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ON FREE MODULAR LATTICES OVER PARTIAL LATTICES WITH FOUR GENERATORS

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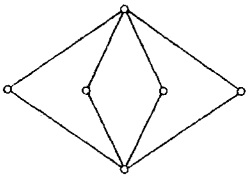
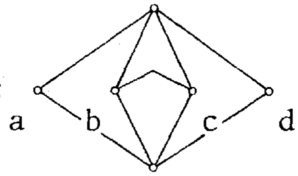
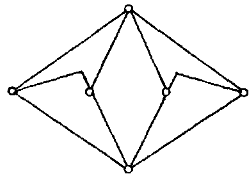
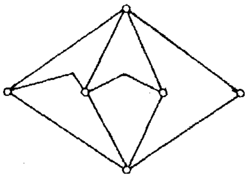
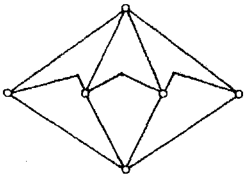
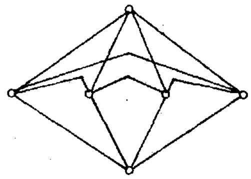
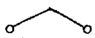
1. Main Results: This paper is a continuation of DAY, HERRMANN, WILLE [2]. It also wants to give some contribution to the word problem for the free modular lattice with four generators by examining free modular lattices over partial lattices with four generators. By a partial lattice we understand a relative sublattice of any lattice, that is a subset together with the restrictions of the operations $\wedge$ and $\vee$ to this subset (e.g. GRÄTZER [3 ; Definition 5.12]). The principal results proved in this paper are the following theorems.

Theorem 1: Let J^4 be a partial lattice $(\{0,g_1,g_2,g_3,g_4,1\};\wedge,\vee)$ with $g_i \vee 1=1$ and $g_i \wedge g_j=0$ for $i \neq j (1 \leq i,j \leq 4)$. Then every modular lattice which has J^4 as generating relative sublattice is freely generated by J^4 if and only if J^4 is described by one of the following diagrams:

M_4:  J_1^4: 

$J_{1,1}^4$: 

J_2^4:  J_3^4:  J_4^4: 

(an angle of the form  means that the join of the connected elements is deleted from the lattice M_4).

By $FM(J^4)$, we denote the free modular lattice over the partial lattice J^4, that is a certain modular lattice freely generated by J^4. In DAY, HERRMANN, WILLE [2] the free modular lattice $FM(J_1^4)$ is extensively examined; especially, it is shown that $FM(J_1^4)$ is an infinite, subdirectly irreducible lattice with one non-trivial congruence relation $\Theta(FM(J_1^4)/\Theta \tilde{=} M_4)$. For the formulation of Theorem 2 we still need some notations:

Z is the set of all integers;

N is the set of all positive integers ($N_o := N \cup \{0\}$);

G is the free abelian group with countably many generators;

$(e_i | i \in Z)$ and $(f_i | i \in N)$ are some basis of G;

S_G is the lattice of all subgroups of G.

<u>Theorem 2:</u> The free modular lattices $FM(J_1^4)$, $FM(J_{1,1}^4)$, $FM(J_2^4)$, $FM(J_3^4)$ and $FM(J_4^4)$ are isomorphic to subdirect powers of $FM(J_1^4)$ and sublattices of S_G; generators of the subdirect powers and sublattices, respectively, are described by the following list:

	n	generators in $FM(J_1^4)^n$	generators in S_G
$FM(J_1^4)$	1	a	$\langle e_{2i} \mid i \varepsilon Z \rangle$
		b	$\langle e_{2i}+e_{2i+1} \mid i \varepsilon Z \rangle$
		c	$\langle e_{2i-1}+e_{2i} \mid i \varepsilon Z \rangle$
		d	$\langle e_{2i-1} \mid i \varepsilon Z \rangle$
$FM(J_{1,1}^4)$	2	(a,c)	$\langle e_{4i}, e_{4i-1}+e_{4i+1} \mid i \varepsilon Z \rangle$
		(d,b)	$\langle e_{4i+2}, e_{4i+1}+e_{4i+3} \mid i \varepsilon Z \rangle$
		(b,d)	$\langle e_{4i+3}, e_{4i-2}+e_{4i} \mid i \varepsilon Z \rangle$
		(c,a)	$\langle e_{4i+1}, e_{4i}+e_{4i+2} \mid i \varepsilon Z \rangle$
$FM(J_2^4)$	2	(a,b)	$\langle f_{2i} \mid i \varepsilon N \rangle$
		(b,c)	$\langle f_{2i}+f_{2i+1} \mid i \varepsilon N \rangle$
		(c,d)	$\langle f_{2i-1}+f_{2i} \mid i \varepsilon N \rangle$
		(d,a)	$\langle f_{2i-1} \mid i \varepsilon N \rangle$
$FM(J_3^4)$	3	(d,a,b)	$\langle e_{2i} \mid i \varepsilon Z \rangle$
		(a,b,c)	$\langle e_{2i}+e_{2i+1}, e_{2j-1}+e_{2j} \mid i>0\geq j \rangle$
		(b,c,d)	$\langle e_{2i-1}+e_{2i}, e_{2j}+e_{2j+1} \mid i>0>j \rangle$
		(c,d,a)	$\langle e_{2i-1} \mid i \varepsilon Z \rangle$
$FM(J_4^4)$	4	(a,b,c,d)	$\langle e_{2i}, e_{2j-1} \mid i>0\geq j \rangle$
		(b,c,d,a)	$\langle e_{2i}+e_{2i+1} \mid i \varepsilon Z \rangle$
		(c,d,a,b)	$\langle e_{2i-1}+e_{2i} \mid i \varepsilon Z \rangle$
		(d,a,b,c)	$\langle e_{2i-1}, e_{2j} \mid i>0>j \rangle$

Theorem 3: The congruence lattice of $FM(J_1^4)$, $FM(J_{1,1}^4)$, $FM(J_2^4)$, $FM(J_3^4)$ and $FM(J_4^4)$, respectively, is a 2^n-element Boolean lattice with a new greatest element where n is the number of undefined joins in the generating partial lattice.

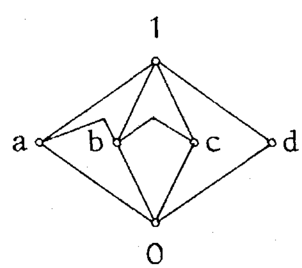
Proof of Theorem 1: Let M be a modular lattice freely generated by J^4, where J^4 as relative sublattice of M is described by one of the listed diagramms. There is a homomorphism ψ from $FM(J^4)$ onto M whose restriction to J^4 is the identity. By Theorem 3, ψ has to be injective (otherwise, there are more joins in ψJ^4 than in J^4). Thus, M is freely generated by J^4. For the converse, we recall that every projective plane Π_p over a prime field is generated by four points a_1, a_2, a_3, a_4 no three on a line. Obviously, the elements 0, a_1, a_2, a_3, a_4 and 1 form a relative sublattice J_6^4 of Π_p in which $a_i \vee a_j$ is not defined for $i \neq j (1 \leq i,j \leq 4)$. Since projective planes over different prime fields are not isomorphic, no such plane is freely generated by J_6^4. This argument can simiarly be applied to the remaining partial lattices $J_3^{3,1}$ (3 undefined joins), $J_{3,1}^{3,1}$ (4 undefined joins) and J_5^4 (5 undefined joins). First, we take a line g in Π_p which does not contain a_1, a_2 and a_3. Then, the relative sublattice $J_3^{3,1}$ consisting of the elements 0, a_1, a_2, a_3, g and 1 generates Π_p, because a_1, a_2, $(a_1 \vee a_3) \wedge g$ and $(a_2 \vee a_3) \wedge g$ are four points no three on a line. $J^{3,1}_{3,1}$ is represented in $\Pi_p \times FM(J^4_1)$ by $\{(0,0),(a_1,d),(a_2,a),(a_3,b),(g,c),(1,1)\}$, and J^4_5 is represented in $\Pi_p \times \Pi_p$ by $\{(0,0),(a_1,g),(a_2,a_3),(a_3,a_2),(g,a_1),(1,1)\}$. By the above argument, no of the described subsets freely generates its generated sublattice. Thus, there are no more partial lattices J^4 with the desired properties.

Proof of Theorem 2: It can be easily seen that the generators together with the smallest and the greatest element of $FM(J^4_1)^n$ and S_G, resp., form a relative sublattice isomorphic to the corresponding partial lattice J^4. Thus, by Theorem 1, the sublattice generated by the described elements is isomorphic to $FM(J^4)$.

Proof of Theorem 3: This proof will cover the rest of the paper. In section 2 the assertion is proved for $FM(J^4_2)$ by solving the word problem for $FM(J^4_2)$. Using these results, the congruence lattice of $FM(J^4_4)$ is determined in section 3. This result immediately gives us the congruence lattices of the remaining lattices.

2. $FM(J_2^4)$: The goal of this section is to solve the word problem for $FM(J_2^4)$ in a similar manner as the word problem is solved for $FM(J_1^4)$ in DAY, HERRMANN, WILLE [2]. The elements of $FM(J_2^4)$ will be represented by quadruples of natural numbers and ∞. By Proposition 19, meets and joins are described in terms of these quadruples. As consequence of Proposition 19 and [2 ; Theorem 4 and Theorem 5], we get Theorem 3 for J_2^4. It should be mentioned that the lattice $FM(J_2^4)$ appears first in BIRKHOFF [1 ;p.70] where the generators in S_G are described as in Theorem 2.

As in DAY, HERRMANN, WILLE [2], the method which makes computations practicable is to introduce suitable endomorphisms of $FM(J_2^4)$. An detailed study of these endomorphisms by several lemmata prepares the proof of Proposition 19. Since it does not make any confusion, we choose the same notation for the elements of J_2^4 as for the elements of J_1^4 :



<u>Lemma 4:</u> There are endomorphisms μ and ν of $FM(J_2^4)$ such that

(1) $\mu 0 = 0$ (2) $\nu 0 = 0$

$\mu a = b$ $\nu a = d \wedge (a \vee b)$

$\mu b = a \wedge (b \vee c)$ $\nu b = c \wedge (a \vee b)$

$\mu c = d \wedge (b \vee c)$ $\nu c = b$

$\mu d = c$ $\nu d = a$

$\mu 1 = b \vee c$ $\nu 1 = a \vee b$

Proof: By modularity, it can be easily seen that (1) and (2) define homomorphisms from J_2^4 into $FM(J_2^4)$. Since $FM(J_2^4)$ is freely generated by J_2^4, those homomorphisms can be (uniquely) extended to endomorphisms μ and ν of $FM(J_2^4)$.

<u>Lemma 5:</u> $\mu\nu = \nu\mu$

Proof: $\mu\nu a = \mu(d \wedge (a \vee b)) = c \wedge (b \vee (a \wedge (b \vee c))) = c \wedge (a \vee b) = \nu b = \nu\mu a$,

$\mu\nu b = \mu(c \wedge (a \vee b)) = (d \wedge (b \vee c)) \wedge (b \vee (a \wedge (b \vee c))) = d \wedge (b \vee c) \wedge (a \vee b)$

$= d \wedge (a \vee b) \wedge ((c \wedge (a \vee b)) \vee b) = \nu(a \wedge (b \vee c)) = \nu\mu b$,

$\mu\nu c = \nu\mu c$ (analogous to $\mu\nu a = \nu\mu a$),

$\mu\nu d = \mu a = b = \nu c = \nu\mu d$.

<u>Lemma 6:</u> $\mu^n x \leq \mu^m x$ and $\nu^n x \leq \nu^m x$ for $x \in J_2^4$ if $n \equiv m \pmod 2$ and $n \geq m$.

Proof: The assertion is an immediate consequence of $\mu^2 x \leq x$ and $\nu^2 x \leq x$.

Lemma 7: Let $n \varepsilon N_o$.

(1) $a \vee \mu^{2n} d = a \vee \mu^{2n+1} d$ (2) $c \vee \nu^{2n} d = c \vee \nu^{2n+1} d$

(3) $d \vee \mu^{2n} a = d \vee \mu^{2n+1} a$ (4) $d \vee \nu^{2n} c = d \vee \nu^{2n+1} c$

Proof: (1): $a \vee \mu^{2n} d = a \vee \mu^{2n} a \vee \mu^{2n} d = a \vee \mu^{2n}(a \vee d) = a \vee \mu^{2n}(a \vee c) = a \vee \mu^{2n} a \vee \mu^{2n} c = a \vee \mu^{2n} c = a \vee \mu^{2n+1} d$; the proofs of (2), (3) and (4) are analogous.

Lemma 8: Let $x \varepsilon J_2^4$, and let $n \varepsilon N_o$.

(1) $x \wedge \mu^{2n} 1 = \mu^{2n} x$ (2) $x \wedge \nu^{2n} 1 = \nu^{2n} x$

(3) $a \wedge \mu^{2n+1} 1 = \mu^{2n+2} a$ (4) $c \wedge \nu^{2n+1} 1 = \nu^{2n+2} c$

(5) $d \wedge \mu^{2n+1} 1 = \mu^{2n+2} d$ (6) $d \wedge \nu^{2n+1} 1 = \nu^{2n+2} d$

(7) $b \wedge \mu^{2n+1} 1 = \mu^{2n} b$ (8) $b \wedge \nu^{2n+1} 1 = \nu^{2n} b$

(9) $c \wedge \mu^{2n+1} 1 = \mu^{2n} c$ (10) $a \wedge \nu^{2n+1} 1 = \nu^{2n} a$

Proof: (1): Let $x' \varepsilon J_2^4$ with $x \vee x' = 1$. Then $x \wedge \mu^{2n} 1 = x \wedge (\mu^{2n} x \vee \mu^{2n} x') = \mu^{2n} x \vee (x \wedge \mu^{2n} x') = \mu^{2n} x \vee 0 = \mu^{2n} x$. (2): analogous to (1).
(3): $a \wedge \mu^{2n+1} 1 = a \wedge \mu^{2n+1}(b \vee d) = a \wedge (\mu^{2n+2} a \vee \mu^{2n+1} d) = \mu^{2n+2} a \vee (a \wedge \mu^{2n+1} d) = \mu^{2n+2} a$; (4), (5), (6): analogous to (3).
(7): $b \wedge \mu^{2n+1} 1 = b \wedge \mu^{2n+1}(a \vee d) = b \wedge (\mu^{2n} b \vee \mu^{2n+1} d) = \mu^{2n} b \vee (b \wedge \mu^{2n+1} d) = \mu^{2n} b$; (8), (9), (10): analogous to (7).

Lemma 9: Let $n \varepsilon N_o$.

(1) $\nu \mu^{n} a = \mu^{n} d \wedge \nu 1$ (2) $\mu \nu^{n} a = \nu^{n} b \wedge \mu 1$

(3) $\nu \mu^{n} b = \mu^{n} c \wedge \nu 1$ (4) $\mu \nu^{n} b = \nu^{n} a \wedge \mu 1$

(5) $\nu \mu^{n} c = \mu^{n} b \wedge \nu 1$ (6) $\mu \nu^{n} c = \nu^{n} d \wedge \mu 1$

(7) $\nu\mu^n d = \mu^n a\wedge\nu 1$ (8) $\mu\nu^n d = \nu^n c\wedge\mu 1$

(9) $\nu\mu^n 1 = \mu^n 1\wedge\nu 1$ (10) $\mu\nu^n 1 = \nu^n 1\wedge\mu 1$

Proof: (1): The case n=0 is proved by $\nu a=d\wedge(a\vee b)=d\wedge\nu 1$. By induction hypothesis, we get for $n>0$: $\nu\mu^n a=\mu\nu\mu^{n-1}a=\mu(\mu^{n-1}d\wedge\nu 1)=\mu^n d\wedge\mu\nu 1=\mu^n d\wedge(a\vee b)\wedge(b\vee c)=\mu^n d\wedge\nu 1\wedge\mu 1=\mu^n d\wedge\nu 1$. The other assertions analogously follow.

Lemma 10: Let $n,i,j\in N_o$.

(1) $\nu^{2n}(\mu^i a\vee\mu^j d)=\nu^{2n}1\wedge(\mu^i a\vee\mu^j d)$ (2) $\mu^{2n}(\nu^i c\vee\nu^j d)=\mu^{2n}1\wedge(\nu^i c\vee\nu^j d)$

(3) $\nu^{2n+1}(\mu^i a\vee\mu^j d)=\nu^{2n+1}1\wedge(\mu^i d\vee\mu^j a)$

(4) $\mu^{2n+1}(\nu^i c\vee\nu^j d)=\mu^{2n+1}1\wedge(\nu^i d\vee\nu^j c)$

Proof: (1): The case n=0 is trivial. The case n=1 is proved by $\nu^2(\mu^i a\vee\mu^j d)=\nu(\nu\mu^i a\vee\nu\mu^j d)=\nu((\mu^i d\wedge\nu 1)\vee(\mu^j a\wedge\nu 1))=\nu((\mu^i d\wedge\nu 1)\vee\mu^j a)=\nu(\nu 1\wedge(\mu^i d\vee\mu^j a))=\nu^2 1\wedge\nu 1\wedge(\mu^i a\vee\mu^j d)=\nu^2 1\wedge(\mu^i a\vee\mu^j d)$. By induction hypothesis, we get for $n>1$: $\nu^{2n}(\mu^i a\vee\mu^j d)=\nu^{2n-2}(\nu^2 1\wedge(\mu^i a\vee\mu^j d))=\nu^{2n}1\wedge\nu^{2n-2}1\wedge(\mu^i a\vee\mu^j d)=\nu^{2n}1\wedge(\mu^i a\vee\mu^j d)$. The other assertions similarly follow.

