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BREADTH TWO MODULAR LATTICES

by

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ABSTRACT: In this paper a characterization of breadth two modular lattices which can be generated by four elements is given. Those which are subdirectly irreducible are listed. An infinite list of coverings in the free modular lattice on four generators is obtained. If V is the variety of lattices generated by all breadth two modular lattices and if L is a lattice freely generated in V by four generators subject to finitely many relations, then the word problem for L is shown to be solvable.

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1. INTRODUCTION

In [3] Day, Hermann, and Wille give a list of subdirectly irreducible modular lattices which can be generated by four elements. Their list consists of projective planes and lattices of breadth two. They ask if their list is complete. In this paper we show that it is complete insofar as it contains all the subdirectly irreducible breadth two four-generated modular lattices. This is done by showing that any four-generated breadth two modular lattice is a homomorphic image of an explicit set of lattices. It is shown that corresponding to all but three of the subdirectly irreducible four-generated breadth two modular lattices there is a covering, $u \succ v$, in the free modular lattice on four generators, $FM(4)$, such that if ψ is the unique maximal congruence on $FM(4)$ separating u from v, then $FM(4)/\psi$ is isomorphic to the subdirectly irreducible breadth two lattice. All of these lattices corresponding to coverings in $FM(4)$ are splitting modular lattices in the sense of McKenzie (definitions given below). Let V be the variety (equational class) of lattices generated by all breadth two modular lattices and let $FL(V,4)$ be the free V-lattice on four generators. Then every nontrivial quotient (interval) of $FL(V,4)$ contains a covering. Finally it is shown that the word problem for four-generated lattices in V is solvable.

Section 2 gives some basic definitions and gives the preliminary reductions. Section 3 gives the main result and Section 4 gives the subdirectly irreducibles. Section 5 presents the coverings and other applications mentioned above.

2. PRELIMINARY REDUCTIONS

Let $a \geq b$ in a lattice L. The sublattice $\{x \in L \mid a \geq x \geq b\}$ is denoted a/b and is called a _quotient_ or _quotient sublattice_ or _interval_. We say that a/b _transposes up_ to c/d and c/d _transposes down_ to a/b if $a \vee d = c$ and $a \wedge d = b$. We denote this by $a/b \nearrow c/d$ and $c/d \searrow a/b$. Two quotients connected by a sequence of transposes are called _projective_. If $a > b$ and there is no x such that $a > x > b$, then we say a covers b, and denote this $a \succ b$.

Recall that a lattice has breadth n if the join of any $n + 1$ elements is redundant and there is an irredundant join of n element.

LEMMA 1: _A modular lattice has breadth_ n _if and only if it has a sublattice isomorphic to the lattice of all subsets of an_ n _element set but no sublattice isomorphic to the lattice of all subsets of an_ $n + 1$ _element set_.

PROOF: If the join of the elements $x_1, \ldots, x_n$ is irredundant, then the elements $\overline{x}_i = x_1 \vee \ldots \vee x_{i-1} \vee x_{i+1} \vee \ldots \vee x_n$, $i = 1, \ldots, n$ generate a sublattice isomorphic to the lattice of subsets of an n element set. The lemma follows easily from this.

LEMMA 2: _Let_ b _cover_ a, $a \prec b$, _in a modular lattice_ L. _Then there exists a unique largest congruence separating_ a

and b, *which is denoted by* $\psi(a,b)$. *Let* $\theta(a,b)$ *denote the smallest congruence of* L *identifying* a *and* b. *Then* $\theta(a,b) \wedge \psi(a,b) = 0$.

PROOF: It follows from Dilworth's basic result on congruences of lattices that $\theta(a,b) \succ 0$ [2], [4], where 0 is the least congruence on L. The lemma follows.

Suppose $u \succ v$ in a free modular lattice F. Let $\psi(u,v)$ be the largest congruence separating u from v and let $K = F/\psi(u,v)$. Now if L is a homomorphic image of F in which the images of u and v are different, then L is a subdirect product of K and a lattice L' which is a homomorphic image of L such that u and v are identified in L'. This is, of course, an immediate corollary to Lemma 2.

For the rest of the paper, L will denote a breadth two modular lattice generated by four distinct generators a, b, c, d and not by any three elements.

LEMMA 3: *Either any three elements of the set* $\{a,b,c,d\}$ *join to the greatest element of* L, $1 = a \vee b \vee c \vee d$, *or* L *is a subdirect product of one or two, two element lattices and a four-generated breadth two modular lattice in which any three of the four generator join to the greatest element.*

PROOF: Suppose the first statement fails, say $b \vee c \vee d < 1$. Since L has breadth two, it follows that $b \vee c \vee d$ is the join of two of the generators, say $c \vee d = b \vee c \vee d$. Now $1 = (a \vee b) \vee c \vee d$ implies that either $c \vee d = 1$ or $a \vee b \vee c = 1$ or $a \vee b \vee d = 1$. The first possibility contradicts $b \vee c \vee d < 1$. Suppose $a \vee b \vee d = 1$, and that $a \vee b \vee c < 1$. Since $c \vee d = b \vee c \vee d$, we have $a \vee c \vee d = 1$.

In the free modular lattice on four generators the join of any three generators is covered by the greatest element. Hence in L, $1 \succ b \vee c \vee d$ and $1 \succ a \vee b \vee c$. Let ψ_1 be the largest congruence on L separating 1 from $b \vee c \vee d$ and ψ_2 the largest congruence separating 1 from $a \vee b \vee c$. Let $\theta_1 = \theta(1, b \vee c \vee d)$ and $\theta_2 = \theta(1, a \vee b \vee c)$ and $\theta = \theta_1 \vee \theta_2$. Since the congruences of lattices distribute, Lemma 2 implies $\theta \wedge \psi_1 \wedge \psi_2 = 0$. Hence L is a subdirect product of L/θ, L/ψ_1 and L/ψ_2. Now $L/\psi_1 \cong L/\psi_2 \cong \underset{\sim}{2}$ the two element lattice. Furthermore, L/θ has the property that any three of its generators join to the greatest element.

If $a \vee b \vee c = 1$, then $\theta_2 = 0$ and $\theta = \theta_1$. In this case L is a subdirect product of L/θ and $L/\psi_1 \cong \underset{\sim}{2}$. As before, L/θ has the desired properties. The remaining cases are handled by symmetry.

Now we impose the additional condition that any three of the four generators of L join to 1 and meet to 0. Since L has breadth two this implies that any three element subset of $\{a,b,c,d\}$ has a two element subset whose elements join to the top. We shall show that all but at most two of the two element subsets of $\{a,b,c,d\}$ join to 1. First we need a lemma.

LEMMA 4: _Let x and y be noncomparable elements in a breadth two modular lattice. Then $x \vee y/x$ and $x \vee y/y$ are both chains._

PROOF: Suppose $x \leq u$, $v \leq x \vee y$ are noncomparable elements. Then it is not hard to check that the elements u, v, $y \wedge (u \vee v)$ generate a lattice isomorphic to the lattice of subsets of a three element set. Now the lemma follows from Lemma 1.

