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Some Remarks on Free Orthomodular Lattices

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This paper contains some preliminary studies of free orthomodular lattices. An orthomodular lattice (abbreviated: OML) is considered here as a (universal) algebra with basic operations $\vee$, $\wedge$, $'$, 0, 1. All general algebraic notions like subalgebra or homomorphism are to be understood in this way.

We assume the basic notions of the theory of OMLs to be known; the reader can find the necessary information in [1] and [4].

In the first chapter we describe a method to present a finitely generated OML as a direct product of a Boolean algebra and an OML of a special type, which we call tightly generated. We use this to describe certain OMLs which are freely generated by some simple partially ordered sets. In the second chapter we construct a special extension of an OML L. Since it is generated by L and one additional element we call it a one-point extension of L. We use this constructio in the last chapter to prove that the free OML generated by a three-element poset consisting of two comparable elements and an element incomparable with both contains an infinite chain. This answers a question posed by D. Foulis.

1. Some simple free OMLs

As is well known every interval of the form $[0, c]$ in an OML L can be made an OML by defining the orthocomplement a^* of an element a in $[0, c]$ by $a^* = a' \wedge c$. If c is in the center of L, i. e. if c commutes with every element of L, then the map $x \longrightarrow x \wedge c$ is a homomorphism of L onto $[0, c]$; moreover, the map $x \longrightarrow (x \wedge c', x \wedge c)$ is in this case an isomorphism between L and the direct product $[0, c'] \times [0, c]$.

We start out by describing a simple but useful such splitting of a finitely generated OML. To simplify notation we define for an element a of an OML L: $a^1 = a'$ and $a^0 = a$. We say that an OML L is tightly generated by a finite set G iff it is generated by G and for every map $\delta \in 2^G$ (i.e. $\delta : G \longrightarrow \{0,1\}$) the equation $\bigwedge_{x \in G} x^{\delta(x)} = 0$ holds.

(1.1) Let L be an OML generated by a finite set G and define $c = \bigwedge_{\delta \in 2^G} \bigvee_{x \in G} x^{\delta(x)}$. Then c is in the center of L, the OML $[0,c']$ is Boolean and the OML $[0,c]$ is tightly generated by $\{x \wedge c \mid x \in G\}$. In particular is every finitely generated OML the direct product of a Boolean algebra and a tightly generated OML.

Proof. The element c obviously commutes with every element of G and hence with every element of L, which means that it is in the center of L. To show that $[0,c']$ is Boolean it is enough to show that any two elements

$x \wedge c'$, $y \wedge c'$ with $x, y \in G$ commute in $[0,c']$, i.e. that

$$((x \wedge c') \wedge (y \wedge c')) \vee ((x \wedge c') \wedge (y \wedge c')^{\#}) = x \wedge c'$$

holds, where $(y \wedge c')^{\#}$ is the orthocomplement of $y \wedge c'$ in $[0,c']$. But

$$((x \wedge c') \wedge (y \wedge c')) \vee ((x \wedge c') \wedge (y \wedge c')^{\#}) =$$

$$(x \wedge y \wedge c') \vee ((x \wedge c') \wedge ((y' \vee c) \wedge c')) =$$

$$(x \wedge y \wedge c') \vee (x \wedge y' \wedge c') =$$

$$(x \wedge y \wedge \bigvee_{\delta \in 2^G} \bigwedge_{z \in G} z^{\delta(z)}) \vee (x \wedge y' \wedge \bigvee_{\delta \in 2^G} \bigwedge_{z \in G} z^{\delta(z)}) =$$

$$(\bigvee_{\substack{\delta \in 2^G \\ \delta(x)=0 \\ \delta(y)=0}} \bigwedge_{z \in G} z^{\delta(z)}) \vee (\bigvee_{\substack{\delta \in 2^G \\ \delta(x)=0 \\ \delta(y)=1}} \bigwedge_{z \in G} z^{\delta(z)}) =$$

$$\bigvee_{\substack{\delta \in 2^G \\ \delta(x)=0}} \bigwedge_{z \in G} z^{\delta(z)} = x \wedge c'.$$

