent to this duality are so restricted so as not to merit discussion here. Presumably, any "canonical" association of a set B, with $\underset{\sim}{I}(L) \subseteq \underset{\sim}{B} \subseteq \underset{\sim}{P}(L)$, with L will produce a new duality, and it is a reasonable question whether every duality arises in this manner.

The theorem below interprets in terms of the $\underset{\sim}{I}(L)$-duality what topological conditions go with some of the lattice-theoretic notion discussed in this paper.

3.1 Theorem: Let L be an algebraic, Brouwer lattice.

i) L is archimedean if and only if each basic compact, open set of $\underset{\sim}{I}(L)$ has in the subspace topology a dense set of points whose closures are singletons.

ii) L is hyper-archimedean if and only if $\underset{\sim}{I}(L)$ is T_1.

iii) L satisfies the CSP if and only if $\underset{\sim}{I}(L)$ is Hausdorff.

iv) $\underset{\sim}{I}(L)$ is discrete if and only if L is Boolean.

For proofs of these consult [10].

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