As remarked above, there is a two element subset of $\{a,b,c\}$ joining to 1; say $a \vee b = 1$. Also, there is a two element subset of $\{a,c,d\}$ joining to 1. If $c \vee d = 1$, then we have two complementary pairs, both of which join to 1. Suppose $a \vee c = 1$. Now consider $\{b,c,d\}$. If either $b \vee d = 1$ or $c \vee d = 1$, then there exists two complementary pairs, both joining to 1. If $b \vee c = 1$, then we have that all pairs not containing d join to 1. In conclusion, either there are two complementary pairs of generators both joining to 1, or there is a generator such that all pairs of generators not including that generator join to 1.

Suppose $a \vee b = 1 = c \vee d$. If a and b were comparable, then one of them would equal 1, contradicting our assumption that L is not generated by three elements. Hence by Lemma 4 $1/a$ is a chain and thus $a \vee c$ and $a \vee d$ must be comparable. By symmetry we may assume $a \vee c \geq a \vee d$. Then $a \vee c = a \vee c \vee d = 1$. Now $1/b$ and $1/d$ are chains by Lemma 4; hence, as above, either $a \vee d = 1$ or $b \vee d = 1$ and either $b \vee c = 1$ or $b \vee d = 1$. Thus either $b \vee d = 1$ or both $b \vee c = 1$ and $a \vee d = 1$. We conclude that if there are two complementary pairs of generators, each pair joining to 1, then at least five of the six pairs of generators join to 1, or four of the six join to 1 and the two pairs that do not join to 1 are complementary.

Let M_5 be the five element length two lattice.

LEMMA 5: *Let L be a breadth two modular lattice generated by a,b,c,d, in which any three of the generators join to 1. Then one of the following must hold.*

(i) *L has the property that at least four of the six pairs of generators join to 1, and if two pairs do not join to 1, they are complementary,*

(ii) *L is a subdirect product of M_5 and a lattice having the property described in (i),*

(iii) *L is a subdirect product of M_5 and a three generated modular lattice.*

PROOF: By symmetry and the remarks preceding Lemma 5 we may assume that $a \vee b = a \vee c = b \vee c = 1$. In order to apply

Lemma 2 we must find elements $v \prec u$ in the free modular lattice on four generators, FM(4), such that if ψ is the maximum congruence separating v from u, then $FM(4)/\psi \cong M_5$. This can easily be done in $FM(x,y,z)$, since it is finite. For example, $x \vee (y \wedge z) \prec x \vee (z \wedge (x \vee y))$ will do.

Let $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ be the generators of FM(4). Then $\mathbf{a} \to x$, $\mathbf{b} \to y$, $\mathbf{c} \to z$, and $\mathbf{d} \to x \wedge y \wedge z$ can be extended to a homomorphism f from FM(4) onto FM(3). It is not difficult to see that if $f(w) = x \vee (y \wedge z)$ then $w \leq \mathbf{a} \vee [(\mathbf{b} \vee \mathbf{d}) \wedge (\mathbf{c} \vee \mathbf{d})]$ and if $f(w) = x \vee (z \wedge (x \vee y))$ then $w \geq \mathbf{a} \vee (\mathbf{c} \wedge (\mathbf{a} \vee \mathbf{b}))$. It follows that in FM(4) $\mathbf{a} \vee [(\mathbf{b} \vee \mathbf{d}) \wedge (\mathbf{c} \vee \mathbf{d})] \prec [\mathbf{a} \vee ((\mathbf{b} \vee \mathbf{d}) \wedge (\mathbf{c} \vee \mathbf{d}))] \vee \mathbf{a} \vee (\mathbf{c} \wedge (\mathbf{a} \vee \mathbf{b})) = \mathbf{a} \vee ((\mathbf{b} \vee \mathbf{d}) \wedge (\mathbf{c} \vee \mathbf{d})) \vee (\mathbf{c} \wedge (\mathbf{a} \vee \mathbf{b}))$ and if ψ is the largest congruence separating these elements then $FM(4)/\psi \cong M_5$.

Hence in L we have

$$a \vee ((b \vee d) \wedge (c \vee d)) \preceq a \vee ((b \vee d) \wedge (c \vee d)) \vee (c \wedge (a \vee b)).$$

Now if we have equality in the above inequality, then

$$a \vee (c \wedge (a \vee b)) \leq a \vee ((b \vee d) \wedge (c \vee d))$$

or

$$(a \vee c) \wedge (a \vee b) \leq a \vee ((b \vee d) \wedge (c \vee d)).$$

Since $a \vee c = a \vee b = 1$ in L the left hand side of this inequality is 1 and hence the right hand side is also. By Lemma 4 either a and $(b \vee d) \wedge (c \vee d)$ are comparable or $b \vee d$ and $c \vee d$ are comparable. If $a \geq (b \vee d) \wedge (c \vee d)$, then $a = a \vee ((b \vee d) \wedge (c \vee d)) = 1$. In this case, L is generated by b, c, and d contrary to our assumption on L. If $(b \vee d) \wedge (c \vee d) \geq a$, then $(b \vee d) \wedge (c \vee d) = 1$ and in this case the conclusion of the lemma holds.

If $b \vee d \geq c \vee d$, then $b \vee d = b \vee c \vee d = 1$. By Lemma 4, $a \vee d$ and $c \vee d$ are comparable, and as above the larger one must be 1. Thus again the conclusion of the lemma holds.

Now we consider the case

$$\begin{gathered} a \vee ((b \vee d) \wedge (c \vee d)) \\ \prec a \vee ((b \vee d) \wedge (c \vee d) \vee (c \wedge (a \vee b)). \end{gathered}$$

Let θ be the smallest congruence on L identifying these elements and ψ_0 be the unique largest congruence separating these elements. By Lemma 2, L is a subdirect product of L/θ and $L/\psi_0 \cong M_5$. Now arguments just as above show that the conclusions of the lemma hold.

3. MAIN THEOREM

By Lemma 5 we may assume that $a \vee c = a \vee d = b \vee c = b \vee d = 1$. By the dual of Lemma 5 we may assume four of the six pairs of generators meet to 0. We first consider the case $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$. Notice that this situation has a large amount of symmetry. If a relation holds in L, then the relations obtained from it under the permutations (ab), (cd), $(ab)(cd)$, $(ac)(bd)$, $(ad)(bc)$ also hold in L. The case when L can be generated by three elements is of course easy. For now we assume that L cannot be generated by any three element. This implies that no two generators can be comparable. If $a \leq c$, for example, then $a \vee c = 1$ implies $c = 1$ contradicting the hypothesis that L is not generated by three elements. If $a \leq b$, then since c is a complement of both a and b, modularity implies $a = b$, again contradicting our assumption. The other cases are handled by symmetry.