In order to show that the OML $[0,c]$ is tightly generated by $\{x \wedge c \mid x \in G\}$ we define for a given $\varepsilon \in 2^G$:
$H = \{x \in G \mid \varepsilon(x) = 0\}$ and $J = \{x \in G \mid \varepsilon(x) = 1\}$. We then have to prove that

$$\bigwedge_{x \in H} (x \wedge c) \wedge \bigwedge_{x \in J} ((x \wedge c)' \wedge c) = 0$$

holds, which is shown by the following little calculation:

$$\bigwedge_{x \in H} (x \wedge c) \wedge \bigwedge_{x \in J} (x \wedge c)' \wedge c =$$

$$c \wedge \bigwedge_{x \in H} x \wedge \bigwedge_{x \in J} (x' \vee c') =$$

$$c \wedge \bigwedge_{x \in H} x \wedge (c' \vee \bigwedge_{x \in J} x') =$$

$$c \wedge \bigwedge_{x \in H} x \wedge \bigwedge_{x \in J} x' =$$

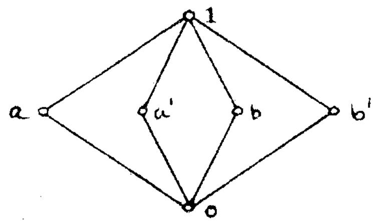
$$\bigwedge_{x \in G} x^{\varepsilon(x)} \wedge \bigwedge_{\delta \in 2^G} \bigvee_{x \in G} x^{\delta(x)} \leq$$

$$\bigwedge_{x \in G} x^{\varepsilon(x)} \wedge \bigvee_{x \in G} x^{1-\varepsilon(x)} = 0,$$

completing the proof.

As a first application of this we characterize the free OML generated by a two-element set. The structure of it is well known, the following simple proof, however, seems to be new.

Let MO2 be the following OML:



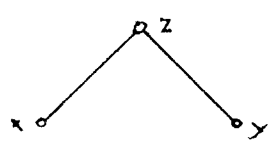
Let p_1, p_2, p_3, p_4 be the atoms of the Boolean algebra 2^4.

(1.2) The OML $2^4 \times$ MO2 is freely generated by the set $\{(p_1 \vee p_2, a), (p_1 \vee p_3, b)\}$.

Proof. Let L be an OML generated by the set $\{x, y\}$. With c having the meaning of (1.1), L is isomorphic with the direct product $[0, c'] \times [0, c]$. Since $[0, c']$ is Boolean and is generated by an at most two-element set it has at most 2^4 elements. Since $[0, c]$ is tightly generated by an at most two-element set it is a homomorphic image of MO2 and, hence, has at most six elements. It follows that L has at most $2^4 . 6 = 96$ elements. But the OML $2^4 \times$ MO2 has 96 elements and is generated by $\{(p_1 \vee p_2, a), (p_1 \vee p_3, b)\}$. It follows that it is freely generated by this set.

In a similar fashion one can determine the structure of the OML which is freely generated by the poset

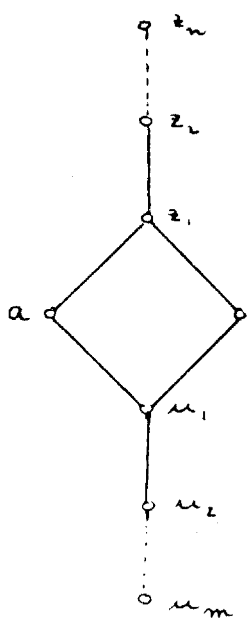
(1)



