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INSIDE FREE SEMILATTICES*

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Abstract. Necessary and sufficient conditions are derived for a given semilattice to be embeddable in a free semilattice.

§0. Introduction

I'd like to talk today about a circle of ideas concerning free semilattices. The problems involved are fairly concrete, and yet in them you will see echoes of several higher-level concepts dealt with in other papers at this conference.

As you well know, the very structure of free lattices and free modular lattices presents some very difficult questions. The basic structure of free distributive lattices is somewhat more transparent, and yet still eludes even a simple count of elements in the finite case.

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In contrast, the structure of free semilattices seems utterly trivial -- so much so, in fact, that it is hard at first to imagine how a free semilattice could give rise to any interesting questions at all.

Specifically, let us consider join-semilattices $(S, \vee)$, not necessarily with a 0-element or a 1-element. An example of such a semilattice is $\mathrm{Fin}(X)$, the semilattice of nonempty finite subsets of an arbitrary nonempty set X, with set-union being the operation. Our basic fact is that, for any nonempty set X of generators, the free semilattice $\mathrm{FSL}(X)$ on X is isomorphic to $\mathrm{Fin}(X)$. The isomorphism is the obvious one: For any $x_1,\ldots,x_n \in X$, the element $x_1 \vee \ldots \vee x_n$ of $\mathrm{FSL}(X)$ corresponds to $\{x_1,\ldots,x_n\} \in \mathrm{Fin}(X)$.

§1. Horn's Problem

A. Horn posed the following tempting "lunch-table problem."

Problem 1. Clearly, $\mathrm{FSL}(X)$ and its subsemilattices obey the condition

(*) every principal ideal is finite.

Is (*) also a sufficient condition for a semilattice S to be isomorphically embeddable in a free semilattice?

One indication pointing in the direction of an affirmative answer is that every semilattice can be isomorphically realized as a semilattice of subsets of itself; therefore the answer is always positive for <u>finite</u> semilattices. In a sense, then, the problem asks whether <u>local</u> embeddability is sufficient for <u>global</u> embeddability.

The answer, interestingly, is <u>no</u>. A counterexample is the "ladder" R depicted in Figure 1a.

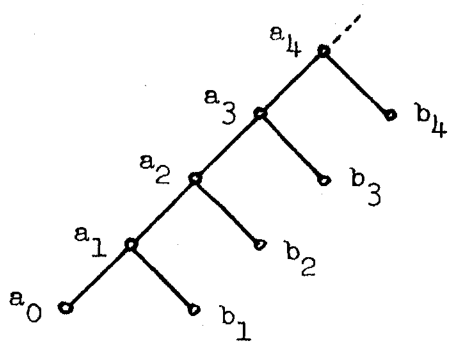

Figure 1a: the "ladder" R

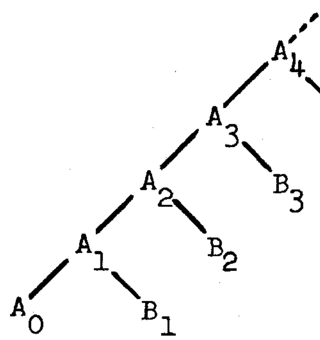

Figure 1b: subsets of X

As a ladder, R has certain deficiencies, but as a semilattice, R will be a useful example throughout this talk.

To show that R is a genuine counterexample, let us suppose, on the contrary, that R could be embedded in $FSL(X)$ for some X. Then there would be a corresponding subsemilattice of $Fin(X)$, consisting of finite subsets of X with the inclusion relations indicated by Figure 1b. For each n, $A_0 \cup B_n = A_n$, so that B_n can differ from A_n by at most a few elements of A_0, a fixed finite set. Thus, if we watch $A_0 \cap B_n$ as n varies, we must arrive at i and j $(i < j)$ such that $A_0 \cap B_i = A_0 \cap B_j$. In other words, to go from A_i to B_i we lose the same elements as in going from A_j to B_j. Since $A_i \subseteq A_j$, we conclude that $B_i \subseteq B_j$, in contradiction to Figure 1b.