Let $a_0 = a^0 = a$, $b_0 = b^0 = b$, $c_0 = c^0 = c$, and $d_0 = d^0 = d$. Define inductively $a_i = a \wedge (c_{i-1} \vee d_{i-1})$, $b_i = b \wedge (c_{i-1} \vee d_{i-1})$, $c_i = c \wedge (a_{i-1} \vee b_{i-1})$, $d_i = d \wedge (a_{i-1} \vee b_{i-1})$ and dually $a^i = a \vee (c^{i-1} \wedge d^{i-1})$, $b^i = b \vee (c^{i-1} \wedge d^{i-1})$, $c^i = c \vee (a^{i-1} \wedge b^{i-1})$, $d^i = d \vee (a^{i-1} \wedge b^{i-1})$. We now derive some formulae concerning these elements

(1) $a_0 = a \geq a_1 \geq a_2 \geq \ldots \quad a^0 = a \leq a^1 \leq a^2 \leq \ldots$ etc.

(2) $a_i = a_{i-1} \wedge (c_{i-1} \vee d_{i-1})$, $a^i = a^{i-1} \vee (c^{i-1} \wedge d^{i-1})$

(3) $a \wedge d^i = a \wedge c^i = a \wedge b^{i-1} \qquad i \geq 1$

(4) $a \vee d_i = a \vee c_i = a \vee b_{i-1} \qquad i \geq 1$

(5) $a_i \vee d_i = a_i \vee c_i = b_i \vee c_i = b_i \vee d_i =$
$(a_{i-1} \vee b_{i-1}) \wedge (c_{i-1} \vee d_{i-1})$.

For example, (4) can be proved with the aid of (2) and induction:

$$\begin{aligned}
a \vee d_i &= a \vee (d_{i-1} \wedge (a_{i-1} \vee b_{i-1})) \\
&= a \vee (d_{i-1} \wedge [(a \wedge (c_{i-2} \vee d_{i-2})) \vee (b \wedge (c_{i-2} \vee d_{i-2}))]) \\
&= a \vee [d_{i-1} \wedge (c_{i-2} \vee d_{i-2}) \wedge (a \vee (b \wedge (c_{i-2} \vee d_{i-2})))] \\
&= a \vee [d_{i-1} \wedge (a \vee (b \wedge (c_{i-2} \vee d_{i-2})))] \\
&= (a \vee d_{i-1}) \wedge [a \vee (b \wedge (c_{i-2} \vee d_{i-2}))] \\
&= (a \vee b_{i-2}) \wedge (a \vee b_{i-1}) \\
&= a \vee b_{i-1}.
\end{aligned}$$

Note that $a_o = a \geq a_1 \geq a_2 \geq \ldots$ is a descending chain in $a/0$ and $0 = a \wedge d \leq a \wedge d^1 \leq a \wedge d^2 \leq \ldots$ is an ascending chain in $a/0$. By Lemma 4, $a/0$ is a chain, and thus each $a \wedge d^j$ must be comparable with each a_i. Let n be the smallest integer such that $a \wedge b \geq a_{n+1}$, if such an integer exists. Joining both sides of $a_{n+1} \leq a \wedge b$ with c_n we obtain

$$(a \wedge b) \vee c_n \geq [a_n \wedge (c_n \vee d_n)] \vee c_n = (a_n \vee c_n) \wedge (c_n \vee d_n).$$

However, (5) tells us $a_n \vee c_n = (a_{n-1} \vee b_{n-1}) \wedge (c_{n-1} \vee d_{n-1})$.

Thus

$$(a \wedge b) \vee c_n \geq (a_{n-1} \vee b_{n-1}) \wedge (c_n \vee d_n)$$

Hence

$$a \wedge b = (a \wedge b) \vee (c_n \wedge b) = [(a \wedge b) \vee c_n] \wedge b \geq$$
$$b \wedge (c_n \vee d_n) \wedge (a_{n-1} \vee b_{n-1}) = b_{n+1} \wedge (a_{n-1} \vee b_{n-1}) = b_{n+1}.$$

Thus $a \wedge b \geq b_{n+1}$. It follows that n is the smallest integer such that $a \wedge b \geq b_{n+1}$. Now observe

$$a_{n+1} = a \wedge (c_n \vee d_n) \leq a \wedge b \wedge (c_n \vee d_n) = a_{n+1} \wedge b_{n+1}$$

Hence $a_{n+1} = b_{n+1}$. Thus

$$c_{n+2} = c \wedge (a_{n+1} \vee b_{n+1}) = c \wedge a_{n+1} = 0.$$

LEMMA 6: *Let* L *be a breadth two modular lattice generated by four noncomparable generators* a,b,c,d *satisfying* $a \vee c = a \vee d = b \vee c = b \vee d = 1$ *and* $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$. *If* $a_n > a \wedge b \geq a_{n+1}$, *then* $b_n > a \wedge b \geq b_{n+1}$ *and* $a_{n+3} = b_{n+3} = c_{n+2} = d_{n+2} = 0$. *Furthermore*, $c_m > c \wedge d \geq c_{m+1}$ *and* $d_m > c \wedge d \geq d_{m+1}$ *where* m *is either* $n - 1$, n, *or* $n + 1$.

PROOF: If $a_n > a \wedge b \geq a_{n+1}$ then $c_{n+2} = 0$, as shown above. Thus $m \leq n + 1$. Similarly, $n \leq m + 1$. The rest of the lemma follows easily from the remarks above.

We shall require a few additional observations.

(6) $(a \wedge b^i) \vee d = (a \wedge d^{i+1}) \vee d = d^{i+1}$.

If $a_{i+1} \geq a \wedge b$ then

(7) $a_i/a_{i+1} \nearrow a_i \vee b_i/a_{i+1} \vee b_i \searrow d_{i+1}/d_{i+2}$.

If $d_{i+1} \geq c \wedge d$ then

(8) $d_i/d_{i+1} \nearrow d_i \vee c_i/d_{i+1} \vee c_i \quad a_{i+1}/a_{i+2}$.

(6) easily follows from (3). To see (7), note that since $a_i \geq a \wedge b$, $b_i \geq a \wedge b$ by Lemma 6. From this it follows that $a_i/a_{i+1} \nearrow a_i \vee b_i/a_{i+1} \vee b_i$. Repeatedly using (4) with the poles of a and d interchanged we obtain

$$\begin{aligned} d_{i+1} \vee a_{i+1} \vee b_i &= [d \wedge (a_i \vee b_i)] \vee a_{i+1} \vee b_i \\ &= (d \vee a_{i+1} \vee b_i) \wedge (a_i \vee b_i) \\ &= (d \vee c_{i-1}) \wedge (a_i \vee b_i) \\ &= (d \vee a_i \vee b_i) \wedge (a_i \vee b_i) \\ &= a_i \vee b_i \end{aligned}$$

and

$$\begin{aligned} d_{i+1} \wedge (a_{i+1} \vee b_i) &= d \wedge (a_i \vee b_i) \wedge (a_{i+1} \vee b_i) \\ &= d \wedge (a_{i+1} \vee b_i) \\ &= d \wedge [(a \wedge (c_i \vee d_i)) \vee (b \wedge (c_{i-1} \vee d_{i-1}))] \end{aligned}$$

$$d_{i+1} \wedge (a_{i+1} \vee b_i) = d \wedge (c_{i-1} \vee d_{i-1}) \wedge (b \vee a_{i+1})$$
$$= [(d \wedge c_{i-1}) \vee d_{i-1}] \wedge (b \vee d_{i+2})$$
$$= d_{i+2} \vee [b \wedge (d_{i-1} \vee (d \wedge c_{i-1}))]$$
$$= d_{i+2}$$