Lemma 11: $\mu^{2n}x\vee\nu^{2m}x = x$ for $x\in J_2^4$ and $n,m\in N_o$.

Proof: The cases n=0 or m=0 are immediate consequences of Lemma 6. The case n=1, m=1 can be easily checked by Lemma 4. By induction hypothesis, we get for $n+m>2$ (w.l.o.g. $m>1$):

$\mu^{2n}x\vee\nu^{2m}x=\mu^{2n}x\vee\nu^{2m}x\vee\mu^{2n}\nu^{2m-2}x=\mu^{2n}x\vee\nu^{2m-2}(\nu^{2}x\vee\mu^{2n}x)=$
$\mu^{2n}x\vee\nu^{2m-2}x=x.$

Lemma 12: Let $n,m\varepsilon N_o$ with $m\leq n$.
(1) $\mu^{2m}a\vee\mu^{2n+1}a=\nu 1\wedge(\mu^{2m}a\vee\mu^{2n}d)$ (2) $\nu^{2m}c\vee\nu^{2n+1}c=\mu 1\wedge(\nu^{2m}c\vee\nu^{2n}d)$

Proof: (1): The case n=m=0 is proved by $a\vee\mu a=a\vee b=\nu 1=\nu 1\wedge(a\vee d)$.
By induction hypothesis, we get for $n>m=0$: $a\vee\mu^{2n+1}a=$
$a\vee\mu^{2}a\vee\mu^{2n+1}a=a\vee\mu^{2}(a\vee\mu^{2n-1}a)=a\vee\mu^{2}(\nu 1\wedge(a\vee\mu^{2n-2}d))=$
$a\vee(\mu^{2}\nu 1\wedge(\mu^{2}a\vee\mu^{2n}d))=a\vee(\mu^{2}1\wedge\nu 1\wedge(\mu^{2}a\vee\mu^{2n}d))=\nu 1\wedge(a\vee\mu^{2}a\vee\mu^{2n}d)=$
$\nu 1\wedge(a\vee\mu^{2n}d)$. Furthermore it follows for $n\geq m>0$: $\mu^{2m}a\vee\mu^{2n+1}a=$
$\mu^{2m}(a\vee\mu^{2n+1-2m}a)=\mu^{2m}(\nu 1\wedge(a\vee\mu^{2n-2m}d))=\mu^{2m}\nu 1\wedge(\mu^{2m}a\vee\mu^{2n}d)=$
$\mu^{2m}1\wedge\nu 1\wedge(\mu^{2m}a\vee\mu^{2n}d)=\nu 1\wedge(\mu^{2m}a\vee\mu^{2n}d)$.

Lemma 13: Let $n\varepsilon N_o$.
(1) $d\vee\mu^{2n+1}d = d\vee\mu^{2n}a$ (2) $d\vee\nu^{2n+1}d = d\vee\nu^{2n}c$

Proof: (1): The case n=0 is proved by $d\vee\mu d=d\vee c=d\vee a$.
By induction hypothesis we get for $n>0$: $d\vee\mu^{2n+1}d=$
$d\vee\mu^{2}d\vee\mu^{2n+1}d=d\vee\mu^{2}(d\vee\mu^{2n-1}d)=d\vee\mu^{2}(d\vee\mu^{2n-2}a)=d\vee\mu^{2}d\vee\mu^{2n}a=$
$d\vee\mu^{2n}a$. (2): analogous to (1).

Lemma 14: Let $n\varepsilon N_o$.
(1) $\mu^{n}a \leq \nu 1$ (2) $\nu^{n}c \leq \mu 1$
(3) $\mu^{n}b \leq \nu 1$ (4) $\nu^{n}b \leq \mu 1$

Proof: (1): $a \geq \mu^{2n}a$ and $b \geq \mu^{2n+1}a$ implies the assertion. (2), (3), (4): analogous to (1).

In the following we write $(i,j)\sim(k,l)$ for $0 \leq i,j,k,l \leq \infty$, if $m \equiv n \pmod 2$ for all $m,n \in \{i,j,k,l\} \setminus \{\infty\}$; furthermore, let $\mu^{\infty}x = 0 = \nu^{\infty}x$ for all $x \in FM(J_2^4)$.

<u>Lemma 15:</u> Let $0 \leq i,j,k,l \leq \infty$, and let $(i,j)\sim(\infty,\infty)$ and $(k,l)\sim(\infty,\infty)$.

(1) $(\mu^i a \vee \mu^j d) \wedge (\mu^k a \vee \mu^l d) =$

$\mu^{\max\{i,k\}} a \vee \mu^{\max\{j,l\}} d$ if $(i,j)\sim(k,l)$,

$\mu^{\max\{i+1,j+1,k\}} a \vee \mu^{\max\{i+1,j+1,l\}} d$

if $(i,j)\not\sim(k,l)$, $\max\{i,j\} \leq \max\{k,l\}$,

$\mu^{\max\{i,k+1,l+1\}} a \vee \mu^{\max\{j,k+1,l+1\}} d$

if $(i,j)\not\sim(k,l)$, $\max\{i,j\} > \max\{k,l\}$;

(2) $(\nu^i c \vee \nu^j d) \wedge (\nu^k c \vee \nu^l d) =$

$\nu^{\max\{i,k\}} c \vee \nu^{\max\{j,l\}} d$ if $(i,j)\sim(k,l)$,

$\nu^{\max\{i+1,j+1,k\}} c \vee \nu^{\max\{i+1,j+1,l\}} d$

if $(i,j)\not\sim(k,l)$, $\max\{i,j\} \leq \max\{k,l\}$,

$\nu^{\max\{i,k+1,l+1\}} c \vee \nu^{\max\{j,k+1,l+1\}} d$

if $(i,j)\not\sim(k,l)$, $\max\{i,j\} > \max\{k,l\}$.

Proof: (1): The proof is divided into the following cases:

1. $(i,j)\sim(k,l)$

  1.1. $i\le k, j\le l$

  1.2. $i\le k, j\ge l$

  1.3. $i\ge k, j\le l$

  1.4. $i\ge k, j\ge l$

2. $(i,j)\nsim(k,l), \max\{i,j\}\le\max\{k,l\}$

  2.1. $i\le j\le k\le l$

  2.2. $i\le k\le j\le l$

  2.3. $k\le i\le j\le l$

  2.4. $i\le j\le l\le k$

  2.5. $i\le l\le j\le k$

  2.6. $l\le i\le j\le k$

  2.7. $j\le i\le k\le l$

  2.8. $j\le k\le i\le l$

  2.9. $k\le j\le i\le l$

  2.10. $j\le i\le l\le k$

  2.11. $j\le l\le i\le k$

  2.12. $l\le j\le i\le k$

By symmetry, the case $(i,j)\nsim(k,l), \max\{i,j\}>\max\{k,l\}$ is analogous to 2..

<u>1.1.:</u> $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=\mu^k a\vee\mu^l d=\mu^{\max\{i,k\}}a\vee\mu^{\max\{j,l\}}d$ (Lemma 6);

<u>1.2.:</u> $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=(\mu^i a\wedge\mu^l d)\vee\mu^k a\vee\mu^j d=0\vee\mu^k a\vee\mu^j d=\mu^{\max\{i,k\}}a\vee\mu^{\max\{j,l\}}d$ (Lemma 6);

<u>1.3.:</u> analogous to 1.2.;

<u>1.4.:</u> analogous to 1.1..

<u>2.1.:</u> $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=(\mu^i a\vee\mu^{j+1}d)\wedge(\mu^k a\vee\mu^l d)=$
$((\mu^i a\vee\mu^{j+1}d)\wedge\mu^k a)\vee\mu^l d=((\mu^i a\vee\mu^j d)\wedge\nu 1\wedge\mu^k a)\vee\mu^l d=$
$((\mu^i a\vee\mu^{j+1}a)\wedge\mu^k a)\vee\mu^l d=\mu^k a\vee\mu^l d$ (Lemma 7,6,14,12);

<u>2.2.:</u> $(\mu^i a\vee\mu^j d)\wedge(\mu^k a\vee\mu^l d)=((\mu^i a\vee\mu^{j+1}a)\wedge\mu^k a)\vee\mu^l d=$
$(\mu^i a\wedge\mu^k a)\vee\mu^{j+1}a\vee\mu^l d=0\vee\mu^{j+1}a\vee\mu^l d=\mu^{j+1}a\vee\mu^l d$
(2.1., Lemma 6);

<u>2.3.:</u> analogous to 2.2.;

2.4.: analogous to 2.1.;

2.5.: $(\mu^i a \vee \mu^j d) \wedge (\mu^k a \vee \mu^l d) = (\mu^i a \vee \mu^j d) \wedge (\mu^{k+1} a \vee \mu^l d) =$
$((\mu^i a \vee \mu^j d) \wedge \mu^l d) \vee \mu^{k+1} a = ((\mu^i a \vee \mu^{j+1} d) \wedge \mu^l d) \vee \mu^{k+1} a =$
$(\mu^i a \wedge \mu^l d) \vee \mu^{j+1} d \vee \mu^{k+1} a = 0 \vee \mu^{k+1} a \vee \mu^{j+1} d = \mu^k a \vee \mu^{j+1} d$
(Lemma 7,6);

2.6.: analogous to 2.5;

2.7.: $(\mu^i a \vee \mu^j d) \wedge (\mu^k a \vee \mu^l d) = (\mu^{i+1} a \vee \mu^j d) \wedge (\mu^k a \vee \mu^{l+1} d) =$
$\mu^k a \vee \mu^{l+1} d = \mu^k a \vee \mu^l d$ (Lemma 7,6);

2.8.: $(\mu^i a \vee \mu^j d) \wedge (\mu^k a \vee \mu^l d) = (\mu^{i+1} a \vee \mu^j d) \wedge (\mu^k a \vee \mu^{l+1} d) =$
$(\mu^k a \wedge \mu^j d) \vee \mu^{i+1} a \vee \mu^{l+1} d = 0 \vee \mu^{i+1} a \vee \mu^{l+1} d = \mu^{i+1} a \vee \mu^l d$
(Lemma 7,6);

2.9.: analogous to 2.8.;

2.10.: $(\mu^i a \vee \mu^j d) \wedge (\mu^k a \vee \mu^l d) = (\mu^{i+1} a \vee \mu^j d) \wedge (\mu^k a \vee \mu^l d) =$
$((\mu^{i+1} a \vee \mu^j d) \wedge \mu^l d) \vee \mu^k a = ((\mu^i a \vee \mu^j d) \wedge \mu^l d) \vee \mu^k a =$
$((\mu^{i+1} d \vee \mu^j d) \wedge \mu^l d) \vee \mu^k a = \mu^k a \vee \mu^l d$ (Lemma 7,6,13);

2.11.: $(\mu^i a \vee \mu^j d) \wedge (\mu^k a \vee \mu^l d) = ((\mu^{i+1} d \vee \mu^j d) \wedge \mu^l d) \vee \mu^k a =$
$(\mu^j d \wedge \mu^l d) \vee \mu^{i+1} d \vee \mu^k a = 0 \vee \mu^{i+1} d \vee \mu^k a = \mu^k a \vee \mu^{i+1} d$ (2.10.,
Lemma 6);

2.12.: analogous to 2.11..

The proof of (2) analogously goes as the proof of (1).

Lemma 16: Let $0 \le i,j,k,l \le \infty$, and let $(i,j)\sim(\infty,\infty),(k,l)\sim(\infty,\infty)$ and $\{j,l\} \neq \{\infty\}$.

(1) $\mu^{i}a \vee \mu^{j}d \vee \mu^{k}a \vee \mu^{l}d =$

$\mu^{\min\{i,k\}}a \vee \mu^{\min\{j,l\}}d$ if $(i,j)\sim(k,l)$,

$\mu^{\min\{i-1,j-1,k\}}a \vee \mu^{\min\{i-1,j-1,l\}}d$

if $(i,j)\nsim(k,l), \min\{i,j\} \ge \min\{k,l\}$,

$\mu^{\min\{i,k-1,l-1\}}a \vee \mu^{\min\{j,k-1,l-1\}}d$

if $(i,j)\nsim(k,l), \min\{i,j\} < \min\{k,l\}$;

(2) $\nu^{i}c \vee \nu^{j}d \vee \nu^{k}c \vee \nu^{l}d =$

$\nu^{\min\{i,k\}}c \vee \nu^{\min\{j,l\}}d$ if $(i,j)\sim(k,l)$,

$\nu^{\min\{i-1,j-1,k\}}c \vee \nu^{\min\{i-1,j-1,l\}}d$

if $(i,j)\nsim(k,l), \min\{i,j\} \ge \min\{k,l\}$,

$\nu^{\min\{i,k-1,l-1\}}c \vee \nu^{\min\{j,k-1,l-1\}}d$

if $(i,j)\nsim(k,l), \min\{i,j\} < \min\{k,l\}$.

Proof: (1): The proof is divided into the following cases:

1. $(i,j)\sim(k,l)$

2. $(i,j)\nsim(k,l), \min\{i,j\} \ge \min\{k,l\}$

2.1. $k \le l \le i \le j$

2.2. $k \le i \le l \le j$ $(l \neq \infty)$

2.3. $k \le i \le j \le l$ $(j \neq \infty)$

2.4. $l \le k \le i \le j$

2.5. $l \le i \le k \le j$

2.6. $l \le i \le j \le k$

2.7. $k \le l \le j \le i$

2.8. $k \le j \le l \le i$

2.9. $k \le j \le i \le l$

2.10. $l \le k \le j \le i$

2.11. $l \le j \le k \le i$

2.12. $l \le j \le i \le k$

By symmetry, the case $(i,j)\nsim(k,l), \min\{i,j\} < \min\{k,l\}$ is analogous to 2..

1. is an immediate consequence of Lemma 6.

2.1.: $\mu^{i}a\vee\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{i-1}a\vee\mu^{j-1}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{k}a\vee\mu^{l}d$ (Lemma 7,6);

2.2.: $\mu^{i}a\vee\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{i}a\vee\mu^{j-1}a\vee\mu^{k}a\vee\mu^{l}d=\mu^{i}a\vee\mu^{k}a\vee\mu^{l}d=$
$\mu^{i}a\vee\mu^{k}a\vee\mu^{l-1}d=\ldots=\mu^{i}a\vee\mu^{k}a\vee\mu^{i}d=\mu^{i}a\vee\mu^{k}a\vee\mu^{i-1}d=$
$\mu^{i-1}a\vee\mu^{k}a\vee\mu^{i-1}d=\mu^{k}a\vee\mu^{i-1}d$ (Lemma 13,6,7);

2.3.: analogous to 2.2.;

2.4.: analogous to 2.1.;

2.5.: $\mu^{i}a\vee\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{i-1}a\vee\mu^{j-1}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{i-1}a\vee\mu^{l}d$
(Lemma 7,6);

2.6.: $\mu^{i}a\vee\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{i}a\vee\mu^{j}d\vee\mu^{k-1}a\vee\mu^{l}d=\mu^{i}a\vee\mu^{j}d\vee\mu^{l}d=$
$\mu^{i}a\vee\mu^{j+1}d\vee\mu^{l}d=\mu^{i}a\vee\mu^{l}d=\mu^{i-1}a\vee\mu^{l}d$ (Lemma 7,6);

2.7.: analogous to 2.1.;

2.8.: $\mu^{i}a\vee\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{i+1}a\vee\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=$
$\mu^{j-1}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{k}a\vee\mu^{j-1}d$ (Lemma 7,6);

2.9.: analogous to 2.8.;

2.10.: analogous to 2.1.;

2.11.: $\mu^{i}a\vee\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{i+1}a\vee\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=\mu^{j}d\vee\mu^{k}a\vee\mu^{l}d=$
$\mu^{j-1}a\vee\mu^{k}a\vee\mu^{l}d=\mu^{j-1}a\vee\mu^{l}d$ (Lemma 7,6,13);

2.12.: analogous to 2.11..

The proof of (2) analogously goes as the proof of (1).

Lemma 17: Let $0 \leq i,j,k,l < \infty$, and let $N,M \in 2N_o$ with $N \geq \max\{i,j\}, M \geq \max\{k,l\}$.

$$(\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d) = \nu^M(\mu^i a \vee \mu^j d) \vee \mu^N(\nu^k c \vee \nu^l d)$$

Proof: $(\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d) = (\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d) \wedge (\mu^N 1 \vee \nu^M 1) = ((\mu^i a \vee \mu^j d) \wedge \nu^M 1) \vee ((\nu^k c \vee \nu^l d) \wedge \mu^N 1) = \nu^M(\mu^i a \vee \mu^j d) \vee \mu^N(\nu^k c \vee \nu^l d)$ (Lemma 11,7,6,10).