This proof settles Problem 1, but it simultaneously raises another question, to be known, out of turn, as

Problem 3. Characterize those semilattices which can be embedded in a free semilattice.

An equivalent problem, of course, is to characterize those semilattices which can be isomorphically represented by finite subsets of some set, under the union operation. A logical setting for an attack on this problem is therefore the general theory of representations of semilattices by sets.

§2. Representations of semilattices

Let us review this theory. Many of the basic ideas are simply semilattice adaptations of the early distributive-lattice set-representations invented by Birkhoff and turned into a pretty, topological duality theory by Stone. Birkhoff and Frink [3] discussed meet-representations of arbitrary lattices, by ideals, which extend naturally to the semilattice case. Bruns [5,6], developed and surveyed these ideas further, placing them in their most natural context. Recently, such ideas have been studied in terms of category theory and duality and there further developed. Several speakers at this conference have followed this approach, although the specific categories used have differed, in varying degrees, from the ones I'll be using implicitly now.

Let S be a join-semilattice and let X be a set. Although our ultimate interest is representations by finite subsets, we must work now with $\mathrm{Pow}(X)$, the set of all subsets [power set] of X. We regard $\mathrm{Pow}(X)$ as a semilattice under $\cup$.

Definition 2.1. A representation of S on X is a semilattice homomorphism $\sigma : S \to \mathrm{Pow}(X)$ such that

(i) the sets $\sigma(s)$ distinguish points of X, i.e. no two distinct elements of X are contained in exactly the same subsets $\sigma(s)$, $s \in S$; and

(ii) the sets $\sigma(s)$ cover X, i.e., $\bigcup_{s\in S} \sigma(s) = X$.

If σ is one-to-one, i.e., an isomorphism, let us call σ "faithful."

Bruns [5,6] does not initially require conditions (i) and (ii), but they will be convenient for our purposes and are not really restrictive. For example, if S can be embedded in a free semilattice on a set Y of generators, then, as we noted, S is isomorphic to a semilattice of finite subsets of Y; if Y is "reduced" to a smaller set X by deleting elements not used and by identifying elements not distinguished by the finite subsets used, then we get a genuine faithful representation of S by finite subsets of X.

For a given semilattice S, there are three "famous representations" of S, all faithful:

1. The "regular" representation, σ_{reg}. Here $X = S$ and $\sigma_{reg}(s) = \{t \in S : s \not\leq t\}$.
2. The "ideal representation," σ_{id}. Here $X = Id(S)$, the set of ideals of S (including $\emptyset$), and $\sigma_{id}(s) = \{I \in Id(S) : s \notin I\}$.

3. The "CMI" representation, σ_{cmi}. Here $X = CMI(Id(S))$, the set of nonempty, completely meet-irreducible (c.m.i.) ideals of S, and again $\sigma_{cmi}(s) = \{I \in CMI(Id(S)) : s \notin I\}$.

(An element $m < 1$ of a complete lattice L is said to be completely (or strictly) meet-irreducible if m is not the meet of any set of strictly larger elements [2, p. 194]. Equivalently, there is a least element $c > m$ in L. Notice that c covers m. $Id(S)$ is an algebraic lattice, so has many c.m.i. elements; in fact, every element of an algebraic lattice is a meet of c.m.i. elements. If S has a 0-element, then $\emptyset$ is a legitimate c.m.i. element of $Id(S)$, but for technical reasons we'll always explicitly exclude $\emptyset$ in discussion of c.m.i. ideals.)

Each of the representations (1), (2), (3), has its own virtues:

(1) is the simplest, most natural representation. (The dual version of (1) is even more natural: Each element of a meet-semilattice is represented by the principal ideal it generates.)

(2) is the ultimate parent representation, in that <u>any</u> representation of S is equivalent to a "subrepresentation" of σ_{id}, obtained by restricting attention to some subset of $Id(S)$.