LEMMA 7: *Let* L *satisfy the hypotheses of Lemma* 6. *Suppose, also that*

$$(9) \quad a_n > a \wedge b \geq a_{n+1} \ \text{and} \ d_{n+1} > c \wedge d \geq d_{n+2}.$$

Then

$$(10) \quad a_i \geq a \wedge b^{n-i+1} \geq a \wedge b^{n-i} \geq a_{i+1}$$
$$i \equiv n(\text{mod } 2), \quad i \leq n$$
$$(11) \quad d_k \geq d \wedge c^{n-k+1} \geq d \wedge c^{n-k} \geq d_{k+1}$$
$$k \equiv n + 1(\text{mod } 2), \ k \leq n.$$

Furthermore, the images of $a \wedge b^{n-i+1}$ *and* $a \wedge b^{n-i}$ *under the projectivity* (7) *are* $d \wedge c^{n-i}$ *and* $d \wedge c^{n-i-1}$. *The images of* $d \wedge c^{n-k+1}$ *and* $d \wedge c^{n-k}$ *under the projectivity* (8) *are* $a \wedge b^{n-k}$ *and* $a \wedge b^{n-k-1}$.

PROOF: First we show that

$$(12) \quad a_i \geq a \wedge b^{n-i+1} \geq a \wedge b^{n-i}$$
$$i \equiv n(\text{mod } 2), \ i \leq n$$

and

$$(13) \quad d_k \geq d \wedge c^{n-k+1} \geq d \wedge c^{n-k}$$
$$k \equiv n + 1(\text{mod } 2), \ k \leq n.$$

We prove these inequalities by induction on $n - i$ and $n - k$. First note that the second inequality in both (12) and (13) follows immediately from the monotome nature of the b^j's and c^j's. Now we show that $a_n \geq a \wedge b^1$. Using (3) we have that

$$\begin{aligned} a \wedge [(c \wedge d) \vee (b \wedge a^1)] &= a \wedge [(d \wedge a^1) \vee (b \wedge a^1)] \\ &= a \wedge a^1 \wedge (d \vee (b \wedge a^1)) \\ &= a \wedge (d \vee (b \wedge a^1)) \\ &= a \wedge (d \vee (b \wedge d^2)) \\ &= a \wedge (d \vee b) \wedge d^2 \\ &= a \wedge d^2 \\ &= a \wedge b^1 \end{aligned}$$

Now, since $a^1 = a \vee (c \wedge d) \leq a \vee d_{n+1}$, we have

$$\begin{aligned} a \wedge b^1 &= a \wedge [(c \wedge d) \vee (b \wedge a^1)] \\ &\leq a \wedge [d_{n+1} \vee (b \wedge (a \vee d_{n+1}))] \\ &= a \wedge [d_{n+1} \vee (a \wedge (b \vee d_{n+1}))] \\ &= (a \wedge d_{n+1}) \vee [a \wedge (b \vee d_{n+1})] \\ &= a \wedge (b \vee d_{n+1}) \\ &= a \wedge (b \vee a_n) \\ &= (a \wedge b) \vee a_n \\ &= a_n \end{aligned}$$

Thus, $a \wedge b^1 \leq a_n$.

Now suppose we have shown that $a_i \geq a \wedge b^{n-i+1}$. We shall show that $d_{i-1} \geq d \wedge c^{n-i+2}$. Observe that

$$d \wedge [(a \wedge b^{n-i+1}) \vee (c \wedge d^{n-i+2})] =$$

$$\begin{aligned}
&= d \wedge [(a \wedge d^{n-i+2}) \vee (c \wedge d^{n-i+2})] \\
&= d \wedge d^{n-i+2} \wedge [a \vee (c \vee d^{n-i+2})] \\
&= d \wedge [a \vee (c \wedge a^{n-i+3})] \\
&= d \wedge (a \vee c) \wedge a^{n-i+3} \\
&= d \wedge a^{n-i+3} \\
&= d \wedge c^{n-i+2}.
\end{aligned}$$

Hence, since $d^{n-i+2} = d \vee (a \wedge b^{n-i+1}) \leq d \vee a_i$,

$$\begin{aligned}
d \wedge c^{n-i+2} &= d \wedge [(a \wedge b^{n-i+1}) \vee (c \wedge d^{n-i+2})] \\
&\leq d \wedge [a_i \vee (c \wedge (d \vee a_i))] \\
&= d \wedge [a_i \vee (d \wedge (c \vee a_i))] \\
&= (d \wedge a_i) \vee [d \wedge (c \vee a_i)] \\
&= d \wedge (c \vee a_i) \\
&= d \wedge (c \vee d_{i-1}) \\
&= (c \wedge d) \vee d_{i-1} \\
&= d_{i-1}.
\end{aligned}$$

Thus $d \wedge c^{n-i+2} \leq d_{i-1}$.

Thus if j is either $n - i + 1$ or $n - i$ then $a_i \geq a \wedge b^j$ and $d_{i+1} \geq d \wedge c^{j-1}$. By way of induction suppose that $d_{i+1} \geq d \wedge c^{j-1} \geq d_{i+2}$ for j as above. Then the image of $a \wedge b^j$ under the projectivity (7) is

$$\begin{aligned}
d_{i+1} \wedge [(a \wedge b^j) \vee a_{i+1} \vee b_i] &= d_{i+1} \wedge [(a \wedge (b^j \vee a_{i+1})) \vee b_i] \\
&= d_{i+1} \wedge (a \vee b_i) \wedge (a_{i+1} \vee b^j)
\end{aligned}$$

$$\begin{aligned} d_{i+1} \wedge [(a \wedge b^j) \vee a_{i+1} \vee b_i] &= d_{i+1} \wedge (a_{i+1} \vee b^j) \\ &= d_{i+1} \wedge (a_{i+1} \vee b \vee b^j) \\ &= d_{i+1} \wedge (d_{i+2} \vee b^j) \\ &= d_{i+2} \vee (d_{i+1} \wedge d \wedge b^j) \\ &= d_{i+2} \vee (d_{i+1} \wedge d \wedge c^{j-1}) \\ &= d_{i+2} \vee (d \wedge c^{j-1}) \\ &= d \wedge c^{j-1} \end{aligned}$$

This shows that $a_i \geq a \wedge b^j \geq a_{i+1}$, which completes the proof of the lemma.

Arguments similar to these prove the following lemma.