i.e. by the set $\{x, y, z\}$ with the relations $x \leq z$ and $y \leq z$. If an OML L is generated by a set of this kind and if c is defined as in (1.1) it is easy to see that $[0,c]$ is still tightly generated by the set $\{x \wedge c, y \wedge c\}$ and hence is a homomorphic image of MO2. The Boolean algebra $[0,c']$ is in this case generated by the set $\{x \wedge c', y \wedge c', z \wedge c'\}$ satisfying $x \wedge c', y \wedge c' \leq z\ c'$. From this it follows easily that $[0,c']$ has at most 2^5 elements. We thus obtain that L has at most $2^5 .6 = 192$ elements. Again, if p_1, p_2, p_3, p_4, p_5 are the atoms of 2^5 and if a,**b** have the meaning of (1.2), it is easy to see that the 192-element OML $2^5 \times$ MO2 is generated by the set $\{(p_1 \vee p_2, a), (p_1 \vee p_3, b), (p_5', 1)\}$, the elements of which are in the appropriate position. We thus have:

(1.3) The free OML generated by the poset (1) is isomorphic with $2^5 \times$ MO2.

By a slightly more elaborate argument but using the same method it is easy to determine the OML which is freely generated by the poset



We leave this to the reader.

2. The one-point extension of an OML

The problem of determining the structure of an OML L freely generated by a poset P becomes considerably more difficult if P contains elements x,y,z, where x is incomparable with both y and z. We are far from being able to solve it´s word problem. The aim of the rest of this paper is to show that every such OML contains an infinite chain.

As a first step towards this goal we describe a special extension of an orthocomplemented lattice (abbreviated: OCL) which we hope might have other applications than the one given in this paper. We start out with a definition.

Definition. A quasi-ideal in an OCL L is a subset A of L which satisfies tha following conditions:

1. $0 \in A$,
2. if $a \in A$ and $b \leq a$ then $b \in A$,
3. if $a \in A$ then $a' \notin A$,
4. if $M \subseteq A$, if $\bigvee M$ exists and if $\bigvee M \notin A$ then $(\bigvee M)' \in A$,
5. for every $x \in L$: $\bigvee([0,x] \cap A)$ exists.

Note that condition 5 is alwais fulfilled if all chains in L have bounded lenght, the only case we are dealing with in this paper.

We want to construct an OCL E which contains L as a sub-poset, has the same zero and unit as L, the orthocomplementation of which extends the orthocomplementation of L and which is generated by L and one additional element.

We do not know whether our extension can be described by some universal property.

Let A be a quasi-ideal in an OCL L. Define $A' = \{a' \mid a \in A\}$. Let s, s' be arbitrary elements. In order to make our construction set-theoretically sound we have to make the somewhat technical assumption that the sets L, $A \times \{s'\}$ and $A' \times \{s\}$ are pairwise disjoint. We then define the underlying set of our extension to be

$$E = L \cup (A \times \{s'\}) \cup (A' \times \{s\}).$$

To avoid confusion we denote the partial ordering of L by "$\leq_L$" and the join-operation in L by "$\vee_L$". We now define a relation $\leq$ in E by:

$a \leq b$ iff one of the following conditions holds:

1. $a, b \in L$ and $a \leq_L b$,
2. $a \in L$, $b = (x, s')$ and $a \leq_L x$,
3. $a \in A$, $b = (x, s)$ and $a \leq_L x$,
4. $a = (x, s')$, $b \in A'$ and $x \leq_L b$,
5. $a = (x, s')$, $b = (y, s')$ and $x \leq_L y$,
6. $a = (x, s)$, $b \in L$ and $x \leq_L b$,
7. $a = (x, s)$, $b = (y, s)$ and $x \leq_L y$.