(I'll clarify this terminology in a moment.) For example, σ_{reg} corresponds to the set of principal ideals, and of course σ_{cmi} corresponds to the set of c.m.i. ideals. The association of each representation with a subset of Id(S) also provides a handy way of comparing the "size" of representations: Informally, we can write "$\sigma \subseteq \tau$" when the associated subsets of Id(S) are so related.

(3) is an especially economical, efficient representation, as Birkhoff and Frink point out in the case of semilattice representation of lattices [2].

Before considering an example, let's clarify the terminology just used: Two representations σ_1, σ_2 of S on sets X_1, X_2 are said to be equivalent if there is a one-to-one correspondence between X_1 and X_2 which makes $\sigma_1(s)$ correspond to $\sigma_2(s)$ for each $s \in S$. For a representation σ of S on X, a subrepresentation of σ is any representation τ of S obtained by taking a subset Y of X and setting $\tau(s) = \sigma(s) \cap Y$. To be more graphic, we can say that "τ is the intersection of σ with Y." Of course, even for faithful σ, it is possible to "lose faith" in passing from σ to τ, if we strip away too many elements of X in forming Y. An obvious necessary and sufficient condition for τ to be faithful is that σ be faithful and that of any two representing sets $\sigma(s_1) \neq \sigma(s_2)$, there be an element of Y in one and not the other. If $\sigma = \sigma_{id}$, this condition is fulfilled if Y is the set of principal ideals or the set of nonempty c.m.i. ideals.

If σ is a representation of S on a set X, the equivalent subrepresentation of σ_{id} is easily constructed: each element $x \in X$ corresponds to the ideal $I_x = \{t \in S : x \notin \sigma(t)\} \in Id(S)$, and Y is the set of such ideals. This same correspondence shows up as the basis of categorical duality theory, where ideals may appear as characters and $Id(S)$ as the dual space of S.

Let's look at all three standard representations in one particular setting.

Example 2.2. Let S be $Fin(X)$ itself, for some set X, and let $\sigma : S \to Pow(X)$ be simply the inclusion map. Thus, the elements of S are finite subsets of X; the ideals of S correspond naturally to arbitrary subsets of X. The subset A of X corresponds to the ideal $I_A = \{F \in Fin(X) : F \subseteq A\}$ of $Fin(X)$. For each element of S, i.e., for each nonempty finite subset F of X, $\sigma_{reg}(F)$ consists of all finite subsets of X which do not contain F; $\sigma_{id}(F)$ consists of ideals corresponding to all subsets of X which do not contain F; and it is not hard to determine that $\sigma_{cmi}(F)$ consists of ideals corresponding to those "cosingleton" subsets $X - \{x\}$ for which $x \in F$. Of the three, only the CMI representation has finite representing subsets even when X is infinite, so its pretense to economy is borne out in this instance.

By the way, one feature of this example, namely, that ideals of S are "represented" by subsets of the same set X, leads to a generalization, in which Id(S) is regarded as a semilattice:

<u>Observation</u> 2.3. If σ is a representation of a semilattice S on a set X, then σ^* is a representation of Id(S) on X, where $\sigma^*(I) = \bigcup_{s \in I} \sigma(s)$ for each $I \in \mathrm{Id}(S)$. Even if σ is faithful, though, σ^* may not be, as can be seen by representing Pow(X) on X by the identity map, for an infinite set X.

§3. Economy of representation.

We have now reviewed the three basic representations of a semilattice S. To judge from the example of the preceding section, the CMI representation, with its economy, will be the most useful for studying representations by finite subsets. In this connection, we have left one question as yet unanswered:

Problem 2. In what sense is the CMI representation the most economical?

Once this problem is settled, we'll be in a stronger position to investigate embeddings in free semilattices.

A natural conjecture in answer to Problem 2 would be that "$\sigma_{cmi} \subseteq \sigma$" for all faithful representations σ of S. A glance at the example of the preceding section shows the falsity of this conjecture, however: For an infinite set X and $S = \mathrm{Fin}(X)$, $\sigma_{cmi} \nsubseteq \sigma_{reg}$, even though $\sigma_{cmi}(s)$ is always finite.