LEMMA 8: *Let* L *satisfy the hypotheses of Lemma* 6. *Suppose also that*

(14) $a_n > a \wedge b \geq a_{n+1}$ *and* $d_n > c \wedge d \geq d_{n+1}$

Then

(15) $a_i \geq a \wedge b^{n-i} \geq a_{i+1}$

(16) $d_i \geq d \wedge c^{n-i} \geq d_{i+1}$

Furthermore, the image of $a \wedge b^{n-i}$ *under the projectivity* (7) *is* $d \wedge c^{n-i-1}$. *The image of* $d \wedge c^{n-i}$ *under the projectivity* (8) *is* $a \wedge b^{n-i-1}$.

Let L_n be the modular lattice freely generated by a,b,c,d subject to the relations $a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$, $a_n \geq a \wedge b \geq a_{n+1}$, and $d_n \geq c \wedge d \geq d_{n+1}$. By the above lemma

(17) $a \geq a \wedge b^n \geq a_1 \geq a \wedge b^{n-1} \geq \ldots$

$\geq a_n \geq a \wedge b \geq a_{n+1} \geq a_{n+2} = 0$

(18) $d \geq d \wedge c^n \geq d_1 \geq d \wedge c^{n-1} \geq \ldots$

$\geq d_n \geq c \wedge d \geq d_{n+1} \geq d_{n+2} = 0.$

For notational convenience define

(19) $e_0 = 0, \quad a_i = e_{2n+3-2i} \qquad 0 \leq i \leq n+1$

$a \wedge b^j = e_{2j+2} \qquad 0 \leq j \leq n.$

Then the chain (17) becomes

(20) $e_{2n+3} \geq e_{2n+2} \geq \ldots \geq e_1 \geq e_0 = 0.$

Similarly, using (18) we define h_i, $i = 0, \ldots, 2n+3$. Moreover, we define f_i to be the element obtained from e_i by interchanging a and b, and g_i to be the element obtained from h_i by interchanging c and d. Let U be the following subset of L_n:

$$U = \{e_i \vee f_j \mid 2 \leq i,j \leq 2n+3\} \cup \{g_i \vee h_j \mid 2 \leq i,j \leq 2n+3\}$$
$$\cup \{e_i \vee h_j \mid 0 \leq i,j \leq 2n+3 \text{ and } |i-j| \leq 2\}$$

We shall show that U is closed under joins and meets and hence $U = L_n$. In addition, we shall evaluate all joins and meets of elements of U thereby describing the lattice L_n. First we require a lemma.

LEMMA 9: *The following formulae hold in* L_n.

(21) $a_i \vee b_j = a_i \vee c_{j+1} = a_i \vee d_{j+1} \qquad i \leq j$

(22) $(a \wedge b^i) \vee (b \wedge a^j) = (a \wedge b^i) \vee (c \wedge d^{j-1})$

$= (a \wedge b^i) \vee (d \wedge c^{j-1})$

$n \geq i \geq j$

$$(23) \quad a_i \vee (b \wedge a^j) = a_i \vee (c \wedge d^{j-1}) = a_i \vee (d \wedge c^{j-1})$$

$$i \le n - j, \quad j \le n$$

$$(24) \quad (a \wedge b^i) \vee b_j = (a \wedge b^i) \vee c_{j+1} = (a \wedge b^i) \vee d_{j+1}$$

$$i \ge n + 1 - j$$

The second equality in (21) also holds for $i \le j + 2$ and the second equality in (23) also holds for $i \le n + 1 - j$, $j \le n + 1$.

PROOF: We prove (21) using (4) and induction on i. Thus assume (21) holds when the subscript of a is less than i and assume also that the corresponding formula obtained by interchanging a and d, and b and c holds when the subscript of d is less than i.

$$\begin{aligned} a_i \wedge b_j &= [a \wedge (c_{i-1} \vee d_{i-1})] \vee b_j \\ &= [a \wedge (b_i \vee d_{i-1})] \vee b_j \\ &= (a \vee b_j) \wedge (b_i \vee d_{i-1}) \\ &= (a \vee d_{j+1}) \wedge (a_i \vee d_{i-1}) \\ &= a_i \vee [d_{i-1} \wedge (a \vee d_{j+1})] \\ &= a_i \vee d_{j+1} \end{aligned}$$

To prove (22) note that since $i,j \le n$ we have $(a \wedge b^i) \vee (b \wedge a^j) = a^j \wedge b^i$. Since $b^i \ge b^j \ge d \wedge b^j = d \wedge a^j$,

$$(a \wedge b^i) \vee (d \wedge c^{j-1}) = (a \wedge b^i) \vee (d \wedge a^j)$$
$$= b^i \wedge [a \vee (d \wedge a^j)]$$
$$= b^i \wedge a^j.$$

To prove (23) note that $i \leq n - j$ and $j \leq n$ imply that $b_i \geq b \wedge a^j$ and $a^j \leq a \vee b$. Thus

$$a_i \vee (b \wedge a^j) = [a \wedge (c_{i-1} \vee d_{i-1})] \vee (b \wedge a^j)$$
$$= [a \wedge (b_i \vee d_{i-1})] \vee (b \wedge a^j)$$
$$= (b_i \vee d_{i-1}) \wedge [a \vee (b \wedge a^j)]$$
$$= (b_i \vee d_{i-1}) \wedge a^j$$
$$= (c_{i-1} \vee d_{i-1}) \wedge [a \vee (d \wedge a^j)]$$
$$= (c_{i-1} \vee d_{i-1}) \wedge [a \vee (d \wedge c^{j-1})]$$
$$= a_i \vee (d \wedge c^{j-1})$$

Since $i > n + 1 - j$, $b^i \geq d \wedge b^{n+1-j} = d \wedge c^{n-j} \geq d_{j+1}$. Thus

$$(a \wedge b^i) \vee b_j = (a \vee b_j) \wedge b^i$$
$$= (a \vee d_{j+1}) \wedge b^i$$
$$= (a \wedge b^i) \vee d_{j+1}.$$

The proof of the last statement of the lemma is similar to above proofs.

The previous lemma can be put into a more compact form.

COROLLARY: <u>The following holds in</u> L_n.

(25) $e_i \vee f_j = e_i \vee g_{j-2} = e_i \vee h_{j-2} \qquad i \geq j$

The joins in U are given by the following.

(26) $(e_i \vee f_j) \vee (e_k \vee f_\ell) = e_p \vee f_g$

$p = \max\{i,k\}, \quad q = \max\{j,\ell\}$

$(g_i \vee h_j) \vee (g_k \vee h_\ell) = g_p \vee h_q$

$(e_i \vee h_j) \vee (e_k \vee h_\ell) = e_p \vee h_q$

If $i \geq j$ and $\ell \geq k$ and $2 \leq i,j,k,\ell \leq 2n + 3$ and $r = \max\{\ell+2,j\}$, $s = \max\{i+2,k\}$ then

$$(27) \quad (e_i \vee f_j) \vee (g_k \vee h_\ell) = \begin{cases} e_i \vee h_\ell & \text{if } |i-\ell| \leq 2 \\ e_i \vee f_r & \text{if } i \geq \ell + 2 \\ g_s \vee h_\ell & \text{if } \ell \geq i + 2 \end{cases}$$

If $j \geq i$ and $\ell \geq k$ then $(e_i \vee f_j) \vee (g_k \vee h_\ell)$ is as above except the roles of e and f are interchanged. The cases $j \geq i$ and $k \geq \ell$, and $i \geq j$ and $k \geq \ell$ are handled similarly.