Lemma 18: $\mu^{2n} x \wedge \nu^{2m} x = \mu^{2n} \nu^{2m} x$ for $x \in J_2^4$ and $n,m \in N_o$.

Proof: The cases n=0 or m=0 are immediate consequences of Lemma 6. The case n=1, m=1 can be easily checked by Lemma 4 and Lemma 9. By induction hypothesis, we get for n+m>2 (w.l.o.g. m>1): $\mu^{2n} x \wedge \nu^{2m} x = \mu^{2n} x \wedge \nu^2 x \wedge \nu^{2m} x = \mu^{2n} \nu^2 x \wedge \nu^{2m} x = \nu^2(\mu^{2n} x \wedge \nu^{2m-2} x) = \nu^2 \mu^{2n} \mu^{2m-2} x = \mu^{2n} \nu^{2m} x$.

We define that a quadrupel (i,j,k,l) satisfies $(*)$ if one of the following conditions hold for (i,j,k,l):

(1) $i,j,k,l \in N_o, (i,j) \sim (\infty,\infty), (k,l) \sim (\infty,\infty)$;

(2) $i=\infty, j \in 2N_o, k=\infty, l \in 2N_o$;

(3) $i=\infty, j \in 2N_o+1, k \in 2N_o, l=\infty$;

(4) $i \in 2N_o, j=\infty, k=\infty, l \in 2N_o+1$;

(5) $i \in 2N_o+1, j=\infty, k \in 2N_o+1, l=\infty$;

(6) $i=j=k=l=\infty$.

In $FM(J_2^4)$ we define $f(i,j,k,l) := (\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d)$ for $0 \leq i,j,k,l \leq \infty$.

<u>Proposition 19:</u> $FM(J_2^4)=\{f(i,j,k,l)\mid(i,j,k,l) \text{ satisfies } (*)\}$

and

(1) $f(i,j,k,l)\wedge f(i',j',k',l') =$

$f(\max\{i,i'\},\max\{j,j'\},\max\{k,k'\},\max\{l,l'\})$ if $(i,j)\sim(i',j')$, $(k,l)\sim(k',l')$,

$f(\max\{i,i'\},\max\{j,j'\},\max\{k+1,l+1,k'\},\max\{k+1,l+1,l'\})$ if $(i,j)\sim(i',j')$, $(k,l)\nsim(k',l')$, $\max\{k,l\}\le\max\{k',l'\}$,

$f(\max\{i,i'\},\max\{j,j'\},\max\{k,k'+1,l'+1\},\max\{l,k'+1,l'+1\})$ if $(i,j)\sim(i',j')$, $(k,l)\nsim(k',l')$, $\max\{k,l\}>\max\{k',l'\}$,

$f(\max\{i+1,j+1,i'\},\max\{i+1,j+1,j'\},\max\{k,k'\},\max\{l,l'\})$ if $(i,j)\nsim(i',j')$, $(k,l)\sim(k',l')$, $\max\{i,j\}\le\max\{i',j'\}$,

$f(\max\{i,i'+1,j'+1\},\max\{j,i'+1,j'+1\},\max\{k,k'\},\max\{l,l'\})$ if $(i,j)\nsim(i',j')$, $(k,l)\sim(k',l')$, $\max\{i,j\}>\max\{i',j'\}$,

$f(\max\{i+1,j+1,i'\},\max\{i+1,j+1,j'\},\max\{k+1,l+1,k'\},\max\{k+1,l+1,l'\})$ if $(i,j)\nsim(i',j')$, $(k,l)\nsim(k',l')$, $\max\{i,j\}\le\max\{i',j'\}$, $\max\{k,l\}\le\max\{k',l'\}$,

$f(\max\{i+1,j+1,i'\},\max\{i+1,j+1,j'\},\max\{k,k'+1,l'+1\},\max\{l,k'+1,l'+1\})$ if $(i,j)\nsim(i',j')$ $(k,l)\nsim(k',l')$, $\max\{i,j\}\le\max\{i',j'\}$, $\max\{k,l\}>\max\{k',l'\}$,

$f(\max\{i,i'+1,j'+1\},\max\{j,i'+1,j'+1\},\max\{k+1,l+1,k'\},\max\{k+1,l+1,l'\})$ if $(i,j)\nsim(i',j')$ $(k,l)\nsim(k',l')$, $\max\{i,j\}>\max\{i',j'\}$, $\max\{k,l\}\le\max\{k',l'\}$,

$f(\max\{i,i'+1,j'+1\},\max\{j,i'+1,j'+1\},\max\{k,k'+1,l'+1\},\max\{l,k'+1,l'+1\})$ if $(i,j)\nsim(i',j')$ $(k,l)\nsim(k',l')$, $\max\{i,j\}>\max\{i',j'\}$, $\max\{k,l\}>\max\{k',l'\}$,

and

(2) $f(i,j,k,l) \vee f(i',j',k',l') =$

$f(\min\{i,i'\},\min\{j,j'\},\min\{k,k'\},\min\{l,l'\})$

if $(i,j)\sim(i',j')$, $(k,l)\sim(k',l')$

$f(\min\{i,i'\},\min\{j,j'\},\min\{k-1,l-1,k'\},\min\{k-1,l-1,l'\})$

if $(i,j)\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\min\{k,l\}\geq\min\{k',l'\}$,

$f(\min\{i,i'\},\min\{j,j'\},\min\{k,k'-1,l'-1\},\min\{l,k'-1,l'-1\})$

if $(i,j)\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\min\{k,l\}<\min\{k',l'\}$

$f(\min\{i-1,j-1,i'\},\min\{i-1,j-1,j'\},\min\{k,k'\},\min\{l,l'\})$

if $(i,j)\not\sim(i',j')$, $(k,l)\sim(k',l')$, $\min\{i,j\}\geq\min\{i',j'\}$,

$f(\min\{i,i'-1,j'-1\},\min\{j,i'-1,j'-1\},\min\{k,k'\},\min\{l,l'\}$

if $(i,j)\not\sim(i',j')$, $(k,l)\sim(k',l')$, $\min\{i,j\}<\min\{i',j'\}$,

$f(\min\{i-1,j-1,i'\},\min\{i-1,j-1,j'\},\min\{k-1,l-1,k'\},\min\{k-1,l-1,l'\})$

if $(i,j)\not\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\min\{i,j\}\geq\min\{i',j'\}$, $\min\{k,l\}\geq\min\{k',l'\}$,

$f(\min\{i-1,j-1,i'\},\min\{i-1,j-1,j'\},\min\{k,k'-1,l'-1\},\min\{l,k'-1,l'-1\})$

if $(i,j)\not\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\min\{i,j\}\geq\min\{i',j'\}$, $\min\{k,l\}<\min\{k',l'\}$,

$f(\min\{i,i'-1,j'-1\},\min\{j,i'-1,j'-1\},\min\{k,k'-1,l'-1\},\min\{l,k'-1,l'-1\})$

if $(i,j)\not\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\min\{i,j\}<\min\{i',j'\}$, $\min\{k,l\}<\min\{k',l'\}$.

$f(\min\{i,i'-1,j'-1\},\min\{j,i'-1,j'-1\},\min\{k-1,l-1,k'\},\min\{k-1,l-1,l'\})$

if $(i,j)\not\sim(i',j')$, $(k,l)\not\sim(k',l')$, $\min\{i,j\}<\min\{i',j'\}$, $\min\{k,l\}\geq\min\{k',l'\}$

Proof: (1) and (2) imply that $\{f(i,j,k,l)\,|\,(i,j,k,l)$ satisfies $(*)\}$ is a sublattice of $FM(J_2^4)$ and, because of $a=f(0,\infty,\infty,1)$, $b=f(1,\infty,1,\infty)$, $c=f(\infty,1,0,\infty)$ and $d=f(\infty,0,\infty,0)$, that it is equal to $FM(J_2^4)$. Thus, what actually has to be proved is (1) and (2).

(1): Because of $((\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d))\wedge((\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d)=$ $((\mu^i a\vee\mu^j d)\wedge(\mu^i a\vee\mu^j d))\wedge((\nu^k c\vee\nu^l d)\wedge(\nu^k c\vee\nu^l d))$, the assertion is an immediate consequence of Lemma 15.

(2): The proof is divided into cases s.t. $(1\leq s,t\leq 6)$, where the case s.t. means that (i,j,k,l) satisfies condition (s) in (*) and (i',j',k',l') satisfies condition (t) in (*). By commutativity of $\vee$, it is sufficient to handle the cases s.t. with $s\leq t$. Furthermore, because of $f(\infty,\infty,\infty,\infty)=0$, all the cases s.6 are trivial.

<u>1.1.:</u> Let $M\in 2N_o$ with $M\geq\max\{i,j,k,l,i',j',k',l'\}$.
$((\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d))\vee((\mu^{i'} a\vee\mu^{j'} d)\wedge(\nu^{k'} c\vee\nu^{l'} d)=$
$\nu^M(\mu^i a\vee\mu^j d)\vee\mu^M(\nu^k c\vee\nu^l d)\vee\nu^M(\mu^{i'} a\vee\mu^{j'} d)\vee\mu^M(\nu^{k'} c\vee\nu^{l'} d)=$
$\nu^M(\mu^i a\vee\mu^j d\vee\mu^{i'} a\vee\mu^{j'} d)\vee\mu^M(\nu^k c\vee\nu^l d\vee\nu^{k'} c\vee\nu^{l'} d)=$
$\nu^M(\mu^{\min\{i,i'\}}a\vee\mu^{\min\{j,j'\}}d)\vee\mu^M(\nu^{\min\{k,k'\}}c\vee\nu^{\min\{l,l'\}}d)=$
$(\mu^{\min\{i,i'\}}a\vee\mu^{\min\{j,j'\}}d)\wedge(\nu^{\min\{k,k'\}}c\vee\nu^{\min\{l,l'\}}d)$
(Lemma 17,16).

<u>1.2.:</u> Let $M\in 2N_o$ with $M\geq\max\{i,j,k,l,j',l'\}$. Because of $j',l'\in 2N_o$, we have $d=\mu^{M-j'}d\vee\nu^{M-l'}d$ by Lemma 11. Together with Lemma 17 and 18 it follows
$((\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d))\vee(\mu^{j'} d\wedge\nu^{l'} d)=$
$\nu^M(\mu^i a\vee\mu^j d)\vee\mu^M(\nu^k c\vee\nu^l d)\vee\mu^{j'}\nu^{l'} d=$
$\nu^M(\mu^i a\vee\mu^j d)\vee\mu^M(\nu^k c\vee\nu^l d)\vee\mu^M\nu^{l'} d\vee\nu^M\mu^{j'} d=$
$\nu^M(\mu^i a\vee\mu^j d\vee\mu^{j'} d)\vee\mu^M(\nu^k c\vee\nu^l d\vee\nu^{l'} d)$. From this we get the

assertion by Lemma 16 and 17.

1.3.: By Lemma 18, $j'\varepsilon 2N_o+1$ and $k'\varepsilon 2N_o$ imply $\mu^{j'}d\wedge\nu^{k'}c=\mu^{j'-1}c\wedge\nu^{k'}c=\mu^{j'-1}\nu^{k'}c$. Then, the proof is analogous to 1.2..

1.4.: analogous to 1.3..

1.5.: By Lemma 18, $i'\varepsilon 2N_o+1$ and $k'\varepsilon 2N_o+1$ imply $\mu^{i'}a\wedge\nu^{k'}c=\mu^{i'-1}b\wedge\nu^{k'-1}b=\mu^{i'-1}\nu^{k'-1}b$. Then, the proof is analogous to 1.2..

2.2.: If $j\leq j'$ and $l>l'$, we get the assertion from $(\mu^{j}d\wedge\nu^{l}d)\vee(\mu^{j'}d\wedge\nu^{l'}d)=\mu^{j}\nu^{l}d\vee\mu^{j'}\nu^{l'}d=\mu^{j}\nu^{l'}(\mu^{j'-j}d\vee\nu^{l-l'}d)=\mu^{j}\nu^{l'}d=\mu^{j}d\wedge\nu^{l'}d=(\mu^{j}d\vee\mu^{j'}d)\wedge(\nu^{l}d\vee\nu^{l'}d)$ by Lemma 16. The other cases are analogous.

2.3.: Let $M\varepsilon 2N_o$ with $M\geq\max\{j,l,j',k'\}$. Using Lemma 6, 11,18,13 and 17, we get $(\mu^{j}d\wedge\nu^{l}d)\vee(\mu^{j'}d\wedge\nu^{k'}c)=$

$(d\wedge\mu^{j}d\wedge\nu^{l}d)\vee(c\wedge\mu^{j'-1}c\wedge\nu^{k'}c)=$

$((\mu^{M}d\vee\nu^{M}d)\wedge\mu^{j}d\wedge\nu^{l}d)\vee((\mu^{M}c\vee\nu^{M}c)\wedge\mu^{j'-1}c\wedge\nu^{k'}c)=$

$((\mu^{M}d\wedge\nu^{l}d)\vee(\mu^{j}d\wedge\nu^{M}d))\vee((\mu^{M}c\wedge\nu^{k'}c)\vee(\mu^{j'-1}c\wedge\nu^{M}c))=$

$\mu^{M}\nu^{l}d\vee\mu^{j}\nu^{M}d\vee\mu^{M}\nu^{k'}c\vee\mu^{j'-1}\nu^{M}c=$

$\nu^{M}(\mu^{j}d\vee\mu^{j'}d)\vee\mu^{M}(\nu^{k'}c\vee\nu^{l}d)=(\mu^{j}d\vee\mu^{j'}d)\wedge(\nu^{k'}c\vee\nu^{l}d)$.

2.4., 2.5., 3.4.: similar to 2.3..

3.3.: Because of $(\mu^{j}d\wedge\nu^{k}c)\vee(\mu^{j'}d\wedge\nu^{k'}c)=(\mu^{j-1}c\wedge\nu^{k}c)\vee(\mu^{j'-1}c\wedge\nu^{k'}c)$, the proof is analogous to 2.2..

4.4., 5.5.: similar to 3.3..

3.5.: Let $M\in 2N_0$ with $M\geq\max\{j,k,i',k'\}$. Using Lemma 6,11, 18,12,10 and 17, we get in the case $k<k'$ that

$(\mu^j d\wedge\nu^k c)\vee(\mu^{i'} a\wedge\nu^{k'} c)=(c\wedge\mu^{j-1}c\wedge\nu^k c)\vee(b\wedge\mu^{i'-1}b\wedge\nu^{k'-1}b)=$
$((\mu^M c\vee\nu^M c)\wedge\mu^{j-1}c\wedge\nu^k c)\vee((\mu^M b\vee\nu^M b)\wedge\mu^{i'-1}b\wedge\nu^{k'-1}b)=$
$((\mu^M c\wedge\nu^k c)\vee(\mu^{j-1}c\wedge\nu^M c))\vee((\mu^M b\wedge\nu^{k'-1}b)\vee(\mu^{i'-1}b\wedge\nu^M b))=$
$\mu^M\nu^k c\vee\mu^{j-1}\nu^M c\vee\mu^M\nu^{k'-1}b\vee\mu^{i'-1}\nu^M b=\nu^M(\mu^j d\vee\mu^{i'} a)\vee\mu^M(\nu^k c\vee\nu^{k'} c)=$
$\nu^M(\mu^j d\vee\mu^{i'} a)\vee\mu^M(\mu 1\wedge(\nu^k c\vee\nu^{k'-1}d))=$
$\mu(\nu^M(\mu^{j-1}d\vee\mu^{i'-1}a)\vee\mu^M(\nu^k d\vee\nu^{k'-1}c))=$
$\mu((\mu^{j-1}d\vee\mu^{i'-1}a)\wedge(\nu^k d\vee\nu^{k'-1}c))=(\mu^j d\vee\mu^{i'} a)\wedge\mu 1\wedge(\nu^k c\vee\nu^{k'-1}d)=$
$(\mu^{i'} a\vee\mu^j d)\wedge(\nu^k c\vee\nu^{k'-1}d)$. The case $k'<k$ analogously follows.

4.5.: analogous to 3.5..

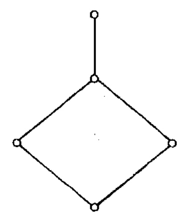
Proposition 20: An isomorphism from $FM(J_2^4)$ onto a subdirect power of $FM(J_1^4)$ is given by $f(i,j,k,l)\longmapsto(e(i,j),e(k,l))$ (the elements $e(i,j)$ of $FM(J_1^4)$ are defined in DAY, HERRMANN, WILLE [2]).

Proof: The assertion is a straightforward consequence of Proposition 19 and of Theorem 4 and 5 in [2].

Proposition 21: Let (i,j,k,l) and (i',j',k',l') satisfy $(*)$. Then $f(i,j,k,l)=f(i',j',k',l')$ if and only if $(i,j,k,l)=(i',j',k',l')$.

Proof: The assertion immediately follows from Proposition 20 and Theorem 5 in [2].