Here's another try. The topological analogue of a finite set is a compact set, and, happily, compact subsets form a semilattice under union, in any topological space. (The intersection of two compact sets may not be compact.) Topological representation theories, on the other hand, most naturally represent structures having a join operation by open subsets. Stone early showed the advantage of performing a marriage of these two properties by considering open compact subsets; among spaces with many such subsets, the prime example - in fact the ideal example - is the Stone representation space of a Boolean algebra [20,21]. The Stone space is Hausdorff; for semilattices, T_0 spaces constitute a natural setting.

An investigation provides the following solution to Problem 2, with a few added frills.

Theorem 3.1. Let $\sigma : S \to \mathrm{Pow}(X)$ be a faithful representation of a semilattice S on a set X. Then the following conditions on σ are equivalent:

(1) "$\sigma_{cmi} \subseteq \sigma$";

(2) under some topology on X, every set $\sigma(s)$ is compact and open;

(3) σ^* is a faithful representation of $\mathrm{Id}(S) \setminus \{\emptyset\}$ on X;

(4) each (nonempty) c.m.i. ideal I of S has the form I_x for some $x \in X$, where $I_x = \{t \in S : x \notin \sigma(t)\}$.

[A proof of Theorem 3.1 is supplied in the Appendix.]

Thus the CMI representation is the smallest faithful representation by open compact subsets.

In particular, every semilattice has a faithful representation by open compact subsets.

The theorem immediately gives a fact, reminiscent of Example 2.2 ($S = \mathrm{Fin}(X)$), which is exactly what we need:

Corollary 3.2. For a semilattice S, the following are equivalent.

(1) S can be embedded in some free semilattice;

(1') S has a faithful representation by finite subsets of some set X;

(2) the CMI representation of S is itself a representation by finite subsets.

The only implication needing proof is $(1') \Rightarrow (2)$. All we have to do for this proof is to give X of $(1')$ the discrete topology and quote $(2) \Rightarrow (1)$ of Theorem 3.1.

The theorem 3.1 gives us useful information even in the case where S is finite. For such an S, all nonempty ideals are principal and so correspond to elements. The nonempty CMI ideals correspond to the "uniquely covered" elements - elements covered by exactly one other element. For convenience, let $\mathrm{NUC}(S)$ denote the Number of Uniquely Covered elements of S. In the CMI representation, then, $\sigma_{cmi}(t)$ consists of ideals corresponding to uniquely covered elements not $\geq t$. It follows that $|\sigma_{cmi}(t)| = NUC(S) - NUC[t,1]$, where 1 is the top element of S, [t,1] denotes the closed interval $\{s \in S : t \leq s \leq 1\}$, and $|A|$ denotes the cardinality of a set A. Thus we obtain the following fact.

Corollary 3.3. Let σ faithfully represent a finite semilattice S on a set X. then for each $t \in S$, $|\sigma(t)| \geq NUC(S) - NUC[t,1]$.

Proof. Again we put the discrete topology on X and quote (2) $\Rightarrow$ (1) of Theorem 3.1. X is necessarily finite.

Here we have implicitly observed that for finite semilattices, the CMI representation really is "contained" in any faithful representation. Of course, the CMI representation for finite semilattices is really nothing more than a dualized version of the familiar expression of lattice elements as joins of join-irreducibles. A direct proof of Corollary 3.3 would not be difficult.

§4. The Characterization.

Recall that our goal has been a solution of

Problem 3. Characterize those semilattices which can be embedded in a free semilattice.

Actually, Corollary 3.2 deserves to be called an answer, in that it gives a criterion which is "intrinsic" to S (namely, that the CMI representation of S is itself a representation by finite subsets). By rephrasing this criterion, we obtain

Solution 1. A semilattice S can be embedded in a free semilattice if and only if each element of S is contained in all except finitely many completely meet-irreducible ideals of S.

In most situations, this criterion would be cumbersome. It does apply nicely, though, to our original "ladder" semilattice R of Figure 1a. There the principal ideal generated by each b_i is plainly c.m.i., and none of these ideals contains a_0. Thus, the condition of Solution 1 fails, and R is not embeddable in a free semilattice. (Actually, Solution 1 was developed first and R was invented to conform to a failure of that criterion.)