If $i \geq j$ then

$$(28) \quad e_i \vee f_j \vee e_k \vee h_\ell = \begin{cases} e_p \vee h_{q'} & \text{if } |p-q'| \leq 2 \\ e_p \vee f_{q'+2} & \text{if } p \geq q' + 2 \end{cases}$$

where $p = \max\{i,k\}$ and $q' = \max\{j-2,\ell\}$. All other joins in U are similar.

The meet operation is given by

(29) $(e_i \vee f_i) \wedge (e_k \vee f_\ell) = e_r \vee f_s$

$r = \min\{i,k\}, \quad s = \min\{i,\ell\}$

$(g_i \vee h_j) \wedge (g_k \vee h_\ell) = g_r \vee h_s$

$(e_i \vee h_j) \wedge (e_k \vee h_\ell) = e_r \vee h_s$

and

(30) $(e_i \vee f_j) \wedge (g_k \vee h_\ell) = (e_p \vee f_q) \vee (g_r \vee h_s)$

where $p = \min\{i,k-2,\ell-2\}$, $q = \min\{j,k-2,\ell-2\}$

$r = \min\{k,i-2,j-2\}$, and $s = \{\ell,i-2,j-2\}$.

If $|k-\ell| \leq 2$ then

(31) $(e_i \vee f_j) \wedge (e_k \vee h_\ell) = (e_{p'} \vee f_{q'}) \vee (g_{r'} \vee h_{s'})$

where $p' = \min\{i,k\}$, $q' = \min\{j,k\}$, $r' = s' = \min\{\ell,i-2,j-2\}$.

THEOREM 1: <u>The set</u> U <u>together with the join and meet given in</u> (26) - (31) <u>is the lattice</u> L_n.

<u>PROOF</u>: (26) follows from modularity. The other equations follow easily from the Corollary.

(FIGURE 1)

The lattices L_o, L_1, L_2 are diagrammed in Figure 1. If we let L_n' be the modular lattice generated by a,b,c,d with $a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$, $a_n \geq a \wedge b \geq a_{n+1}$, and $d_{n+1} \geq c \wedge d \geq d_{n+2}$ then an analysis similar to that of L_n can be carried out. The lattices L_o', L_1', L_2' are diagrammed in Figure 2.

(FIGURE 2)

Now let L_∞ be the modular lattice generated by a,b,c,d with $a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$, $a_i \geq a \wedge b$, $i = 0, 1, 2, \ldots.$

It follows that $a_i \geq a \wedge b^j$ and $d_i \geq d \wedge c^j$ for all $i,j \geq 0$.

It remains to consider the case when L is generated by a,b,c,d with $a \vee c = a \vee d = b \vee c = b \vee d = 1$ and all pairs of generators meeting to 0 except for two complementary pairs. By symmetry we may assume $a \wedge b = a \wedge c = b \wedge d = c \wedge d = 0$. Call this lattice L'_∞. We define a_i, b_i, c_i, d_i as before. However we now define $a^i = a \vee (b^{i-1} \wedge c^{i-1})$, $b^i = b \vee (a^{i-1} \wedge d^{i-1})$, $c^i = c \vee (a^{i-1} \wedge d^{i-1})$, $d^i = d \vee (b^{i-1} \wedge c^{i-1})$. We shall show that for all i and j

(32) $a_i \geq a \wedge d^j, \quad c_i \geq c \wedge b^j$

We need two equations. The proofs of these are left to the reader.

(33) $a_i = a \wedge (d \vee c_{i-1})$

(34) $c \wedge b^i = c \wedge d^{i+1}$

To prove (32) it is sufficient to prove that $a_i \geq a \wedge d^i$ and $c_i \geq c \wedge b^i$ for all i. This is obvious for $i = 0$. Assume the equations hold for $i = 1, \ldots, n$. Then

$$\begin{aligned} a_{n+1} &= a \wedge (d \vee c_n) \\ &\geq a \wedge (d \vee (c \wedge b^n)) \\ &= a \wedge (d \vee (c \wedge d^{n+1})) \\ &= a \wedge d^{n+1}. \end{aligned}$$

The last step uses that fact that $d^{n+1} \leq d \vee c$, which is easily proved by induction. Hence the following chain of elements lies below a.

$$a \geq a_1 \geq a_2 \geq \ldots \geq a \wedge d^2 \geq a \wedge d^1 \geq a \wedge d \geq 0$$

With this information an analysis similar to that for L_n can be carried out.

Combining the above information we obtain the following theorem.

THEOREM 2: <u>If L is a breadth two four-generated modular lattice then L is a homomorphic image of a subdirect product of four copies of $\underset{\sim}{2}$, two copies of M_5 and either a three-generated modular lattice or L_n or L_n' for some</u> n, $0 \leq n \leq \infty$.

Not all four-generated subdirect products of L_n or L_n' with four copies of $\underset{\sim}{2}$ and two copies of M_5 are breadth two. However, it is possible to make a list of lattices such that L is a breadth two four-generated modular lattice if and only if L is a homomorphic image of a lattice from this list. This shall not be done here. In Figure 3 we give an example of a breadth two four-generated modular lattice which is maximal in the sense that it is not a homomorphic image of a properly larger breadth two, four-generated modular lattice.

(FIGURE 3)

4. SUBDIRECTLY IRREDUCIBLES

The utility of Theorem 2 is that the lattices in that theorem have only finitely many homomorphic images. With the aid of this fact we shall now characterize all subdirectly irreducible, four-generated breadth two modular lattices by actually listing them. Let L be such a lattice. Then it follows from Theorem 2 and the distributivity of congruence lattices of lattices that L is either $\underset{\sim}{2}$, M_5, or a homomorphic image of L_n or L_n' for some n, $0 \leq n \leq \infty$. The following lemma shows each L_n and L_n', $1 \leq n < \infty$ is the subdirect product of four subdirectly irreducible lattices.

LEMMA 10: <u>If</u> u/v <u>is a prime quotient in</u> L_n <u>or</u> L_n', $1 \leq n < \infty$, <u>then</u> u/v <u>is projective to a subquotient of</u> a/a_2.