It requires some tedious checking that this is indeed a partial ordering of E. It is obvious that this partial ordering extends the partial ordering of L and that the bounds of L are also the bounds of E. We omit the proof that this partial ordering makes E a lattice. For the convenience of the reader we list explicitely all the joins of elements of E. The meets are obtained dually. In the following, x and y are elements of L and a,b are elements of E. The joins are then given by:

$a \vee b = x \vee_L y$ if $a = x$, $b = y$ and $x \vee_L y \in A$ or
if $a = x$, $b = y$ and ($x \notin A$ or $y \notin A$) or
if $a = x$, $b = (y,s')$ and $x \vee_L y \in A'$ or
if $a = x \in A$ and $b = (y,s)$ or
if $a = (x,s')$, $b = (y,s')$ and $x \vee_L y \notin A$ or
if $a = (x,s')$ and $b = (y,s)$,

$a \vee b = (x \vee_L y, s)$ if $a = x \in A$, $b = y \in A$ and $x \vee_L y \notin A$ or
if $a = x \in A$ and $b = (y,s)$ or
if $a = (x,s)$ and $b = (y,s)$,

$a \vee b = (x \vee_L y, s')$ if $a = x$, $b = (y,s')$ and $x \vee_L y \in A$ or
if $a = (x,s')$, $b = (y,s')$ and $x \vee_L y \in A$,

$a \vee b = \bigwedge([x \vee_L y, 1] \cap A')$ if $a = x$, $b = (y,s')$ and $x \vee_L y \notin A, A'$

It is important to note that the join in L of two elements $x,y \in L$ differs from their join in E iff $x,y \in A$ and $x \vee_L y \notin A$, and dually.

It is now easy to see that the orthocomplementation of L extends to an orthocomplementation of E by the definition:

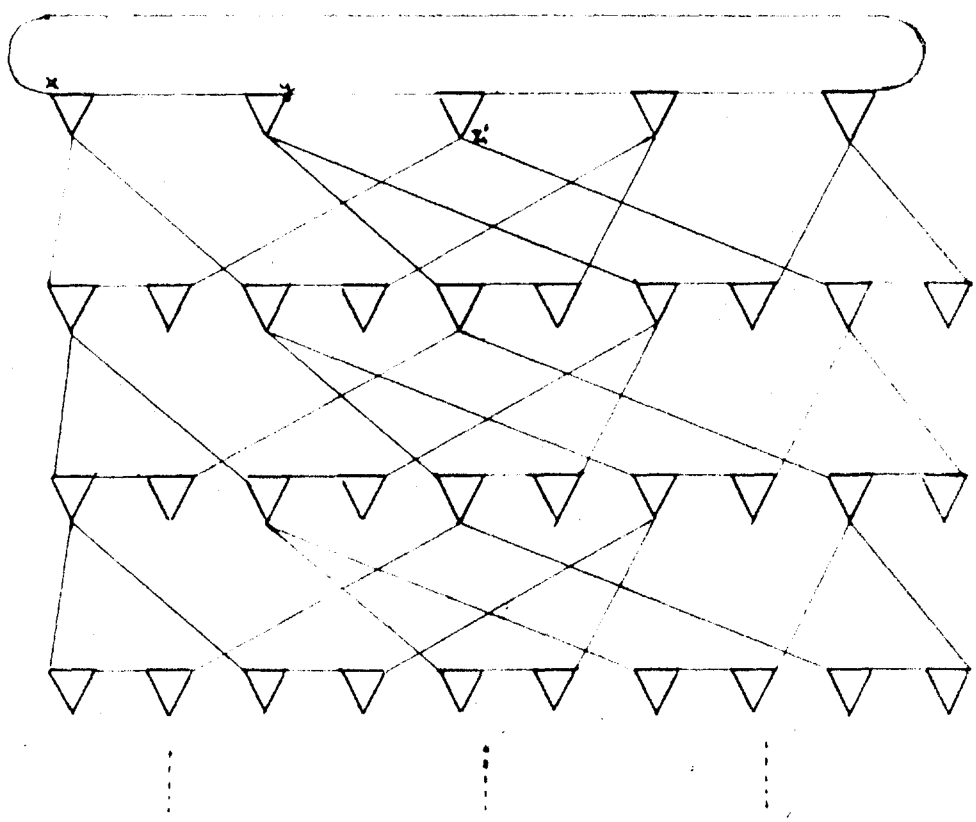
$$(x,s)' = (x',s') \text{ and } (x,s')' = (x',s).$$

Since for every $x \in A$: $x \vee (0,s') = (x,s')$ and dually for every $x \in A'$: $x \wedge (1,s) = (x,s)$ it follows that every element of E is the join of an element of L and the element $(0,s')$ or the complement of such join, in particular, that

E is generated by the set $L \cup \{(1,s)\}$. For the application we have in mind it is finally important to observe that E is an OML if L is an OML. The proof of this is again left to the reader.