One ingredient is missing from Solution 1: The requirement that all principal ideals be finite. This property is especially useful, because Corollary 3.3 gives us potentially relevant information about faithful representations of such a finite ideal, if not the whole semilattice. The following conjecture is natural: For each element $t \in S$, look at representations of the various principal ideals containing t, regarded as semilattices in their own right. (For each principal ideal, choose the most economical faithful representation possible.) If the size of subsets representing t remains bounded as the principal ideals get larger and larger, then S should have a faithful representation which represents t by a finite set. If not, t should not be so representable.

Let us incorporate this conjecture, for all $t \in S$, into a proposed solution, using the estimate of Corollary 3.3. The principal ideal generated by an element s can be denoted by $(s]$.

<u>Solution</u> 2. A semilattice S can be embedded in a free semilattice if and only if the following two conditions are met:

(a) Every principal ideal of S is finite, and

(b) for each $t \in S$, $NUC(s] - NUC[t,s]$ is bounded as s runs through $\{s : s \geq t\}$.

This conjectured solution is true. Half of the proof, at least, is immediate: Suppose S <u>can</u> be embedded in a free semilattice. Then S has a representation σ by finite subsets of a set X. For any $t \in S$ and $s \geq t$, σ restricted to $(s]$ is an isomorphism of $(s]$ into $Pow(X)$. This restriction might not meet our technical requirements for being a representation, but by discarding some elements of X and identifying others, as discussed in Section 2, we get a genuine representation $\sigma^{(s)}$ of $(s]$ on a "smaller" set X_0. By Corollary 3.3, $NUC(s] - NUC[t,s] \leq |\sigma^{(s)}(t)|$, which is at most $|\sigma(t)|$, a bound not depending on s.

For the other half of the proof, I'd like to describe a method which is simple and pretty, if a knowledge of ultraproducts is presupposed: Suppose S satisfies (a) and (b) (and is not itself finite). For each $s \in S$, let $\sigma^{(s)}$ be a representation of the ideal $(s]$ on a set $X^{(s)}$, with $\sigma^{(s)}$ being equivalent to the CMI representation of $(s]$. S can be embedded in an ultraproduct of its principal ideals by taking a suitable ultrafilter $\mathcal{U}$ on S (one among whose members are all principal dual ideals of S [10, Corollary, p. 27]); thus $S \hookrightarrow \Pi_s (s]/\mathcal{U}$. The corresponding ultraproduct of the representation $\sigma^{(s)}$ is a faithful representation σ of $\Pi_s(s]/\mathcal{U}$ on the set $X = \Pi_s X^{(s)}/\mathcal{U}$. "Restricted" to S, σ becomes a faithful representation, with $\sigma(t)$ being essentially $\Pi_{s \geq t}\sigma^{(s)}(t)/\mathcal{U}$. Since $|\sigma^{(s)}(t)| = NUC(s] - NUC[t,s]$, which is bounded as s runs through $\{s : s \geq t\}$, the ultraproduct expression for $\sigma(t)$ yields a finite set.

(Does there exist an alternate proof which constructs the representation of S explicitly, while avoiding any form of the axiom of choice?)

§5. Applications.

Let's apply Solution # 2 in several cases.

Example 5.1. Let S be the "ladder" semilattice R of Figure 1a. For $t = a_0$, s runs through the a_n. $NUC(a_n] - NUC[a_0, a_n] = 2n - n = n$, which is unbounded. Therefore R is not embeddable, as we know.

Example 5.2. Let S be the semilattice depicted in Figure 2. S is really a modular lattice consisting of $N \times N$ ($N = \{0,1,2,\ldots\}$) with additional elements c_i adjoined.