<u>Proposition 22:</u> The congruence lattice of $FM(J_2^4)$ is described by the following diagram :



Proof: By Proposition 20, the intersection of the congruence relations $\Theta(a \vee b, 1)$ and $\Theta(b \vee c, 1)$ is the identity; furthermore, by the Homomorphism Theorem and Corollary 8 in [2], there is only one non-trivial congruence relation greater than $\Theta(a \vee b, 1)$ and $\Theta(b \vee c, 1)$, respectively, namely $\Theta(a \vee b, b \vee c)$. Therefore, the distributivity of the congruence lattice gives us the assertion.

3. $FM(J_4^4)$: In this section we show the existence of epimorphisms α and β from $FM(J_4^4)$ onto $FM(J_2^4)$ (Lemma 23), which separate $FM(J_4^4)$; that is, $\ker \alpha \cap \ker \beta = \omega := \{(X,X) \mid X \in FM(J_4^4)\}$ (Proposition 49). To establish the proof, we define monomorphisms $\underline{\alpha}$ and $\underline{\beta}$ from $FM(J_2^4)$ into $FM(J_4^4)$ (Lemma 26) and meet-morphisms $\bar{\alpha}$ and $\bar{\beta}$ from $FM(J_2^4)$ into $FM(J_4^4)$ (Lemma 43), such that $\underline{\alpha}x \leq \bar{\alpha}x$ and $\underline{\beta}x \leq \bar{\beta}x$ for all $x \in FM(J_2^4)$. We prove that the intervals $[\underline{\alpha}x, \bar{\alpha}x]$ and $[\underline{\beta}x, \bar{\beta}x]$ are the congruence classes of $\ker \alpha$ and $\ker \beta$, respectively (Lemma 48). Thereby, we get $FM(J_4^4)$ as a subdirect power of $FM(J_2^4)$. Both Proposition 22 and Proposition 49 give us the proof of Theorem 3. Furthermore, the word problem can be solved for $FM(J_4^4)$.

For the preparation of the main-assertions, several lemmata have to be proved and are listed in the beginning of part 3. For simplification in the following we choose lower case letters for the generators of J_2^4 and upper case letters for the generators of J_4^4.

Lemma 23: There are epimorphisms α and β of $FM(J_4^4)$ onto $FM(J_2^4)$ such that

(1)	$\alpha 0 = 0$	(2)	$\beta 0 = 0$
	$\alpha A = c$		$\beta A = a$
	$\alpha B = d$		$\beta B = b$

$\alpha C = a$ $\qquad$ $\beta C = c$

$\alpha D = b$ $\qquad$ $\beta D = d$

$\alpha 1 = 1$ $\qquad$ $\beta 1 = 1$

Proof: Since $FM(J_4^4)$ and $FM(J_2^4)$ are lattices freely generated by J_4^4 and J_2^4, respectively, the homomorphisms from J_4^4 into J_2^4 can be extended to epimorphisms from $FM(J_4^4)$ onto $FM(J_2^4)$.

Lemma 24: There are endomorphisms ϕ and ψ of $FM(J_4^4)$ such that

(1)		(2)	
	$\phi 0 = 0$		$\psi 0 = 0$
	$\phi A = D\wedge(A\vee B)$		$\psi A = B\wedge(A\vee D)$
	$\phi B = C\wedge(A\vee B)$		$\psi B = A\wedge(B\vee C)$
	$\phi C = B\wedge(C\vee D)$		$\psi C = D\wedge(B\vee C)$
	$\phi D = A\wedge(C\vee D)$		$\psi D = C\wedge(A\vee D)$
	$\phi 1 = (A\vee B)\wedge(C\vee D)$		$\psi 1 = (A\vee D)\wedge(B\vee C)$

Proof: It can be easily seen by modularity that (1) and (2) define homomorphisms from J_4^4 into $FM(J_4^4)$. Thus, the freeness of $FM(J_4^4)$ gives as the assertion.

Lemma 25: Let μ and ν the endomorphisms of $FM(J_2^4)$ defined in Lemma 4.

(1) $\phi\psi = \psi\phi$ $\quad$ (2) $\alpha\psi = \mu\alpha$ $\quad$ (3) $\beta\phi = \nu\beta$

(4) $\alpha\phi = \nu\alpha$ $\quad$ (5) $\beta\psi = \mu\beta$

Proof: (1): $\phi\psi A=\phi(B\wedge(A\vee D))=(C\wedge(B\vee D))\wedge((D\wedge(A\vee B))\vee(A\wedge(C\vee D)))$

$=C\wedge(A\vee B)\wedge(A\vee(D\wedge(A\vee B)))\wedge(C\vee D)=C\wedge(A\vee B)\wedge(A\vee D)\wedge(A\vee B)\wedge(C\vee D)$

$=C\wedge(A\vee B)\wedge(A\vee D)\wedge(B\vee C)=(C\wedge(A\vee D))\wedge(A\vee(B\wedge(A\vee D)))\wedge(B\vee C)$

$=(C\wedge(A\vee D))\wedge((A\wedge(B\vee C))\vee(B\wedge(A\vee D)))=\psi(D\wedge(B\vee A))=\psi\phi A$;

$\phi\psi B=\psi\phi B$, $\phi\psi C=\psi\phi C$, $\phi\psi D=\psi\phi D$ (analogous to $\phi\psi A=\psi\phi A$).

(2) $\alpha\psi A=\alpha(B\wedge(A\vee D))=d\wedge(c\vee b)=\mu c=\mu\alpha A$,

$\alpha\psi B=\alpha(A\wedge(B\vee C))=c\wedge(d\vee a)=c=\mu d=\mu\alpha B$,

$\alpha\psi C=\alpha(D\wedge(B\vee C))=b\wedge(d\vee a)=b=\mu a=\mu\alpha C$,

$\alpha\psi D=\alpha(C\wedge(A\vee D))=a\wedge(c\vee b)=\mu b=\mu\alpha D$.

The other assertions analogously follow.

By Lemma 25, the following diagram commutes:

$$\begin{array}{ccc} FM(J_4^4) & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & FM(J_2^4) \\ \phi\downarrow\ \downarrow\psi & & \mu\downarrow\ \downarrow\nu \\ FM(J_4^4) & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & FM(J_2^4) \end{array}$$

<u>Lemma 26:</u> There are monomorphisms $\underline{\alpha}$ and $\underline{\beta}$ from $FM(J_2^4)$ into $FM(J_4^4)$ such that

(1)	$\underline{\alpha}0 = 0$	(2)	$\underline{\beta}0 = 0$
	$\underline{\alpha}a = C\wedge(A\vee B)$		$\underline{\beta}a = A\wedge(C\vee D)$
	$\underline{\alpha}b = D\wedge(A\vee B)\wedge(B\vee C)$		$\underline{\beta}b = B\wedge(A\vee D)\wedge(C\vee D)$
	$\underline{\alpha}c = A\wedge(B\vee C)$		$\underline{\beta}c = C\wedge(A\vee D)$
	$\underline{\alpha}d = B$		$\underline{\beta}d = D$
	$\underline{\alpha}1 = (A\vee B)\wedge(B\vee C)$		$\underline{\beta}1 = (A\vee D)\wedge(C\vee D)$

Proof: (1) Since $FM(J_2^4)$ is freely generated by J_2^4, it is enough to proof the following statements:

$$\underline{\alpha}a \vee \underline{\alpha}c = (C\wedge(A\vee B))\vee(A\wedge(B\vee C))=(C\vee(A\wedge(B\vee C)))\wedge(A\vee B)$$
$$=(C\vee A)\wedge(B\vee C)\wedge(A\vee B)=(B\vee C)\wedge(A\vee B)=\underline{\alpha}1$$

$$\underline{\alpha}a \vee \underline{\alpha}d = (C\wedge(A\vee B))\vee B=(B\vee C)\wedge(A\vee B)=\underline{\alpha}1$$

$$\underline{\alpha}b \vee \underline{\alpha}d = (D\wedge(A\vee B)\wedge(B\vee C))\vee B=(B\vee D)\wedge(A\vee B)\wedge(B\vee C)=(A\vee B)\wedge(B\vee C)=\underline{\alpha}1$$

$$\underline{\alpha}c \vee \underline{\alpha}d = (A\wedge(B\vee C))\vee B=(A\vee B)\wedge(B\vee C)=\underline{\alpha}1$$

$$\underline{\alpha}a \wedge \underline{\alpha}b = \underline{\alpha}a\wedge\underline{\alpha}c=\underline{\alpha}a\wedge\underline{\alpha}d=\underline{\alpha}b\wedge\underline{\alpha}c=\underline{\alpha}b\wedge\underline{\alpha}d=\underline{\alpha}c\wedge\underline{\alpha}d=0=\underline{\alpha}0.$$

(2): analogous to (1).

Lemma 27:

(1) $\alpha\underline{\alpha}=\beta\underline{\beta}=id_{FM(J_2^4)}$ (2) $\alpha\underline{\beta}=\beta\underline{\alpha}=\mu\nu$

(3) $\underline{\alpha}\mu=\psi\underline{\alpha}$ (4) $\underline{\alpha}\nu=\phi\underline{\alpha}$

(5) $\underline{\beta}\mu=\psi\underline{\beta}$ (6) $\underline{\beta}\nu=\phi\underline{\beta}$

Proof: (1): $\alpha\underline{\alpha}a=\alpha(C\wedge(A\vee B))=a\wedge(c\vee d)=a=\beta(A\wedge(C\vee D))=\beta\underline{\beta}a$,

$\alpha\underline{\alpha}b=\alpha(D\wedge(A\vee B)\wedge(B\vee C))=b\wedge(c\vee d)\wedge(a\vee d)=b=$

$\beta=(b\wedge(c\vee d)\wedge(a\vee d))=\beta\underline{\beta}b$,

$\alpha\underline{\alpha}c=\alpha(A\wedge(B\vee C))=c\wedge(a\vee d)=c=\beta(C\wedge(A\vee D))=\beta\underline{\beta}c$,

$\alpha\underline{\alpha}d=\alpha B=d=\beta D=\beta\underline{\beta}d$.

(2): $\alpha\underline{\beta}a=\alpha(A\wedge(C\vee D))=c\wedge(a\vee b)=\nu b=\nu\mu a=\mu\nu a=c\wedge(a\vee b)$

$=\beta(C\wedge(A\vee B))=\beta\underline{\alpha}a$,

$\alpha\underline{\beta}b=\alpha(B\wedge(A\vee D)\wedge(C\vee D))=d\wedge(c\vee b)\wedge(a\vee b)=\mu(c\wedge(a\vee b))$

$=\mu\nu b=\beta(D\wedge(A\vee B)\wedge(B\vee C)=\beta\underline{\alpha}b$,

$\alpha\underline{\beta}c=\alpha(C\wedge(A\vee D))=a\wedge(c\vee b)=\mu b=\mu\nu c=\beta(A\wedge(C\vee B))=\beta\underline{\alpha}c$,

$\alpha\underline{\beta}d=\alpha D=b=\mu a=\mu\nu d=\beta B=\beta\underline{\alpha}d$.

(3) $\underline{\alpha}\mu a=\underline{\alpha}b=D\wedge(A\vee B)\wedge(B\vee C)=D\wedge(B\vee C)\wedge(A\vee B)\wedge(A\vee D)\wedge(B\vee C)$

$=D\wedge(B\vee C)\wedge(A\vee(B\wedge(A\vee D)))\wedge(B\vee C)=$

$=D\wedge(B\vee C)\wedge((B\wedge(A\vee D))\vee(A\wedge(B\vee C)))=\psi(C\wedge(A\vee B))=\psi\underline{\alpha}a$,

$\underline{\alpha}\mu b=\underline{\alpha}(a\wedge(b\vee c))=C\wedge(A\vee B)\wedge((D\wedge(A\vee B)\wedge(B\vee C))\vee(A\wedge(B\vee C)))$

$=C\wedge(A\vee B)\wedge((A\wedge(B\vee C))\vee(D\wedge(B\vee C)))\wedge(A\vee B)=$

$=C\wedge(A\vee B)\wedge(D\vee(A\vee(B\vee C)))\wedge(A\vee D)$

$=C\wedge(A\vee D)\wedge(A\vee B)\wedge(B\vee C)\wedge(A\vee D)\wedge((D\vee(A\wedge(B\vee C)))\wedge(B\vee C))$

$=\psi(D\wedge(A\vee B)\wedge(B\vee C))=\psi\underline{\alpha}b$,

$\underline{\alpha}\mu c=\psi\underline{\alpha}c$ (analogous to $\underline{\alpha}\mu a=\psi\underline{\alpha}a$),

$\underline{\alpha}\mu d=\underline{\alpha}c=A\wedge(B\vee C)=\psi B=\psi\underline{\alpha}d$.

(4), (5), (6): analogous to (3).

<u>Lemma 28:</u> (1): $\phi^m X\leq\phi^n X$ and $\psi^m X\leq\psi^n X$ for $X\varepsilon J_4^4$ if $m\equiv n(\text{mod }2)$ and $m\geq n$.

(2): $\phi^m 1\leq\phi^n 1$ and $\psi^m 1\leq\psi^n 1$ if $m\geq n$.

Proof: The assertions are consequences of $\phi^2 X\leq X$, $\psi^2 X\leq X$ and $\phi 1\leq 1$, $\psi 1\leq 1$, respectively.

<u>Lemma 29:</u> (1) $\phi^m X\wedge\phi^n Y=0$ for $X,Y\varepsilon J_4^4\setminus\{1\}$ if (i): $X\neq Y$ and $m\equiv n$ (mod 2) or (ii): $X\varepsilon\{0,A,D\}$ and $Y\varepsilon\{0,B,C\}$ or (iii): $X=Y$ and $m\not\equiv n(\text{mod }2)$.

(2) $\psi^m X \wedge \psi^n Y = 0$ for $X,Y \in J_4^4 \setminus \{1\}$ if (i): $X \neq Y$ and $m \equiv n \pmod 2$ or (ii): $X \in \{0,A,B\}$ and $Y \in \{0,C,D\}$ or (iii): $X=Y$ and $m \not\equiv n \pmod 2$.

Proof: $X \wedge Y = 0$ for $X,Y \in J_4^4 \setminus \{1\}$ and $X \neq Y$ implies the assertions.

Lemma 30: Let $n \in N$.

(1) $\phi^{2n}A = A \wedge (C \vee \phi^{2n-1}A) = A \wedge (D \vee \phi^{2n-1}B)$

(2) $\phi^{2n-1}A = D \wedge (A \vee \phi^{2n-2}B) = D \wedge (B \vee \phi^{2n-2}A)$

(3) $\phi^{2n}B = B \wedge (C \vee \phi^{2n-1}A) = B \wedge (D \vee \phi^{2n-1}B)$

(4) $\phi^{2n-1}B = C \wedge (A \vee \phi^{2n-2}B) = C \wedge (B \vee \phi^{2n-2}A)$

(5) $\phi^{2n}C = C \wedge (A \vee \phi^{2n-1}C) = C \wedge (B \vee \phi^{2n-1}D)$

(6) $\phi^{2n-1}C = B \wedge (C \vee \phi^{2n-2}D) = B \wedge (D \vee \phi^{2n-2}C)$

(7) $\phi^{2n}D = D \wedge (A \vee \phi^{2n-1}C) = D \wedge (B \vee \phi^{2n-1}D)$

(8) $\phi^{2n-1}D = A \wedge (C \vee \phi^{2n-2}D) = A \wedge (D \vee \phi^{2n-2}C)$

(9) $\psi^{2n}A = A \wedge (C \vee \psi^{2n-1}A) = A \wedge (B \vee \psi^{2n-1}D)$

(10) $\psi^{2n-1}A = B \wedge (A \vee \psi^{2n-2}D) = B \wedge (D \vee \psi^{2n-2}A)$

(11) $\psi^{2n}B = B \wedge (A \vee \psi^{2n-1}C) = B \wedge (D \vee \psi^{2n-1}B)$

(12) $\psi^{2n-1}B = A \wedge (B \vee \psi^{2n-2}C) = A \wedge (C \vee \psi^{2n-2}B)$

(13) $\psi^{2n}C = C \wedge (A \vee \psi^{2n-1}C) = C \wedge (D \vee \psi^{2n-1}B)$

(14) $\psi^{2n-1}C = D \wedge (B \vee \psi^{2n-2}C) = D \wedge (C \vee \psi^{2n-2}B)$

(15) $\psi^{2n}D = D \wedge (C \vee \psi^{2n-1}A) = D \wedge (B \vee \psi^{2n-1}D)$

(16) $\psi^{2n-1}D = C \wedge (A \vee \psi^{2n-2}D) = C \wedge (D \vee \psi^{2n-2}A)$.