3. Existence of infinite chains

In this chapter we sketch a proof of the existence of an OML L which is generated by a three-element poset $P = \{x,y,z\}$ satisfying $y < z$ and which contains an infinite chain. As a first step we construct an infinite OML L generated by such a poset P, in which all maximal chains have four elements. Instead of giving an explicite set-theoretical construction, we modify Greechie´s method [3] for graphical representations of OMLs and simply draw a "graph" of such an OML L. Here it is:



This graph is to be understood in the follwing way. The vertices of each triangle represent the atoms of an eight-element Boolean algebra. The bounds 0,1 of each of these Boolean algebras are "identified" and whenever two vertices of two triangles are connected by a line the atoms represented by the connected vertices are "identified" and so are their complements. Our construction is a special case of "Greechie's paste job" and it follows easily from [3] that our graph if interpreted this way represents indeed an OML. It is finally easy to see that this OML is generated by the elements x,y,z' indicated in the graph and hence also by the elements x,y,z, which are in the appropriate position.

From the graph it is obvious that there exists a countably infinite sequence $b_o, b_1, \ldots b_n, \ldots$ of co-atoms in L which satisfy the following conditions:

(A1) If $0 < a \leq b_i$, $0 < b \leq b_j$ and $i \neq j$ then $a \vee b = 1$,

(A2) if $0 < b \leq b_j$ and $i \neq j$ then $b \vee b_i' = 1$.

From (A2) it follows:

(1) if $i \neq j$ then $b_i' \nleq b_j$,

and from (A1) we obtain:

(2) if $i \neq j$ then $[0,b_i] \cap [0,b_j] = \{0\}$.

Put

$$A_o = [0,b_o] \cup [0,b_1].$$

This is obviously a quasi-ideal. Let

$$L_1 = L \cup (A_0 \times \{s_1'\}) \cup (A_0' \times \{s_1\})$$

be the one-point extension corresponding to it. We now define recursively a sequence $(L_n)_{n<\omega}$ of OMLs and a sequence $(A_n)_{n<\omega}$ where A_n is the quasi-ideal

$$A_n = [0,(1,s_n)]_{L_n} \cup [0,b_{n+1}]_{L_n}$$

in L_n and

$$L_{n+1} = L_n \cup (A_n \times \{s_{n+1}'\}) \cup (A_n' \times \{s_{n+1}\}).$$

It is easy to prove by induction that these sequences have the following properties:

(B1) If $0 <_{L_n} a \leq_{L_n} (1,s_n)$, $0 <_{L_n} b \leq_{L_n} b_j$ and $n+1 \leq j$ then $a \vee_{L_n} b = 1$

(B2) if $0 <_{L_n} b \leq_{L_n} b_j$ and $n+1 \leq j$ then $b \vee_{L_n} (0,s_n') = 1$,

(B3) if $n+1 \leq j$ then $[0,(1,s_n)]_{L_n} \cap [0,b_j]_{L_n} = \{0\}$.

It follows from these properties that for every n, A_n is indeed a quasi-ideal of L_n and that for elements $a,b \in L_i$, $a \vee_{L_n} b \neq a \vee_{L_{n+1}} b$ only holds if $a \vee_{L_n} b = 1$ and dually for meets. This means that every generating set of L is also a generating set of every L_n, in particular that every L_n is generated by P. This then is also true for the direct limit of the family $(L_n)_{n<\omega}$, defined in the obvious fashion. But this direct limit contains the infinite chain $\{(1,s_n) \mid n < \omega\}$, proving that the OML which is freely generated by the poset P contains an infinite chain.

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