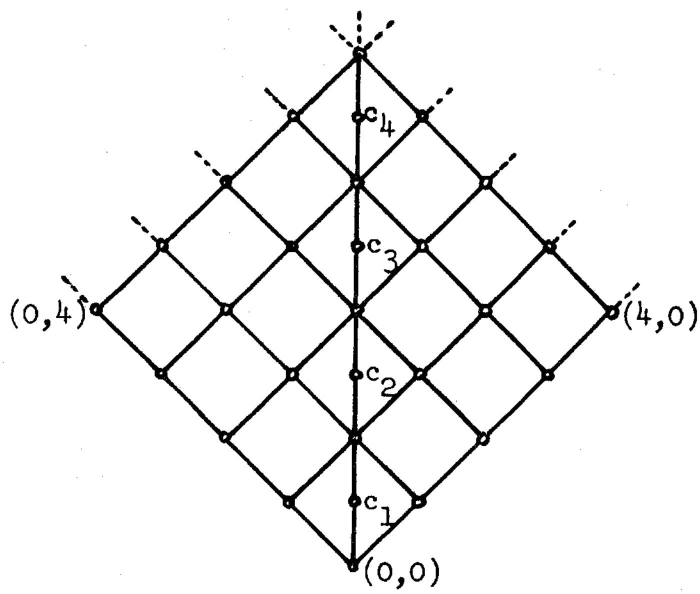

Figure 2

For $t = (m,n)$ and $s = (m',n')$, we get $NUC(s] - NUC[t,s] = (m' + n' + [\text{a certain number of } c_i]) - ([m' - m] + [n' - n] + [\text{a certain number of } c_i]) = m + n + |\{c_i : c_i \leq (m',n'), c_i \not\leq (m,n)\}| \leq m + n + |\{c_i : c_i \not\leq (m,n)\}| = m + n + \max(m,n)$. The computations where s and/or t is among the c_i differ by at most 1 from the same answer, for suitable $m = n$ or $m' = n'$. Thus $NUC(s] - NUC[t,s]$ is bounded, for each t, and the semilattice of Figure 2 is embeddable in a free semilattice.

Example 5.3. Let V be an infinite-dimensional vector space over a finite field $GF(q)$, and let S be its (semi-)lattice of finite-dimensional subspaces. Because $(s]$ (i..e, $[0,s]$) is relatively complemented, the only uniquely covered elements are its "coatoms." Since $[0,s]$ is self-dual, we can count its atoms (one-dimensional subspaces) instead; if s is a space of dimension n, this count is $(q^n-1)/(q-1)$, the number of nonzero vectors divided by the number of vectors in a one-dimensional subspace. If t is k-dimensional, $[t,s]$ is isomorphic to the subspace lattice of an $(n-k)$-dimensional vector space, so that the same kind of calculation applies. Thus $NUC(s] - NUC[t,s] = [(q^n-1)/(q-1)] - [(q^{n-k}-1)/(q-1)] = q^{n-k}(q^k-1)/(q-1)$, which is unbounded for fixed k as $n \to \infty$. Therefore S cannot be embedded in a free semilattice, even though its principal ideals are finite.

Further examples, for which the calculations are interesting but will not be carried out here, are these:

Example 5.4. Let T be an infinite set, and let S be the (semi-) lattice consisting of those partitions of T which have only finitely many nontrivial classes. In other words, S is the semilattice of compact elements of the full partition lattice of T.

Example 5.5. Again let T be an infinite set and let S be the dual of the meet-semilattice of "cocompact" partitions of T; i.e., the partitions of T into finitely many pieces.

Finally, let us consider this case:

Example 5.6. Let S be any distributive lattice in which all principal ideals are finite. In a finite distributive lattice D, the number of meet-irreducible elements equals the length $\ell(D)$; therefore $NUC(s] - NUC[t,s] = \ell([0,s]) - \ell([t,s]) = \ell([0,t])$, a fixed, hence bounded, quantity as s varies. Thus such a lattice, regarded as a semilattice, can always be embedded in a free semilattice. (Horn and Kimura [12] have shown that any distributive lattice of this type is projective as a semilattice, from which the embeddability is also immediate.)

Appendix: Proof of Theorem 3.1.