(1): The case $n=1$ is proved by

$\phi^2 A=\phi(D\wedge(A\vee B))=A\wedge(C\vee D)\wedge((D\wedge(A\vee B))\vee(C\wedge(A\vee B)))$

$=A\wedge(C\vee D)\wedge(C\vee(D\wedge(A\vee B)))\wedge(A\vee B)=A\wedge(C\vee(D\wedge(A\vee B)))=A\wedge(C\vee\phi A)$ and

$A\wedge(C\vee D)\wedge((D\wedge(A\vee B))\vee(C\wedge(A\vee B)))=A\wedge(C\vee D)\wedge(D\vee(C\wedge(A\vee B)))\wedge(A\vee B)$

$=A\wedge(D\vee(C\wedge(A\vee B)))=A\wedge(D\vee\phi B)$. By induction hypothesis, we get

for $n>1$: $\phi^{2n}A=\phi^2\phi^{2n-2}A=\phi^2(A\wedge(C\vee\phi^{2n-3}A))=$

$\phi(D\wedge(A\vee B)\wedge((B\wedge(C\vee D))\vee\phi^{2n-2}A)=\phi(D\wedge((B\wedge(C\vee D))\vee(A\wedge(C\vee\phi^{2n-3}A)))$

$=\phi(D\wedge(B\vee(A\wedge(C\vee\phi^{2n-3}A)))\wedge(C\vee D))=\phi(D\wedge(B\vee(A\wedge(C\vee\phi^{2n-3}A))))$

$=\phi(D\wedge(B\vee\phi^{2n-2}A))=A\wedge(C\vee D)\wedge((C\wedge(A\vee B))\vee(D\wedge(B\vee\phi^{2n-2}A)))$

$=A\wedge(C\vee D)\wedge(C\vee(D\wedge(B\vee\phi^{2n-2}A)))\wedge(A\vee B)=A\wedge(C\vee(D\wedge(B\vee\phi^{2n-2}A)))$

$=A\wedge(C\vee\phi(A\wedge(C\vee\phi^{2n-3}A)))=A\wedge(C\vee\phi^{2n-1}A)$ and furthermore

$\phi^{2n}A=\phi^2(A\wedge(D\vee\phi^{2n-3}B))=\phi(D\wedge(A\vee B)\wedge((A\wedge(C\vee D))\vee\phi^{2n-2}B))$

$=\phi(D\wedge((A\wedge(C\vee D))\vee(B\wedge(D\vee\phi^{2n-3}B))))=\phi(D\wedge(A\vee(B\wedge(D\vee\phi^{2n-3}B)))\wedge(C\vee D))$

$=\phi(D\wedge(A\vee\phi^{2n-2}B))=A\wedge(C\vee D)\wedge((D\wedge(A\vee B))\vee\phi^{2n-1}B)$

$=A\wedge((D\wedge(A\vee B))\vee(\phi^{2n-2}C\wedge\phi^{2n-2}(A\vee B)))$

$=A\wedge(D\vee(\phi^{2n-2}C\wedge\phi^{2n-2}(A\vee B))\wedge(A\vee B)=A\wedge(D\vee\phi^{2n-2}(C\wedge(A\vee B)))$

$=A\wedge(D\vee\phi^{2n-1}B)$ (Lemma 28).

All other cases of Lemma 30 can be proved in a similar way.

Lemma 31: Let $m,n\in N_o$ and $m\le n$.

(1) $\phi^{2m}A\vee\phi^{2n}C=\phi^{2m}A\vee\phi^{2n-1}C$ (2) $\phi^{2m}B\vee\phi^{2n}D=\phi^{2m}B\vee\phi^{2n-1}D$

(3) $\phi^{2m}C\vee\phi^{2n}A=\phi^{2m}C\vee\phi^{2n-1}A$ (4) $\phi^{2m}D\vee\phi^{2n}B=\phi^{2m}D\vee\phi^{2n-1}B$

(5) $\psi^{2m}A\vee\psi^{2n}C=\psi^{2m}A\vee\psi^{2n-1}C$ (6) $\psi^{2m}B\vee\psi^{2n}D=\psi^{2m}B\vee\psi^{2n-1}D$

(7) $\psi^{2m}C\vee\psi^{2n}A=\psi^{2m}C\vee\psi^{2n-1}A$ (8) $\psi^{2m}D\vee\psi^{2n}B=\psi^{2m}D\vee\psi^{2n-1}B$.

Proof: (1): $\phi^{2m}A\vee\phi^{2n}C=\phi^{2m}(A\vee\phi^{2(n-m)}C)=\phi^{2m}(A\vee(C\wedge(A\vee\phi^{2(n-m)-1}C)))$
$=\phi^{2m}((A\vee C)\wedge(A\vee\phi^{2(n-m)-1}C)=\phi^{2m}A\vee\phi^{2n-1}C$ (Lemma 30).

(2), (3), ..., (8): analogous to (1).

<u>Lemma 32:</u> Let $m,n\varepsilon N_o$ and $m\leq n$.

(1) $\phi^{2m}A\vee\phi^{2n+1}B=\phi^{2m}A\vee\phi^{2n}B$ (2) $\phi^{2m}B\vee\phi^{2n+1}A=\phi^{2m}B\vee\phi^{2n}A$

(3) $\phi^{2m}C\vee\phi^{2n+1}D=\phi^{2m}C\vee\phi^{2n}D$ (4) $\phi^{2m}D\vee\phi^{2n+1}C=\phi^{2m}D\vee\phi^{2n}C$

(5) $\psi^{2m}A\vee\psi^{2n+1}D=\psi^{2m}A\vee\psi^{2n}D$ (6) $\psi^{2m}D\vee\psi^{2n+1}A=\psi^{2m}D\vee\psi^{2n}A$

(7) $\psi^{2m}B\vee\psi^{2n+1}C=\psi^{2m}B\vee\psi^{2n}C$ (8) $\psi^{2m}C\vee\psi^{2n+1}B=\psi^{2m}C\vee\psi^{2n}B$

Proof: (1): $\phi^{2m}A\vee\phi^{2n+1}B=\phi^{2m}(A\vee\phi^{2(n-m)+1}B)=\phi^{2m}(A\vee(C\wedge(A\vee\phi^{2(n-m)}B)))$
$=\phi^{2m}((A\vee C)\wedge(A\vee\phi^{2(n-m)}B))=\phi^{2m}A\vee\phi^{2n}B$. (Lemma 30).

(2), (3), ..., (8): analogous to (1).

<u>Lemma 33:</u> Let $m,n\varepsilon N_o$.

(1) $\phi^{2m+1}A\vee\psi^{2n+1}C=D$ (2) $\phi^{2m+1}B\vee\psi^{2n+1}D=C$

(3) $\phi^{2m+1}C\vee\psi^{2n+1}A=B$ (4) $\phi^{2m+1}D\vee\psi^{2n+1}B=A$

Proof: The case m=n=0 can be easily checked by Lemma 24.
By induction hypothesis, we get for m>n=0: $\phi^{2m+1}A\vee\psi C$
$=\phi^{2m+1}A\vee\psi C\vee\psi\phi^{2}C=\phi^{2}(\phi^{2m-1}A\vee\psi C)\vee\psi C=\phi^{2}D\vee\psi C$
$=(D\wedge(B\vee(A\wedge(C\vee D))))\vee(D\wedge(B\vee C))=((D\wedge(B\vee C))\vee B\vee(A\wedge(C\vee D)))\wedge D$
$=(B\vee C\vee(A\wedge(C\vee D)))\wedge D=(B\vee C\vee D)\wedge D=D$ (Lemma 24). Now let n>0.
We get: $\phi^{2m+1}A\vee\phi^{2n+1}C=\phi^{2m+1}A\vee\phi^{2m+1}\psi^{2}A\vee\psi^{2n+1}C$
$=\phi^{2m+1}A\vee\psi^{2}(\phi^{2m+1}A\vee\psi^{2n-1}C)=\phi^{2m+1}A\vee\psi^{2}D=\phi^{2m+1}A\vee\phi^{2m+1}\psi B\vee\psi^{2}D$

$=\phi^{2m+1}A\vee\psi(\phi^{2m+1}B\vee\psi D)=\phi^{2m+1}A\vee\psi C=D$ (Lemma 28);

(2), (3), (4): analogous to (1).

<u>Lemma 34:</u> Let $m,n\varepsilon N_o$.

(1) $\phi^{2m}A\vee\psi^{2n+1}B=A$ (2) $\phi^{2m}B\vee\psi^{2n+1}A=B$

(3) $\phi^{2m}C\vee\psi^{2n+1}D=C$ (4) $\phi^{2m}D\vee\psi^{2n+1}C=D$

(5) $\phi^{2m+1}A\vee\psi^{2n}D=D$ (6) $\phi^{2m+1}B\vee\psi^{2n}C=C$

(7) $\phi^{2m+1}C\vee\psi^{2n}B=B$ (8) $\phi^{2m+1}D\vee\psi^{2n}A=A$

Proof: (1): Case $m=0$ is a consequence of Lemma 28. For $m>0$ we get by Lemma 33:
$A=\phi^{2m-1}D\vee\psi^{2n+1}B\geq\phi^{2m}A\vee\psi^{2n+1}B\geq\phi^{2m+1}D\vee\psi^{2n+1}B=A$. The other assertions similarly follow.

<u>Lemma 35:</u> Let $m,n\varepsilon N_o$.

(1) $\phi^{2m}A\vee\psi^{2n}A=A$ (2) $\phi^{2m}B\vee\psi^{2n}B=B$

(3) $\phi^{2m}C\vee\psi^{2n}C=C$ (4) $\phi^{2m}D\vee\psi^{2n}D=D$

(5) $\phi^{m}1\vee\phi^{n}1=1$

Proof: (1): For $m=0$ or $n=0$ the assertion follows by Lemma 28. If $m,n>0$ we get
$A=\phi^{2m-1}D\vee\psi^{2n-1}B\geq\phi^{2m}A\vee\psi^{2n}A\geq\phi^{2m+1}D\vee\psi^{2n+1}B=A$ by Lemma 33.
(2), (3), (4): analogous to (1). (5) we get by $1=A\vee C=B\vee D$ using Lemma 34 and (1), ..., (4).

In the following let $\phi^{\infty}X=0=\psi^{\infty}X$ for all $X\varepsilon J_4^4$.

Lemma 36: Let $n\varepsilon N_o\cup\{\infty\}$.

(1) $\phi\psi^{n}A=\phi 1\wedge\psi^{n}D=(A\vee B)\wedge\psi^{n}D$ (2) $\phi\psi^{n}B=\phi 1\wedge\psi^{n}C=(A\vee B)\wedge\psi^{n}C$

(3) $\phi\psi^{n}C=\phi 1\wedge\psi^{n}B=(C\vee D)\wedge\psi^{n}B$ (4) $\phi\psi^{n}D=\phi 1\wedge\psi^{n}A=(C\vee D)\wedge\psi^{n}A$

(5) $\psi\phi^{n}A=\psi 1\wedge\phi^{n}B=(A\vee D)\wedge\phi^{n}B$ (6) $\psi\phi^{n}B=\psi 1\wedge\phi^{n}A=(B\vee C)\wedge\phi^{n}A$

(7) $\psi\phi^{n}C=\psi 1\wedge\phi^{n}D=(B\vee C)\wedge\phi^{n}D$ (8) $\psi\phi^{n}D=\psi 1\wedge\phi^{n}C=(A\vee D)\wedge\phi^{n}C$

Proof: (1): The case $n=0$ is trivial. For $n>0$ we get by induction hypothesis and Lemma 25: $\phi\psi^{n}A=\psi((A\vee B)\wedge\psi^{n-1}D)$ $=((B\wedge(A\vee D))\vee(A\wedge(B\vee C)))\wedge\psi^{n}D=(A\vee B)\wedge(A\vee D)\wedge(B\vee C)\wedge\psi^{n}D$ $=(A\vee B)\wedge\psi^{n}D=(A\vee B)\wedge(C\vee D)\wedge\psi^{n}D=\phi 1\wedge\psi^{n}D$. The other assertions analogously follow.

Lemma 37: Let $m,n\varepsilon N_o\cup\{\infty\}$.

(1) $\phi(\psi^{m}A\vee\psi^{n}C)=\phi 1\wedge(\psi^{m}D\vee\psi^{n}B)$ (2) $\phi(\psi^{m}A\vee\psi^{n}D)=\phi 1\wedge(\psi^{m}D\vee\psi^{n}A)$

(3) $\phi(\psi^{m}B\vee\psi^{n}C)=\phi 1\wedge(\psi^{m}C\vee\psi^{n}B)$ (4) $\phi(\psi^{m}B\vee\psi^{n}D)=\phi 1\wedge(\psi^{m}C\vee\psi^{n}A)$

(5) $\psi(\phi^{m}A\vee\phi^{n}B)=\psi 1\wedge(\phi^{m}B\vee\phi^{n}A)$ (6) $\psi(\phi^{m}A\vee\phi^{n}C)=\psi 1\wedge(\phi^{m}B\vee\phi^{n}D)$

(7) $\psi(\phi^{m}B\vee\phi^{n}D)=\psi 1\wedge(\phi^{m}A\vee\phi^{n}C)$ (8) $\psi(\phi^{m}C\vee\phi^{n}D)=\psi 1\wedge(\phi^{m}D\vee\phi^{n}C)$

Proof: (1): $\phi(\psi^{m}A\vee\psi^{n}C)=((A\vee B)\wedge\psi^{m}D)\vee((C\vee D)\wedge\psi^{n}B)$ $=(((A\vee B)\wedge\psi^{m}D)\vee\psi^{n}B)\wedge(C\vee D)=(\psi^{m}D\vee\psi^{n}B)\wedge(A\vee B)\wedge(C\vee D)$ $=\phi 1\wedge(\psi^{k}D\vee\psi^{l}B)$ (Lemma 36). All other cases similarly follow.

Lemma 38: Let $k,1,m \in N_o \cup \{\infty\}$.

(1) $\phi^{2m}(\psi^k A \vee \psi^1 C) = \phi^{2m} 1 \wedge (\psi^k A \vee \psi^1 C)$

(2) $\phi^{2m}(\psi^k A \vee \psi^1 D) = \phi^{2m} 1 \wedge (\psi^k A \vee \psi^1 D)$

(3) $\phi^2 m(\psi^k B \vee \psi^1 C) = \phi^{2m} 1 \wedge (\psi^k B \vee \psi^1 C)$

(4) $\phi^{2m}(\psi^k B \vee \psi^1 D) = \phi^{2m} 1 \wedge (\psi^k B \vee \psi^1 D)$

(5) $\psi^{2m}(\phi^k A \vee \phi^1 B) = \psi^{2m} 1 \wedge (\phi^k A \vee \phi^1 B)$

(6) $\psi^{2m}(\phi^k A \vee \phi^1 C) = \phi^{2m} 1 \wedge (\phi^k A \vee \phi^1 C)$

(7) $\psi^{2m}(\phi^k B \vee \phi^1 D) = \psi^{2m} 1 \wedge (\phi^k B \vee \phi^1 D)$

(8) $\psi^{2m}(\phi^k C \vee \phi^1 D) = \phi^{2m} 1 \wedge (\phi^k C \vee \phi^1 D)$

(9) $\phi^k \psi^1 1 = \phi^k 1 \wedge \psi^1 1$

Proof: (1): The case $m=0$ is trivial. By induction hypothesis we get for $m>0$: $\phi^{2m}(\psi^k A \vee \psi^1 C) = \phi^2(\phi^{2m-2} 1 \wedge (\psi^k A \vee \psi^1 C))$ $= \phi^{2m} 1 \wedge \phi^2 1 \wedge \phi 1 \wedge (\psi^k A \vee \psi^1 C) = \phi^{2m} 1 \wedge (\psi^k A \vee \psi^1 C)$ (Lemma 37, 28). (2), (3), ..., (8): analogous to (1). (9) immediately follows by (1) with $k=1=0$ and Lemma 37.

Lemma 39: Let $i,j,k,1 \in N_o$, and let $M,N \in 2N_o$ with $M \geq \max\{k,1\}$, $N \geq \max\{i,j\}$.

(1) $(\psi^i B \vee \psi^j C) \wedge (\phi^k A \vee \phi^1 B) = \phi^M(\psi^i B \vee \psi^j C) \vee \psi^N(\phi^k A \vee \phi^1 B)$

(2) $(\psi^i A \vee \psi^j D) \wedge (\phi^k C \vee \phi^1 D) = \phi^M(\psi^i A \vee \psi^j D) \vee \psi^N(\phi^k C \vee \phi^1 D)$

(3) $(\psi^i A \vee \psi^j C) \wedge (\phi^k B \vee \phi^1 D) = \phi^M(\psi^i A \vee \psi^j C) \vee \psi^N(\phi^k B \vee \phi^1 D)$

(4) $(\psi^i B \vee \psi^j D) \wedge (\phi^k A \vee \phi^1 C) = \phi^M(\psi^i B \vee \psi^j D) \vee \psi^N(\phi^k A \vee \phi^1 C)$

Proof: (1): $(\psi^i B \vee \psi^j C) \wedge (\phi^k A \vee \phi^l B)$

$=(\psi^i B \vee \psi^j C) \wedge (\phi^k A \vee \phi^l B) \wedge (\phi^M 1 \vee \phi^N 1)$

$=(\psi^N 1 \vee (\phi^M 1 \wedge (\psi^i B \vee \psi^j C))) \wedge (\phi^k A \vee \phi^l B)$

$=(\phi^M 1 \wedge (\psi^i B \vee \psi^j C)) \vee (\psi^M 1 \wedge (\phi^k A \vee \phi^l B))$

$=\phi^M(\psi^i B \vee \psi^j C) \vee \psi^N(\phi^k A \vee \phi^l B))$ (Lemma 35, 38).