<u>PROOF</u>: Since L_n and L_n' are finite dimensional lattices every prime quotient is projective with a subquotient of either $a/0$ or of $1/a$. Hence it suffices to show that every prime quotient of $a/0$ and of $1/a$ is projective to a subquotient of a/a_2. Suppose u/v is a subquotient for a_i/a_{i+1} with $i \leq n$. By (7) and (8) a_i/a_{i+1} is projective to a_{i-2k}/a_{i-2k+1}, $i = 0, 1, \ldots, \frac{[i]}{2}$. Hence the lemma holds in this case. If u/v is a subquotient of $a/0$ but not of a_i/a_{i+1} for all $i \leq n$ then $u = a_{n+1}$ and $v = 0$. In this case, since $n \geq 1$,

$$u/v \nearrow c_n \vee d_n/c_n \searrow d_n/c \wedge d \nearrow a_{n-1} \vee b_{n-1}/b_{n-1} \vee (c \wedge d) \searrow a_{n-1}/a \wedge b^1$$

Since $a_{n-1}/a\wedge b^1$ is a subquotient of a_{n-1}/a_n, u/v is projective to a subquotient of a/a_2 by the above remarks. By the dual argument every prime subquotient of $1/a$ is projective to a subquotient of a^2/a. Now if $n \geq 2$ then the duals of (7) and (8) tell us that a^2/a is projective to d^3/d^1 which transposes down to $a\wedge d^3/a\wedge d^1 = a\wedge b^2/a\wedge b$. Now we may argue as above. The case $n = 1$ has to be argued separately and is left to the reader. Arguments similar to the above prove the lemma for L_n'.

Lemma 10 has the corollary that L_n and L_n' are each subdirect products of four subdirectly irreducible lattices, $i \leq n < \infty$. More specifically, let $L_{n1} = L_n/\theta(a,a\wedge b^{n-1})$, $L_{n2} = L_n/\theta(a,a_1)\vee\theta(a\wedge b^n,a_2)$, $L_{n3} = L_n/\theta(a,a\wedge b^n)\vee\theta(a_1,a_2)$, $L_{n4} = L_n/\theta(a,a\wedge b^{n-1})$. Since L_n is the modular lattice freely generated by a,b,c,d satisfying the relations $a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge c = a \wedge d = b \wedge c = b \wedge d = 0$, $a_n \geq a \wedge b \geq a_{n+1}$, $d_n \geq c \wedge d \geq d_{n+1}$, L_{n1} is the modular lattice freely generated by a,b,c,d satisfying the above relations and also satisfying $a = a \wedge b^n = a_1 = a \wedge b^{n-1}$. Similarly, L_{n2} is the modular lattice freely generated by a,b,c,d subject to the relations of L_n and to the additional relations $a = a \wedge b^n = a_1$, $a \wedge b^{n-1} = a_2$, L_{n3} to the additional relations $a = a \wedge b^n$, $a_1 = a \wedge b^{n-1} = a_2$, L_{n4} to the additional relations $a \wedge b^n = a_1 = a \wedge b^{n-1} = a_2$.

Since the permutation (ad)(bc) generates an automorphism of L_n and since $a \wedge b^{n-1}/a_2$ is projective to $d \wedge c^n/a_1$ we have that L_{n1} is isomorphic to L_{n3}. Similarly L_{n2} and L_{n4} are isomorphic. Furthermore, L_{n2} is isomorphic to $L_{n+1,1}$. To see this, one shows that L_{n2} satisfies the defining relations of $L_{n+1,1}$ and vice versa. This can be done with the use of Lemma 8, and is left to the reader. Similar arguments give that L_n' is a subdirect product of L_{n1}, $L_{n+2,1}$, and two copies of $L_{n+1,1}$.

It follows from (17) that L_n has length $4n + 6$. Using Lemma 8 it follows that L_{n1} has length $n + 1$ and L_{n2} has length $n + 2$. Let $S_1 = \underset{\sim}{2}$ $S_2 = M_5$ and $S_{n+1} = L_{n1}$ $n \geq 2$.

In L_∞ $a^2 \geq a^1 \geq a \geq a_1 \geq a_2$ and by (7) and (8) and their duals every prime quotient of L_∞ is projective to a nontrivial subquotient of a^2/a or a/a_2. If we identify a^2 with a and a_1 with a_2 then we get the modular lattice freely generated by a,b,c,d subject to these relations and the relations of L_∞. These relations are equivalent to $a \vee b = a \vee c = a \vee d = b \vee c = b \vee d = 1$, $a \wedge b = a \wedge c = a \wedge d = b \wedge c = b \wedge d = c \wedge d = 0$. This is the lattice studied in [3]. We denote it by S_∞. Examining the other congruences on L_∞ yield that L_∞ is a subdirect product of two copies of S_∞ and two copies of S_∞^d, its dual. The same statement holds for L_∞'. These facts together imply that L_n and L_n', $0 \leq n \leq \infty$, are each a subdirect product of four subdirectly irreducibles chosen from $\{S_n \mid 1 \leq n \leq \infty\} \vee \{S_\infty^d\}$. It follows from the distributivity of lattice congruences that any subdirectly irreducible, breadth two, four-generated modular lattice is a homomorphic image of one of the S_n or S_∞^d. For $n < \infty$, S_n is finite and hence simple. Thus S_n, $n < \infty$ has no nontrivial homomorphic images. S_∞ and S_∞^d have only one nontrivial homomorphic image: the six element length two lattice, M_6 [3]. Consequently

THEOREM 3: <u>The subdirectly irreducible, breadth two, four-generated modular lattices are precisely the set</u> $\{S_n \mid 1 \leq n \leq \infty\} \vee \{S_\infty^d, M_6\}$.

In [3] the word problem for S_∞ is solved. If one takes the sublattice K_n of S_∞ generated by $a \vee d_n$, $b \vee d_n$, $c \vee a_n$, $d \vee a_n$ if n is even and by $a \vee d_{n-1}$, $b \vee d_{n-1}$, $c \vee a_{n+1}$, $d \vee a_{n+1}$ if n is odd, then using the above mentioned solution to the word problem in S_∞, one can show that K_n satisfies the relations defining S_n. Since S_n is simple it follows that K_n is isomorphic to S_n. This shows that the lattices of Theorem 3 are precisely the breadth two lattices considered in [3]. See Figures 4 and 5.

(FIGURE 4)

(FIGURE 5)

5. COVERINGS IN FM(4)

It is apparent from Lemma 2 that coverings in free modular lattices have important consequences in the study of the structure of modular lattices. Moreover, McKenzie has investigated the connections of coverings in a free lattice to the theory of lattice varieties. In view of these applications we give some examples of coverings in FM(4). In particular, we give an infinite list of covering in FM(4), $u_i \succ v_i$ inequivalent in the strong sense that if $\psi(u_i, v_i)$ is the unique maximal congruence separating u_i from v_i then the $FM(4)/\psi(u_i, v_i)$'s are pairwise nonisomorphic. In fact, there is a covering corresponding to each S_n, $1 \leq n < \infty$.