Let's follow the order $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (3)$: By assumption, σ is equivalent to the intersection of σ_{id} with some subset Y of $Id(S)$ such that $CMI(Id(S)) \subseteq Y$. Then σ^* is equivalent to "$\sigma_{id}{}^* \cap Y$." ($\sigma_{id}{}^*$ is nothing more than the regular representation of $Id(S)$.) Since each nonempty ideal I of S is an intersection of nonempty c.m.i. ideals and so is uniquely identifiable by which c.m.i. ideals do or do not contain I, $\sigma_{id}{}^* \cap Y$, and hence σ^*, is one-to-one on $Id(S) \setminus \{\emptyset\}$.

$(3) \Rightarrow (2)$: Let X be given the topology for which the sets $\sigma(s)$ themselves form a subbase for the open sets. Since σ^* is a complete join -isomorphism, taking joins in $Id(S) \setminus \{\emptyset\}$ to unions in $Pow(X) \setminus \{\emptyset\}$, the fact that the principal ideals $(s]$ are compact elements of S [10; Lemma 2, p. 21] translates into the statement that any covering of one of the chosen subbasic sets by other subbasic sets has a finite subcover. Alexander's Subbase Theorem [13, p. 139] then asserts that each subbasic set $\sigma(s)$ is compact in the generated topology adopted for X.

$(2) \Rightarrow (4)$: Without loss of generality, we may assume that S consists of open compact subsets of the topological space X; the members of S cover X. Let I be a nonempty c.m.i. ideal of S. We must find an $x \in X$ such that $I = I_x$, where $I_x = \{s : x \notin s\}$. Let I^+ be the unique smallest ideal properly containing I, and let s_0 be an element of I^+ not in I. The members of I do not cover s_0; if they did, the union of the members of some (nonempty) finite subcover would contain s_0 and would also be in I, forcing $s_0 \in I$, contrary to assumption. Let x, then, be a point of s_0 not covered by any member of I. By definition, $I_x \supseteq I$. To prove $I_x = I$, let us consider $s \notin I$ and show $s \notin I_x$, i.e., $x \in s$: The join of I and the principal ideal $(s]$, $I \vee (s]$, properly contains I, so $I^+ \subseteq I \vee (s]$. $s_0 \in I^+$ implies that $s_0 \in I \vee (s]$, in other words, that $s_0 \subseteq t \cup s$ for some $t \in S$. Since $x \in s_0$ and $x \notin t$, we must have $x \in s$, as desired.

$(4) \Rightarrow (1)$: It suffices to consider the case where $X \subseteq Id(S)$ and $\sigma = \sigma_{id} \cap X$. But in this case, for each x, the ideal I_x coincides with x itself. Thus the condition of (1), that X include all c.m.i. ideals, reduces to (4).

<u>Remark</u>. Our choice of conventions regarding $\emptyset$ as an ideal, etc., become relevant in the proof just concluded. To have σ^* be an isomorphism on all of Id(S) in (3) of Theorem 3.1, for instance, we could either (a) include $\emptyset$ as a c.m.i. ideal, or (b) exclude $\emptyset$ as an ideal. If (a), then representing sets $\sigma(s)$ cannot be allowed to be empty, or else (1) fails; furthermore, c.m.i. ideals no longer correspond only to uniquely covered elements in the case of a finite lattice, so that the "NUC" calculations must be altered. If (b), then σ_{id} no longer contains all representations, unless the representing sets $\sigma(s)$, $s \in S$, are <u>required</u> to have empty intersection - a condition with other side effects. Of course, the conventions adopted do, unhappily, give Id(S) one more element than S when S is a finite lattice.

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References:

General references for lattice theory are [2] and [11]; for universal algebra, [10]. Relevant category - theoretical ideas are to be found in [5] and [6] (implicitly), in [9], [14], [16], [18], [19], and in the Conference talks of B. Banaschewski and K. H. Hofmann. Ideas related to spaces of ideals are treated in [14], and in the Conference talks of J. Martinez, R. Mena, and H. Werner. For further references, see the bibliographies of [5], [16], and [18].

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