(2), (3), (4): analogous to (1).

Lemma 40: $(\psi^i \phi X \vee \psi^j X) \wedge (\phi^k \psi X \vee \phi^l X) \leq \phi\psi 1$ for all $i,j,k,l \varepsilon N \cup \{\infty\}$ and for all $X \varepsilon J_4^4$.

Proof: The assertion follows by Lemma 38 and by $\psi 1 \geq \psi(\psi^{i-1}\phi X \vee \psi^{j-1} X)$ and $\phi 1 \geq \phi(\phi^{k-1}\psi X \vee \phi^{l-1} X)$.

Lemma 41: Let $i,j \varepsilon N_o$ with $i \equiv j (\mathrm{mod}\ 2)$; $M \varepsilon 2N$ with $M > \max\{i,j\}$ and let $X \varepsilon J_4^4$.

(1) $\psi^i \phi X \vee \psi^j X \geq \psi^M \phi 1$ (2) $\phi^i \psi X \vee \phi^j X \geq \phi^M \psi 1$

Proof: (1): $\psi^i \phi A \vee \psi^j A \geq \psi^i \phi A \vee \psi^j \phi D = \phi(\psi^i A \vee \psi^j D)$

$=\phi 1 \wedge (\psi^i D \vee \psi^j A) \geq \phi 1 \wedge (\psi^{\max\{i,j\}} D \vee \psi^{\max\{i,j\}} A) \geq \phi 1 \wedge \psi^{\max\{i,j\}+1} 1$

$\geq \phi 1 \wedge \psi^M 1 = \phi\psi^M 1$ (Lemma 37, 38). The proofs for B,C,D analogously go. For 0 or 1 the assertions are trivial.

(2): analogous to (1).

Lemma 42: Let $i,j,k,l \in N$; $M,N \in 2N$ with $M > \max\{k,l\}$ and $N > \max\{i,j\}$.

(1) $(\psi^i\phi A\vee\psi^j A)\wedge(\phi^k\psi A\vee\phi^l A)=\phi^M(\psi^i D\vee\psi^j A)\vee\psi^N(\phi^k B\vee\phi^l A)$

(2) $(\psi^i\phi B\vee\psi^j B)\wedge(\phi^k\psi B\vee\phi^l B)=\phi^M(\psi^i C\vee\psi^j B)\vee\psi^N(\phi^k A\vee\phi^l B)$

(3) $(\psi^i\phi C\vee\psi^j C)\wedge(\phi^k\psi C\vee\phi^l C)=\phi^M(\psi^i B\vee\psi^j C)\vee\psi^N(\phi^k D\vee\phi^l C)$

(4) $(\psi^i\phi D\vee\psi^j D)\wedge(\phi^k\psi D\vee\phi^l D)=\phi^M(\psi^i A\vee\psi^j D)\vee\psi^N(\phi^k C\vee\phi^l D)$

Proof: (1): $(\psi^i\phi A\vee\psi^j A)\wedge(\phi^k\psi A\vee\phi^l A)$

$=(\psi^i\phi A\vee\psi^j A)\wedge(\phi^k\psi A\vee\phi^l A)\wedge\psi\phi 1$

$=(\psi^i\phi A\vee\psi^j A)\wedge(\phi^k\psi A\vee\phi^l A)\wedge(\psi^N\phi 1\vee\phi^M\psi 1)$

$=(((\psi^i\phi A\vee\psi^j A)\wedge\phi^M\psi 1)\vee\psi^N\phi 1)\wedge(\phi^k\psi A\vee\phi^l A)$

$=((\psi^i\phi A\vee\psi^j A)\wedge\phi^M\psi 1)\vee((\phi^k\psi A\vee\phi^l A)\wedge\psi^N\phi 1)$

$=\psi((\psi^{i-1}\phi A\vee\psi^{j-1}A)\wedge\phi^M 1)\vee\phi((\phi^{k-1}\psi A\vee\phi^{l-1}A)\wedge\psi^N 1)$

$=\psi((((A\vee B)\wedge\psi^{i-1}D)\vee\psi^{j-1}A)\wedge\phi^M 1)\vee\phi((((A\vee D)\wedge\phi^{k-1}B)\vee\phi^{l-1}A)\wedge\psi^M 1)$

$=\psi((\psi^{i-1}D\vee\psi^{j-1}A)\wedge(A\vee B)\wedge\phi^M 1)\vee\phi((\phi^{k-1}B\vee\phi^{l-1}A)\wedge(A\vee D)\wedge\psi^N 1)$

$=\psi((\psi^{i-1}D\vee\psi^{j-1}A)\wedge\phi^M 1)\vee\phi((\phi^{k-1}B\vee\phi^{l-1}A)\wedge\psi^N 1)$

$=\psi\phi^M(\psi^{i-1}D\vee\psi^{j-1}A)\vee\phi\psi^N(\phi^{k-1}B\vee\phi^{l-1}A)$

$=\phi^M(\psi^i D\vee\psi^j A)\vee\psi^N(\phi^k B\vee\phi^l A)$ (Lemma 40, 35, 41, 37, 38).

(2), (3), (4): analogous to (1).

Before we define meet-morphisms, let us recall that the elements of $FM(J_2^4)$ have a representation as quadrupels $f(i,j,k,l)=(\mu^i a\vee\mu^j d)\wedge(\nu^k c\vee\nu^l d)$ with (i,j,k,l) satisfying condition (*) in section 2 (Proposition 19).

Furthermore, let us define for $i\varepsilon N_o$:

$\psi^{-1}A\vee\psi^{i}D:=B\vee\psi^{i}D$, $\psi^{-1}D\vee\psi^{i}A:=C\vee\psi^{i}A$, $\psi^{-1}B\vee\psi^{i}C:=A\vee\psi^{i}C$,
$\psi^{-1}C\vee\psi^{i}B:=D\vee\psi^{i}B$, $\psi^{-1}A\vee\psi^{-1}D:=1$, $\psi^{-1}B\vee\psi^{-1}C:=1$, and
$\phi^{-1}A\vee\phi^{i}B:=D\vee\phi^{i}B$, $\phi^{-1}B\vee\phi^{i}A:=C\vee\phi^{i}A$, $\phi^{-1}C\vee\phi^{i}D:=B\vee\phi^{i}D$,
$\phi^{-1}D\vee\phi^{i}C:=A\vee\phi^{i}B$, $\phi^{-1}A\vee\phi^{-1}B:=1$, $\phi^{-1}C\vee\phi^{-1}D:=1$.

<u>Lemma 43:</u> There are meet-morphisms $\bar{\alpha}$, $\bar{\beta}$ from $FM(J_2^4)$ into $FM(J_4^4)$ such that

(1) $\bar{\alpha}f(i,j,k,l)=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)$

(2) $\bar{\beta}f(i,j,k,l)=(\psi^{i-1}B\vee\psi^{j-1}C)\wedge(\phi^{k-1}B\vee\phi^{l-1}A)$

Proof: (1): What actually has to be proved is

(i) $\bar{\alpha}f(i,j,k,l)=\bar{\alpha}f(i,j,0,0)\wedge\bar{\alpha}f(0,0,k,l)$

(ii) $\bar{\alpha}(f(i,j,0,0)\wedge f(r,s,0,0))=\bar{\alpha}f(i,j,0,0)\wedge\bar{\alpha}f(r,s,0,0)$

(iii) $\bar{\alpha}(f(0,0,k,l)\wedge f(0,0,t,u))=\bar{\alpha}f(0,0,k,l)\wedge\bar{\alpha}f(0,0,t,u)$

since by Proposition 19, (i), (ii) and (iii) we get:
$\bar{\alpha}(f(i,j,k,l)\wedge f(r,s,t,u))=\bar{\alpha}f(w,x,y,z)=\bar{\alpha}f(w,x,0,0)\wedge\bar{\alpha}f(0,0,y,z)$
$=\bar{\alpha}(f(i,j,0,0)\wedge f(r,s,0,0))\wedge\bar{\alpha}(f(0,0,k,l)\wedge f(0,0,t,u))$
$=\bar{\alpha}f(i,j,0,0)\wedge\bar{\alpha}f(r,s,0,0)\wedge\bar{\alpha}f(0,0,k,l)\wedge\bar{\alpha}f(0,0,t,u)$
$=\bar{\alpha}f(i,j,k,l)\wedge\bar{\alpha}f(r,s,t,u)$, if $0\leq i,j,k,l,r,s,t,u<\infty$. If some $n\varepsilon\{i,j,k,l,r,s,t,u\}$ equals ∞, the proof easily can be checked by definition of $\bar{\alpha}$.

(i): $\bar{\alpha}f(i,j,0,0)\wedge\bar{\alpha}f(0,0,k,l)=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge 1\wedge 1\wedge(\phi^{k-1}D\vee\phi^{l-1}C)$
$=(\psi^{i-1}D\vee\psi^{j-1})\wedge(\phi^{k-1}D\vee\phi^{l-1}C)=\bar{\alpha}f(i,j,k,l)$

(ii): The proof is divided into the following cases ($(i,j)\sim(r,s)$ means that $m\equiv n \pmod 2$ for all $m,n\in\{i,j,r,s\}\setminus\{\infty\}$ and $i,j,r,s\in N_0\cup\{\infty\}$):

1. $(i,j)\sim(r,s)$	2. $(i,j)\nsim(r,s), \max\{i,j\}\le\max\{r,s\}$	
1.1. $i\le r,\ j\le s$	2.1. $0<i\le j\le r\le s$	2.8. $0<r\le j\le i\le s$
1.2. $i\le r,\ j\ge s$	2.2. $0<i\le j\le s\le r$	2.9. $0<i\le r\le j\le s$
1.3. $i\ge r,\ j\le s$	2.3. $0<j\le i\le r\le s$	2.10. $0<r\le i\le j\le s$
1.4. $i\ge r,\ j\ge s$	2.4. $0<j\le i\le s\le r$	2.11. $0<j\le s\le i\le r$
	2.5. $0<i\le s\le j\le r$	2.12. $0<s\le j\le i\le r$
	2.6. $0<s\le i\le j\le r$	2.13. $i\cdot j=0$ and $i+j>0$ or $r\cdot s=0$ and $r+s>0$
	2.7. $0<j\le r\le i\le s$	
		2.14. $i=j=0$ or $r=s=0$

By symmetry, the case $(i,j)\nsim(r,s)$, $\max\{i,j\}>\max\{r,s\}$ is analogous to 2.

1.1. and 1.4. are immediate consequences of part 2, Lemma 6 and Lemma 28.

1.2. $\bar{\alpha}((\mu^i a\vee\mu^j d)\wedge(\mu^r a\vee\mu^s d))=\bar{\alpha}(\mu^r a\vee\mu^j d)=\psi^{r-1}D\vee\psi^{j-1}A$

$=(\psi^{s-1}A\wedge\psi^{i-1}D)\vee\psi^{j-1}A\wedge\psi^{r-1}D=(\psi^{s-1}A\wedge(\psi^{i-1}D\vee\psi^{j-1}A))\vee\psi^{r-1}D$

$=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\psi^{r-1}D\vee\psi^{s-1}A)=\bar{\alpha}(\mu^i a\vee\mu^j d)\wedge\bar{\alpha}(\mu^r a\vee\mu^s d)$

(Lemma 29,28);

1.3. analogous to 1.2..

2. $\bar{\alpha}((\mu^i a \vee \mu^j d) \wedge (\mu^r a \vee \mu^s d))$

$= \bar{\alpha}(\mu^{\max\{i+1,j+1,r\}} a \vee \mu^{\max\{i+1,j+1,s\}} d)$

$= \psi^{\max\{i,j,r-1\}} D \vee \psi^{\max\{i,j,s-1\}} A$

<u>2.1.</u> $\bar{\alpha}(\mu^i a \vee \mu^j d) \wedge \bar{\alpha}(\mu^r a \vee \mu^s d) = (\psi^{i-1} D \vee \psi^{j-1} A) \wedge (\psi^{r-1} D \vee \psi^{s-1} A)$

$(\psi^{i-1} D \vee \psi^j A) \wedge (\psi^{r-1} D \vee \psi^{s-1} A) = ((\psi^{i-1} D \vee \psi^j A) \wedge \psi^{r-1} D) \vee \psi^{s-1} A$

$= \psi^{i-1}((D \wedge \psi^{j-i+1} A) \wedge C \wedge (D \vee \psi^{r-i-1} A)) \vee \psi^{s-1} A$

$= \psi^{i-1}((((\psi^{j-i+1} A \vee D) \wedge \psi^{r-i-1} A) \vee D) \wedge C) \vee \psi^{s-1} A$

$= \psi^{i-1}((((\psi^{j-i} A \vee D) \wedge \psi^{r-i-1} A) \vee D) \wedge C) \vee \psi^{s-1} A$

$= \psi^{i-1}((\psi^{r-i-1} A \vee D) \wedge C) \vee \psi^{s-1} A = \psi^{i-1} \psi^{r-i} D \vee \psi^{s-1} A$

$= \psi^{r-1} D \vee \psi^{s-1} A$ (Lemma 32,28,30);

<u>2.2.,2.3,2.4.:</u> analogous to 2.1.(by symmetry of J_4^4 and commutativity of $\wedge$);

<u>2.5.</u> $\bar{\alpha}(\mu^i a \vee \mu^j d) \wedge \bar{\alpha}(\mu^r a \vee \mu^s d) = (\psi^{i-1} D \vee \psi^{j-1} A) \wedge (\psi^{r-1} D \vee \psi^{s-1} A)$

$= (\psi^{i-1} D \vee \psi^j A) \wedge (\psi^r D \vee \psi^{s-1} A) = \psi^r D \vee (\psi^{s-1} A \wedge (\psi^{i-1} D \vee \psi^j A))$

$= \psi^r D \vee \psi^j A \wedge (\psi^{s-1} A \wedge \psi^{i-1} D) = \psi^r D \vee \psi^j A = \psi^{r-1} D \vee \psi^j A$

(Lemma 32,28,29);

<u>2.6.,2.7.,2.8.:</u> analogous to 2.5.;

<u>2.9.</u> $\bar{\alpha}(\mu^i a \vee \mu^j d) \wedge \bar{\alpha}(\mu^r a \vee \mu^s d) = (\psi^{i-1} D \vee \psi^{j-1} A) \wedge (\psi^{r-1} D \vee \psi^{s-1} A)$

$= ((\psi^{i-1} D \vee \psi^j A) \wedge \psi^{r-1} D) \vee \psi^{s-1} A = \psi^{i-1}((\psi^{j-i+1} A \vee D) \wedge \psi^{r-i} D) \vee \psi^{s-1} A$

$= \psi^{i-1}((\psi^{j-1} A \vee D) \wedge C) \vee \psi^{s-1} A = \psi^{i-1} \psi^{j-i+1} D \vee \psi^{t-1} A$

$= \psi^j D \vee \psi^{t-1} A$ (2.1., Lemma 30,32);

<u>2.10.,2.11.,2.12.:</u> analogous to 2.9.

2.13. Let $0=i<j\le r\le s$. $\bar{\alpha} f(0,j,0,0) \wedge \bar{\alpha} f(0,0,r,s)$

$= (C \vee \psi^{j-1} A) \wedge (\psi^{r-1} D \vee \psi^{s-1} A) = ((C \vee \psi^{j-1} A) \wedge \psi^{r-1} D) \vee \psi^s A$

$= ((C \vee \psi^{j-1} A) \wedge D \wedge (C \vee \psi^{r-2} A)) \vee \psi^s A = (D \wedge (C \vee \psi^{r-2} A)) \vee \psi^s A$

$= \psi^{r-1} D \vee \psi^s A = \psi^{r-1} D \vee \psi^{s-1} A$ (Lemma 32).

Corresponding 2.2. until 2.12., all other cases can be easily proved in a similar way.

2.14.: The assertion is trivial because of $\bar{\alpha}f(0,0,0,0)=1$.

(iii): By symmetry of definition of ϕ and ψ , assertion (iii) analogously as (ii) follows.

Since the proof of (2) is analogous to (1), Lemma 43 is completely proved.

Lemma 44: $\alpha\bar{\alpha}=id_{FM(J_2^4)}=\beta\bar{\beta}$

Proof: 1. Let $x=f(i,j,k,l)\varepsilon FM(J_2^4)$ with $i,j,k,l>0$.

$$\alpha\bar{\alpha}x=\alpha((\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C))$$
$$=(\mu^{i-1}\alpha D\vee\mu^{j-1}\alpha A)\wedge(\nu^{k-1}\alpha D\vee\nu^{l-1}\alpha C)$$
$$=(\mu^{i-1}b\vee\mu^{j-1}c)\wedge(\nu^{k-1}b\vee\nu^{l-1}a)$$
$$=(\mu^{i}a\vee\mu^{j}d)\wedge(\nu^{k}c\vee\nu^{l}d)=f(i,j,k,l) \text{ (Lemma 25)}.$$

2. Let $x=f(i,j,k,l)\varepsilon FM(J_2^4)$ with some of i,j,k,l equals 0. We have to use the definition for expressions with ψ^{-1} or ϕ^{-1}; then the proof is analogous to 1..

$\beta\bar{\beta}=id_{FM(J_2^4)}$ can analogously be shown.

Lemma 45: $\underline{\alpha}x \le \bar{\alpha}x$ and $\underline{\beta}x \le \bar{\beta}x$ for all $x \in FM(J_2^4)$

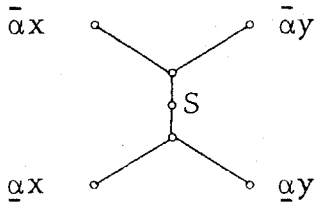
Proof: $\underline{\alpha}f(i,j,k,l) = \underline{\alpha}((\mu^i a \vee \mu^j d) \wedge (\nu^k c \vee \nu^l d))$
$= (\psi^i(C \wedge (A \vee B)) \vee \psi^j B) \wedge (\phi^k(A \wedge (B \vee C)) \vee \phi^l B)$
$\le (\psi^{i-1} D \vee \psi^{j-1} A) \wedge (\phi^{k-1} D \vee \phi^{l-1} C) = \bar{\alpha}f(i,j,k,l)$ (Lemma 37).
$\underline{\beta}x \le \bar{\beta}x$ (analogous to $\underline{\alpha}x \le \bar{\alpha}x$).

Lemma 46: Let I_α and I_β be the set of all intervals $[\underline{\alpha}x, \bar{\alpha}x]$ and $[\underline{\beta}x, \bar{\beta}x]$ with $x \in FM(J_2^4)$, respectively.
(1) $\bigcup I_\alpha = FM(J_4^4)$ (2) $\bigcup I_\beta = FM(J_4^4)$

Proof: (1): Since every meet-morphism is isotone, we get $\bar{\alpha}x \le \bar{\alpha}(x \vee y)$, $\bar{\alpha}y \le \bar{\alpha}(x \vee y)$ and this implies $\bar{\alpha}x \vee \bar{\alpha}y \le \bar{\alpha}(x \vee y)$.
Now, let $S \in [\underline{\alpha}x, \bar{\alpha}x]$ and $T \in [\underline{\alpha}y, \bar{\alpha}y]$.
By $\underline{\alpha}(x \vee y) = \underline{\alpha}x \vee \underline{\alpha}y \le S \vee T \le \bar{\alpha}x \vee \bar{\alpha}y \le \bar{\alpha}(x \vee y)$ and
$\underline{\alpha}(x \wedge y) = \underline{\alpha}x \wedge \underline{\alpha}y \le S \wedge T \le \bar{\alpha}x \wedge \bar{\alpha}y \le \bar{\alpha}(x \wedge y)$, we get $S \vee T \in [\underline{\alpha}(x \vee y), \bar{\alpha}(x \vee y)]$ and $S \wedge T \in [\underline{\alpha}(x \wedge y), \bar{\alpha}(x \wedge y)]$. Since $A \in [A \wedge (B \vee C), A] = [\underline{\alpha}c, \bar{\alpha}c]$, $B \in [B, B] = [\underline{\alpha}D, \bar{\alpha}D]$, $C \in [C \wedge (A \vee B), C] = [\underline{\alpha}a, \bar{\alpha}a]$, $D \in [D \wedge (B \vee A) \wedge (B \vee C), D] = [\underline{\alpha}b, \bar{\alpha}b]$ and since $FM(J_4^4)$ is freely generated by J_4^4, the assertion follows.
(2): analogous to (1).

Lemma 47: I_α and I_β are partitions on $FM(J_4^4)$.

Proof:



$[\underline{\alpha}x,\bar{\alpha}x]\cap[\underline{\alpha}y,\bar{\alpha}y]\neq\emptyset$ implies $\underline{\alpha}x\vee\underline{\alpha}y\leq\bar{\alpha}x\wedge\bar{\alpha}y$. Using Lemma 27 and Lemma 44, we get $x\vee y\leq x\wedge y$ by $\alpha\underline{\alpha}(x\vee y)\leq\alpha\bar{\alpha}(x\wedge y)$. Thus, $x=y$ and the assertion follows by Lemma 46. The assertion for I_β can be proved by the same arguments.

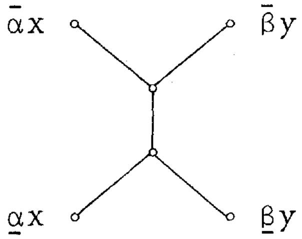
<u>Lemma 48:</u> Let $\ker\alpha_x := \{X\varepsilon FM(J_4^4)\,|\,\alpha X=x\}$ and $\ker\beta_x := \{X\varepsilon FM(J_4^4)\,|\,\beta X=x\}$ for $x\varepsilon FM(J_2^4)$.

(1) $\ker\alpha_x=[\underline{\alpha}x,\bar{\alpha}x]$ (2) $\ker\beta_x=[\underline{\beta}x,\bar{\beta}x]$.

Proof: Lemma 48 is an immediate consequence of Lemma 47 together with Lemma 27 and 44.

<u>Proposition 49:</u> α and β are separating homomorphisms from $FM(J_4^4)$ onto $FM(J_2^4)$.

Proof: By Lemma 48 it is enough to show that the meet of two intervals $[\underline{\alpha}x,\bar{\alpha}x]$ and $[\underline{\beta}y,\bar{\beta}y]$ is empty or consists of only one element. We choose a fixed element $x=f(i,j,k,l)$ of $FM(J_2^4)$. Our first goal is to determine all elements $y\varepsilon FM(J_2^4)$ for which $I_{xy}:=[\underline{\alpha}x,\bar{\alpha}x]\cap[\underline{\beta}y,\bar{\beta}y]\neq\emptyset$.



Suppose $I_{xy}\neq\emptyset$. Then $\underline{\alpha}x\vee\underline{\beta}y\le\bar{\alpha}x\wedge\bar{\beta}y$ and $\beta\underline{\alpha}x\vee y=\beta\underline{\alpha}x\vee\beta\underline{\beta}y\le\beta\bar{\alpha}x\wedge\beta\bar{\beta}y=\beta\bar{\alpha}x\wedge y$ (Lemma 27,44). Therefore we get $\mu\nu x=\beta\underline{\alpha}x\le y\le\beta\bar{\alpha}x$ (Lemma 27). By Lemma 4 and Proposition 19, $\mu\nu x=f(j+1,i+1,l+1,k+1)$. For determining $\beta\bar{\alpha}x$ we have to distinguish in

(i) $i,j,k,l>0$: $\beta\bar{\alpha}f(i,j,k,l)=\beta((\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C))$
$=(\mu^{i-1}\beta D\vee\mu^{j-1}\beta A)\wedge(\nu^{k-1}\beta D\vee\nu^{l-1}\beta C)$
$=(\mu^{j-1}a\vee\mu^{i-1}d)\wedge(\nu^{l-1}c\vee\nu^{k-1}d)=f(j-1,i-1,l-1,k-1)$
(Lemma 25).

(ii) $i\cdot j=0$ and $i+j>0$ or $k\cdot l=0$ and $k+l>0$: Let w.l.o.g. $0=i<j,k,l$.
$\beta\bar{\alpha}f(0,j,k,l)=\beta((C\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)$
$=(c\vee\mu^{j-1}a)\wedge(\nu^{k-1}d\vee\nu^{l-1}c)=(\mu^{j-1}a\vee\mu d)\wedge(\nu^{l-1}c\vee\nu^{k-1}d)$
$=f(j-1,1,l-1,k-1)$ (Lemma 25).

(iii) $i=j=0$ or $k=l=0$: Let w.l.o.g. $0=i=j<k,l$.
$\beta\bar{\alpha}f(0,0,k,l)=\beta(1\wedge(\phi^{k-1}D\vee\phi^{l-1}C))=1\wedge(\nu^{l-1}c\vee\nu^{k-1}d)$
$=f(0,0,l-1,k-1)$ (Lemma 25).

For summarizing these results, we define:
$i^*:=j,\ j^*:=i,\ k^*:=l,\ l^*:=k$

and for all $m\in\{i,j,k,l\}$: $\bar{m}:=\begin{cases} m-1 & \text{if } m>0 \\ 1 & \text{if } m=0 \text{ and } m^*\neq 0 \\ 0 & \text{if } m=m^*=0 \end{cases}$

It follows: $I_{xy} \neq \emptyset \Longleftrightarrow y \varepsilon [f(j+1,i+1,l+1,k+1), f(\bar{j},\bar{i},\bar{l},\bar{k})]$
$\Longleftrightarrow y \varepsilon \{f(\hat{j},\hat{i},\hat{l},\hat{k}) \varepsilon FM(J_2^4) \mid \hat{m} \varepsilon \{\bar{m},m,m+1\}$ and $\hat{m}=m \Longleftrightarrow \hat{m}^*=m$ with $m \varepsilon \{i,j,k,l\}\}$ (Proposition 19).

Let us say, y satisfies (**), if y is an element of the set just mentioned.

In the following we have to prove $\bar{\alpha}x \wedge \bar{\beta}y = \underline{\alpha}x \vee \underline{\beta}y$ for a fixed x and all y satisfying condition (**). The proof will be divided in several cases, since we have some possibilities for the choice of y and since the cases with some of i,j,k,l equals 0 or equals ∞ have to be treated separately. By symmetry of J_4^4 and by commutativity of meet and join, it is enough to prove the following cases:

1.1.: $x=f(i,j,k,l)$, $y=f(j+1,i+1,l+1,k+1)$ for $0<i,j,k,l<\infty$
1.2.: $x=f(i,j,k,l)$, $y=f(j-1,i+1,l+1,k+1)$ for $0<i,j,k,l<\infty$
1.3.: $x=f(i,j,k,l)$, $y=f(j-1,i-1,l+1,k+1)$ for $0<i,j,k,l<\infty$
1.4.: $x=f(i,i,k,l)$, $y=f(i,i,l+1,k+1)$ for $0<i,k,l<\infty$
1.5.: $x=f(\infty,2m,\infty,2n)$, $y=f(2m+1,\infty,2n-1,\infty)$ for $0<m,n<\infty$
2.1.: $x=f(0,j,k,l)$, $y=f(j+1,1,l+1,k+1)$ for $0<j,k,l<\infty$
2.2.: $x=f(0,\infty,\infty,2n+1)$, $y=f(\infty,1,2n,\infty)$ for $0\le n<\infty$
3.1.: $x=f(0,0,k,l)$, $y=f(1,1,l+1,k+1)$ for $0<k,l<\infty$

1.1.: $\bar{\alpha}x\wedge\bar{\beta}y=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{j}B\vee\psi^{i}C)\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{j}B\vee\psi^{i}C)\wedge(\phi^{l}B\vee\phi^{k}A)=\phi^{M}(\psi^{i}C\vee\psi^{j}B)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)$

$=\phi^{M}(\psi^{i}C\vee\psi^{i+1}D\vee\psi^{j}B\vee\psi^{j+1}A)\vee\psi^{N}(\phi^{k}A\vee\phi^{k+1}D\vee\phi^{l}B\vee\phi^{l+1}C)$

$=\phi^{M}(\psi^{i}C\vee\psi^{j}B)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)\vee\phi^{M}(\psi^{j+1}A\vee\psi^{i+1}D)\vee\psi^{N}(\phi^{l+1}C\vee\phi^{k+1}D)$

$=((\psi^{i}\phi B\vee\psi^{j}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{j+1}\phi D\vee\psi^{i+1}D)\wedge(\phi^{l-1}\psi D\vee\phi^{k+1}D))$

$=\underline{\alpha}x\vee\underline{\beta}x$ (Lemma 39,42).

1.2.: $\bar{\alpha}x\wedge\bar{\beta}y=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{j-2}B\vee\psi^{i}C)\wedge(\phi^{l}B\wedge\phi^{k}A)$

$=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\psi^{j-2}B\vee\psi^{i}C)\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{i}C\vee(\psi^{j-2}B\wedge(\psi^{i-1}D\vee\psi^{j-1}A)))\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{i}C\vee\psi^{j-1}A\vee(\psi^{j-2}B\wedge\psi^{i-1}D))\wedge(\phi^{l}B\vee\phi^{k}A)$

$=(\psi^{i}C\vee\psi^{j-1}A)\wedge(\phi^{l}B\vee\phi^{k}A)=\phi^{M}(\psi^{i}C\vee\psi^{j-1}A)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)$

$=\phi^{M}(\psi^{i}C\vee\psi^{j}B\vee\psi^{j-1}A\vee\psi^{i+1}D)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B\vee\phi^{l+1}C\vee\phi^{k+1}D)$

$=((\psi^{i}\phi B\vee\psi^{j}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{j-1}\phi D\vee\psi^{i+1}D)\wedge(\phi^{l+1}\psi D\vee\phi^{k+1}D)$

$=\underline{\alpha}x\vee\underline{\beta}x$ (Lemma 29,39,42).

1.3.: $\bar{\alpha}x\wedge\bar{\beta}y=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{j-2}B\vee\psi^{i-2}C)\wedge(\phi^{l}B\wedge\phi^{k}A)$

$=(\psi^{i-1}D\vee\psi^{j-1}A)\wedge(\phi^{l}B\vee\phi^{k}A)=\phi^{M}(\psi^{i-1}D\vee\psi^{j-1}A)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B)$

$=\phi^{M}(\psi^{i}C\vee\psi^{j}B\vee\psi^{j-1}\vee\psi^{i-1}D)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B\vee\phi^{l+1}C\vee\phi^{k+1}D)$

$=((\psi^{i}\phi B\vee\psi^{j}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{j-1}\phi D\vee\psi^{i-1}D)\wedge(\phi^{l+1}\psi D\vee\phi^{k+1}D))$

$=\underline{\alpha}x\vee\underline{\beta}x$ (Lemma 39,42).

1.4.: $\bar{\alpha}x\wedge\bar{\beta}y=(\psi^{i-1}D\vee\psi^{i-1}A)\wedge(\phi^{k-1}D\vee\phi^{l-1}C)\wedge(\psi^{i-1}B\vee\psi^{i-1}C)\wedge(\phi^{l}B\vee\phi^{k-2}A)$

$=\psi^{i-1}((D\vee A)\wedge(B\vee C))\wedge(\phi^{l}B\vee(\phi^{k-2}A\wedge(\phi^{k-1}D\vee\phi^{l-1}C)))$

$=\psi^{i-1}\psi 1\wedge(\phi^{l}B\vee\phi^{k-1}D\vee(\phi^{l-1}D\wedge\phi^{k-2}A)=\psi^{i}1\wedge(\phi^{k-1}D\vee\phi^{l}B)$

$=\phi^{M}\psi^{i}1\vee\psi^{N}(\phi^{k-1}D\vee\phi^{l}B)$

$=\phi^{M}(\psi^{i}C\vee\psi^{i}B\vee\psi^{i}A\vee\psi^{i}D)\vee\psi^{N}(\phi^{k}A\vee\phi^{l}B\vee\phi^{l+1}C\vee\phi^{k-1}D)$

$=((\psi^{i}\phi B\vee\psi^{i}B)\wedge(\phi^{k}\psi B\vee\phi^{l}B))\vee((\psi^{i}\phi D\vee\psi^{i}D)\wedge(\phi^{l+1}\psi D\vee\phi^{k-1}D)$

$=\underline{\alpha}x\vee\underline{\beta}x$ (Lemma 29,39,42).

1.5.: $\bar{\alpha}x\wedge\bar{\beta}y=\psi^{2m-1}A\wedge\phi^{2n-1}C\wedge\psi^{2m}B\wedge\phi^{2n-2}B=\psi^{2m}B\wedge\phi^{2n-1}C$

$$=B\wedge\phi^{2n-1}C\wedge\psi^{2m}B=(\psi^{2m+1}A\vee\phi^{2n}B)\wedge\phi^{2n-1}C\wedge\psi^{2m}B$$

$$=((\psi^{2m+1}A\wedge\phi^{2n-1}C)\vee\phi^{2n}B)\wedge\psi^{2m}B$$

$$=((\psi^{2m+1}A\wedge\phi 1\wedge\phi^{2n-1}C\wedge\psi 1)\vee\phi^{2n}B)\wedge\psi^{2m}B$$

$$=((\psi^{2m+1}\phi D\wedge\phi^{2n-1}\psi D)\vee\phi^{2n}B)\wedge\psi^{2m}B$$

$$=(\psi^{2m}B\wedge\phi^{2n}B)\vee(\psi^{2m+1}\phi D\wedge\phi^{2n-1}\psi D)=\underline{\alpha}x\vee\underline{\beta}y \text{ (Lemma 34,37).}$$

2.1.: $\bar{\alpha}x\wedge\bar{\beta}y=(C\vee\psi^{j-1}A)\wedge(\phi^{k-1}D\vee\phi^{1-1}C)\wedge(\psi^{j}B\vee C)\wedge(\phi^{1}B\vee\phi^{k}A)$

$$=(\psi^{j}B\vee C)\wedge(\phi^{1}B\vee\phi^{k}A)=\phi^{M}(C\vee\psi^{j}B)\vee\psi^{N}(\phi^{k}A\vee\phi^{1}B)$$

$$=\phi^{M}(C\vee\psi^{j}B\vee\psi^{j+1}A\vee\psi D)\vee\psi^{N}(\phi^{k}A\vee\phi^{1}B\vee\phi^{1+1}C\vee\phi^{k+1}D)$$

$$=((C\vee\psi^{j}B)\wedge(\phi^{k}A\vee\phi^{1}B))\vee\phi^{M}(\psi^{j+1}A\vee\psi D)\vee\psi^{N}(\phi^{1+1}C\vee\phi^{k+1}D)$$

$$=((C\vee\psi^{j}B)\wedge(\phi^{k}A\vee\phi^{1}B)\wedge(B\vee C))\vee\phi^{M}(\psi^{j+1}A\vee\psi D)\vee\psi^{N}(\phi^{1+1}C\vee\phi^{k+1}D)$$

$$=((C\vee\psi^{j}B)\wedge((\phi^{k}A\wedge\psi 1)\vee\phi^{1}B))\vee\phi^{M}(\psi^{j+1}A\vee\psi D)\vee\psi^{N}(\phi^{1+1}C\vee\phi^{k+1}D)$$

$$=((\phi B\vee\psi^{j}B)\wedge(\phi^{k}\psi B\vee\phi^{1}B))\vee((\psi^{j+1}\phi D\vee\psi D)\wedge(\phi^{1+1}\psi D\vee\phi^{k+1}D)$$

$$=\underline{\alpha}x\vee\underline{\beta}y \text{ (Lemma 39,37,42).}$$

2.2.: $\bar{\alpha}x\wedge\bar{\beta}y=C\wedge\phi^{2n}C\wedge C\wedge\phi^{2n-1}B=\phi^{2n}C=\phi^{2n}(\phi B\vee\psi D)=\phi^{2n+1}B\vee\phi^{2n}\psi D$

$$=(\phi B\wedge\phi^{2n+1}B)\vee(\psi D\wedge\phi^{2n}\psi D)=\underline{\alpha}x\vee\underline{\beta}y \text{ (Lemma 33).}$$

3.1.: $\bar{\alpha}x\wedge\bar{\beta}y=1\wedge(\phi^{k-1}D\vee\phi^{1-1}C)\wedge(B\vee C)\wedge(\phi^{1}B\vee\phi^{k}A)$

$$=(B\vee C)\wedge(\phi^{1}B\vee\phi^{k}A)=\phi^{M}(B\vee C)\vee\psi^{N}(\phi^{k}A\vee\phi^{1}B)$$

$$=\phi^{M}(B\vee C\vee\psi A\vee\psi D)\vee\psi^{N}(\phi^{k}A\vee\phi^{1}B\vee\phi^{1+1}C\vee\phi^{k+1}D)$$

$$=((B\vee C)\wedge(\phi^{k}A\vee\phi^{1}B))\vee\phi^{M}(\psi A\vee\psi D)\vee\psi^{N}(\phi^{1+1}C\vee\phi^{k+1}D)$$

$$=((B\vee C)\wedge(A\vee B)\wedge(\phi^{k}A\vee\phi^{1}B)\wedge(B\vee C))\vee\phi^{M}(\psi A\vee\psi D)\vee\psi^{N}(\phi^{1+1}C\vee\phi^{k+1}D)$$

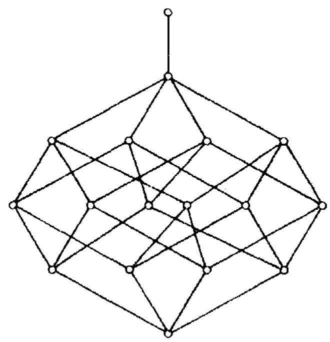
$$=((B\vee C)\wedge(A\vee B)\wedge((\phi^{k}A\wedge\psi 1)\vee\phi^{1}B))\vee\phi^{M}(\psi A\vee\psi D)\vee\psi^{N}(\phi^{1+1}C\vee\phi^{k+1}D)$$

$$=((\phi B\vee B)\wedge(\phi^{k}\psi B\vee\phi^{1}B))\vee((\psi\phi D\vee\psi D)\wedge(\phi^{1+1}D\vee\phi^{k+1}D)$$

$$=\underline{\alpha}x\vee\underline{\beta}y \text{ (Lemma 39,30,37,42).}$$

This completes the proof of Proposition 49.

Proposition 50: The congruence lattice of $FM(J_4^4)$ is described by the following diagram :



Proof: By Proposition 49, the intersection of the congruence relations $\ker\alpha$ and $\ker\beta$ is the identity; furthermore by the Homomorphism Theorem and Proposition 22, there is only one coatom greater than $\ker\alpha$ and $\ker\beta$, respectively, namely $\Theta(A\vee B, B\vee C, C\vee D, D\vee A)$. Therefore, Proposition 22 and Proposition 49 together with the distributivity of the congruence lattice gives us the assertion.

Proof of Theorem 3: By the Homomorphism Theorem, the assertions are immediate consequences of Proposition 50.

Another immediate consequence of Proposition 49 is the following Proposition which (together with Proposition 19 and 21) solves the word problem for $FM(J_4^4)$. For stating the Proposition, we define

$g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k}) := \bar{\alpha}f(i,j,k,l) \wedge \bar{\beta}f(\hat{j},\hat{i},\hat{l},\hat{k})$

for $f(i,j,k,l), f(\hat{j},\hat{i},\hat{l},\hat{k}) \in FM(J_2^4)$.

Proposition 51: The elements of $FM(J_4^4)$ can be uniquely represented as octuples $g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})$ such that (i,j,k,l) satisfies $(*)$ and $f(\hat{j},\hat{i},\hat{l},\hat{k})$ satisfies $(**)$ with respect to $f(i,j,k,l)$; furthermore, for $\circ=\vee$ and $\circ=\wedge$, respectively, we have

$$g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})\circ g(m,n,p,q,\hat{n},\hat{m},\hat{q},\hat{p})=g(r,s,t,u,\hat{s},\hat{r},\hat{u},\hat{t})$$

if and only if

$$f(i,j,k,l)\circ f(m,n,p,q)=f(r,s,t,u) \text{ and}$$

$$f(\hat{j},\hat{i},\hat{l},\hat{k})\circ f(\hat{n},\hat{m},\hat{q},\hat{p})=f(\hat{s},\hat{r},\hat{u},\hat{t}) .$$

Now we can solve the word problems for $FM(J_3^4)$ and $FM(J_{1,1}^4)$.

There exist epimorphisms γ and δ of $FM(J_3^4)$ such that

(1)	$\gamma 0=0$	(2)	$\delta 0=0$
	$\gamma a=d\wedge(a\vee b)$		$\delta a=b$
	$\gamma b=c\wedge(a\vee b)$		$\delta b=a\wedge(b\vee c)$
	$\gamma c=b\wedge(c\vee d)$		$\delta c=d\wedge(b\vee c)$
	$\gamma d=a\wedge(c\vee d)$		$\delta d=c$
	$\gamma 1=(a\vee b)\wedge(c\vee d)$		$\delta 1=b\vee c$

and endomorphisms ρ and σ of $FM(J_{1,1}^4)$ such that

(1)	$\rho 0=0$	(2)	$\sigma 0=0$
	$\rho a=d\wedge(a\vee b)$		$\sigma a=b$
	$\rho b=c\wedge(a\vee b)$		$\sigma b=a$
	$\rho c=b\wedge(c\vee d)$		$\sigma c=d$
	$\rho d=a\wedge(c\vee d)$		$\sigma d=c$
	$\rho 1=(a\vee b)\wedge(c\vee d)$		$\sigma 1=1$

These endomorphisms give us representations of $FM(J_3^4)$ and $FM(J_{1,1}^4)$.

<u>Proposition 52:</u>

$FM(J_3^4)=\{h(k,l,\hat{j},\hat{i},\hat{l},\hat{k})\mid g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})\varepsilon FM(J_4^4)\}$ with

$h(k,l,\hat{j},\hat{i},\hat{l},\hat{k}) := (\gamma^{k-1}d\vee\gamma^{l-1}c)\wedge(\delta^{\hat{j}-1}b\vee\delta^{\hat{i}-1}c)\wedge(\gamma^{\hat{l}-1}b\vee\gamma^{\hat{k}-1}a)$

for $0\leq\hat{i},\hat{j},\hat{k},\hat{l},k,l\leq\infty$ and

$\gamma^{-1}x$, $\delta^{-1}x$ analogously defined as in Lemma 43.

Proof: Let τ the epimorphism from $FM(J_4^4)$ onto $FM(J_3^4)$ with $\tau X=x$ for all $X\varepsilon J_4^4$, $x\varepsilon J_3^4$. We get $\tau\psi=\delta\tau$ and $\tau\phi=\gamma\tau$. Since $i-1\leq\hat{i}$ and $j-1\leq\hat{j}$ it follows $\delta^{i}d\vee\delta^{j}a=\delta^{i-1}c\vee\delta^{j-1}b\geq\delta^{\hat{i}}c\vee\delta^{\hat{j}}b$ and $\tau g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})$

$=(\delta^{i-1}d\vee\delta^{j-1}a)\wedge(\gamma^{k-1}d\vee\gamma^{l-1}c)\wedge(\delta^{\hat{j}-1}b\vee\delta^{\hat{i}-1}c)\wedge(\gamma^{\hat{l}-1}b\vee\gamma^{\hat{k}-1}a)$

$=(\gamma^{k-1}d\vee\gamma^{l-1}c)\wedge(\delta^{\hat{j}-1}b\vee\delta^{\hat{i}-1}c)\wedge(\gamma^{\hat{l}-1}b\vee\gamma^{\hat{k}-1}a)=h(k,l,\hat{j},\hat{i},\hat{l},\hat{k})$

<u>Corollary 53:</u> For $\circ=\vee$ and $\circ=\wedge$, respectively, we have

$h(k,l,\hat{j},\hat{i},\hat{l},\hat{k})\circ h(p,q,\hat{n},\hat{m},\hat{q},\hat{p})=h(t,u,\hat{s},\hat{r},\hat{u},\hat{t})$

if and only if $e(k,l)\circ e(p,q)=e(t,u)$ and

$f(\hat{j},\hat{i},\hat{k},\hat{l})\circ f(\hat{n},\hat{m},\hat{q},\hat{p})=f(\hat{s},\hat{r},\hat{u},\hat{t})$

(The elements $e(i,j)$ of $FM(J_1^4)$ are defined in DAY, HERRMANN, WILLE [2], and $f(i,j,k,l)\varepsilon FM(J_2^4)$).

Proposition 54:

$FM(J^4_{1,1})=\{p(k,l,\hat{l},\hat{k}) \mid g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})\in FM(J^4_4)\}$ with $p(k,l,\hat{l},\hat{k}) := (\rho^{k-1}d\vee\rho^{l-1}c)\wedge(\rho^{\hat{l}-1}b\vee\rho^{\hat{k}-1}a)$ for $0\le k,l,\hat{k},\hat{l}\le\infty$ and $\rho^{-1}x$ analogously defined as in Lemma 43.

Proof: Let η the epimorphism from $FM(J^4_4)$ onto $FM(J^4_{1,1})$ with $\eta X=x$ for all $X\in J^4_4$ and $x\in J^4_{1,1}$.
We get $\rho\eta=\eta\phi$ and $\sigma\eta=\eta\psi$. Since $\sigma^{i-1}d\vee\sigma^{j-1}a=\sigma^{\hat{j}-1}b\vee\sigma^{\hat{i}-1}c=a\vee d=b\vee c=1$ it follows

$\eta g(i,j,k,l,\hat{j},\hat{i},\hat{l},\hat{k})$
$=(\sigma^{i-1}d\vee\sigma^{j-1}a)\wedge(\rho^{k-1}d\vee\rho^{l-1}c)\wedge(\sigma^{\hat{j}-1}b\vee\sigma^{\hat{i}-1}c)\wedge(\rho^{\hat{l}-1}b\vee\rho^{\hat{k}-1}a)$
$=(\rho^{k-1}d\vee\rho^{l-1}c)\wedge(\rho^{\hat{l}-1}b\vee\rho^{\hat{k}-1}a)=p(k,l,\hat{l},\hat{k})$

Corollary 55: For $\circ=\vee$ and $\circ=\wedge$, respectively, we have $p(k,l,\hat{l},\hat{k})\circ p(m,n,\hat{n},\hat{m})=p(r,s,\hat{s},\hat{t})$ if and only if $e(k,l)\circ e(m,n)=e(r,s)$ and $e(\hat{l},\hat{k})\circ e(\hat{n},\hat{m})=e(\hat{s},\hat{r})$.

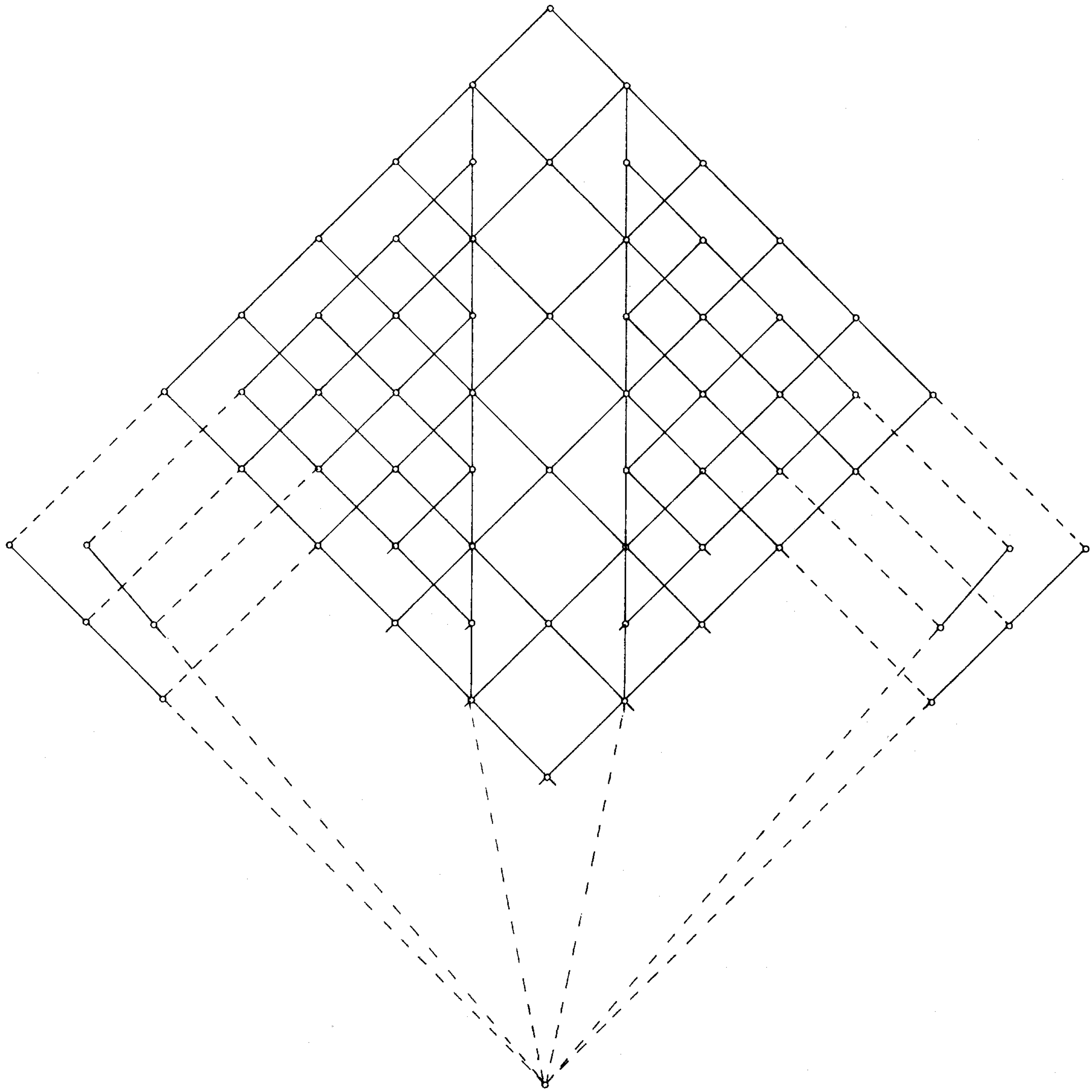
Corollary 56: (1) $FM(J^4_1)$ is a sublattice of $FM(J^4_{1,1})$
(2) $FM(J^4_2)$ is a sublattice of $FM(J^4_4)$.

Proof: (1): There exists a monomorphism λ from $FM(J^4_1)$ into $FM(J^4_{1,1})$ such that

$\lambda 0=0$	$\lambda c=a\wedge(c\vee d)$
$\lambda a=c$	$\lambda d=d$
$\lambda b=b\wedge(c\vee d)$	$\lambda 1=c\vee d$

(2) is an immediate consequence of Lemma 26.

The diagram of $FM(J^4_{1,1})$ is shown below.



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