Let f map $FM(n)$ homomorphically onto L. Then f is called <u>upper bounded</u> if for each $x \in L$ there is an element $u \in FM(4)$ such that $f(u) = x$ and $f(v) = x$ implies $v \leq u$. If the dual property holds then f is <u>lower bounded</u>. If f is both upper and lower bounded then f is <u>bounded</u>. If u is as above we call u the <u>maximal inverse image</u> of x. The minimal inverse image is defined dually. Note that if $f : FM(n) \twoheadrightarrow L$ is bounded and $y \succ x$ in L, and if u is the maximum inverse image of x and v is the minimal inverse image of y, then $u \vee v \succ u$ and $u \wedge v \prec v$ in $FM(n)$. These concepts were defined and studied by R. McKenzie [6]. When L is finite McKenzie gives the following process for deciding if f is bounded. For each $x \in L$ define $M(x)$ to be the family of two elements subsets $\{y, z\}$ of L such that $x \geq y \wedge z$, $x \not\geq y$, $x \not\geq z$ and if $y_0 \geq y$, $z_0 \geq z_0$ and $x \geq y_0 \wedge z_0$ then $y_0 = y$ and $z_0 = z$. Choose $\alpha_0 : L \to FM(n)$ such that α_0 is monotine and $f(\alpha_0(x)) = x$ for all $x \in L$, and such that $\underset{\sim}{a} \leq \alpha_0 f(\underset{\sim}{a})$ for each generator of $FM(n)$. Now define

$$(35) \quad \alpha_i(x) = \alpha_{i-1}(x) \vee \bigvee_{\{x,\ y\} \in M(x)} (\alpha_{i-1}(y) \wedge \alpha_{i-1}(z))$$

Now if $f(u) = x$ then $u \leq \alpha_i(x)$ for some i [6]. Thus f is upper bounded if and only if $\alpha_i = \alpha_{i+1}$ for some i.

Let $FM(4)$ be freely generated by $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$. Let $S_{2n+1} = L_{2n,1}$ be the lattice defined above. Let the generators of S_{2n+1} be a, b, c, d. Let $f : FM(4) \to S_{2n+1}$ be the unique extension of the map $f(\underset{\sim}{a}) = a$, $f(\underset{\sim}{b}) = b$, $f(\underset{\sim}{c}) = c$, $f(\underset{\sim}{d}) = d$. Note that since the maximal inverse image function, when it exists, preserves meets and S_{2n+1} has breadth two we may restrict our attention to the meet irreducibles in S_{2n+1} in calculating the α_i's. The meet irreducibles of S_{2n+1} consist of

$$\begin{array}{l} a \leq a^1 = a^2 \leq a^3 = a^4 \leq \ldots \leq a^{2n-3} = a^{2n-2} \leq a \vee b \\ b \leq b^1 = b^2 \leq b^3 = b^4 \leq \ldots \leq b^{2n-3} = b^{2n-2} \leq a \vee b \\ \quad c = c^1 \leq c^2 = c^3 \leq \ldots \leq c^{2n-2} = c^{2n-1} \\ \quad d = d^1 \leq d^2 = d^3 \leq \ldots \leq d^{2n-2} = d^{2n-1} \end{array}$$

Now $M(a) = \{\{b, c^1\}, \{b, d^1\}\}$, $M(a^{2i}) = \{\{b^{2i}, c^{2i+1}\}, \{b^{2i}, d^{2i+1}\}, \{c^{2i-1}, d^{2i-1}\}\}$, $i = 1, \ldots, n-1$.

$M(a \vee b) = \{\{c^{2n-1}, d^{2n-1}\}\}$, $M(c^{2i+1}) = \{\{d^{2i+1}, a^{2i+2}\}, \{d^{2i+1}, b^{2i+2}\}, \{a^{2i}, b^{2i}\}\}$, $i = 0, \ldots, n-2$, $M(c^{2n-1}) = \{\{a \vee b, d^{2n-1}\}\}$. The definition of $M(b^{2i})$ is similar to $M(a^{2i})$ and $M(d^{2i+1})$ to $M(c^{2i+1})$.

With these definitions one can choose an appropriate definition of α_o and compute α_k by (35). For large enough k, $\alpha_k = \alpha_{k+1}$. We shall only give this final function. In $FM(4)$ with generators $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$ let

$$\text{(36)} \quad \underset{\sim}{a}^i = \underset{\sim}{a} \vee (\underset{\sim}{c}^{i-1} \wedge \underset{\sim}{d}^{i-1}) \qquad \underset{\sim}{b}^i = \underset{\sim}{b} \vee (\underset{\sim}{c}^{i-1} \vee \underset{\sim}{d}^{i-1})$$

$$\underset{\sim}{c}^i = \underset{\sim}{c} \vee (\underset{\sim}{a}^{i-1} \vee \underset{\sim}{b}^{i-1}) \qquad \underset{\sim}{d}^i = \underset{\sim}{d} \vee (\underset{\sim}{a}^{i-1} \vee \underset{\sim}{b}^{i-1})$$

Define $g : S_{2n+1} \to FM(4)$ inductively as follows

$$g(a \vee b) = \underset{\sim}{a} \vee \underset{\sim}{b} \vee (\underset{\sim}{c}^{2n-1} \wedge \underset{\sim}{d}^{2n-1})$$

$$g(c^{2n-1}) = \underset{\sim}{c}^{2n-1} \vee (d^{2n-1} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b})) \qquad g(d^{2n-1}) = \underset{\sim}{d}^{2n-1} \vee (\underset{\sim}{c}^{2n-1} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b}))$$

$$g(a^{2i}) = \underset{\sim}{a}^{2i} \vee (\underset{\sim}{b}^{2i} \wedge g(c^{2i+1})) \qquad g(b^{2i}) = \underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(c^{2i+1}))$$

$$g(c^{2i+1}) = \underset{\sim}{c}^{2i+1} \vee (\underset{\sim}{d}^{2i+1} \wedge g(a^{2i+2})) \qquad g(d^{2i+1}) = \underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))$$

To see that g is the final function we must show that if we let $\alpha_o = g$ in (35) then $\alpha_1 = g$. The following identities in $FM(4)$ may be proved by induction, starting with $i = n - 1$ and working down.

$$g(a^{2i}) = \underset{\sim}{a}^{2i} \vee (\underset{\sim}{b}^{2i} \wedge g(c^{2i+1})) = \underset{\sim}{a}^{2i} \vee (\underset{\sim}{b}^{2i} \wedge g(d^{2i+1}))$$

$$g(c^{2i-1}) = \underset{\sim}{c}^{2i-1} \vee (\underset{\sim}{d}^{2i-1} \wedge g(a^{2i})) = \underset{\sim}{c}^{2i-1} \vee (\underset{\sim}{d}^{2i-1} \wedge g(b^{2i}))$$

Let $\alpha_o = g$ we have

$$\begin{aligned}\alpha_1(a^{2i}) = {} & \underset{\sim}{a}^{2i} \vee (\underset{\sim}{b}^{2i} \wedge g(c^{2i+1})) \\ & \vee \{[\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(c^{2i+1}))] \wedge [\underset{\sim}{c}^{2i+1} \vee (\underset{\sim}{d}^{2i+1} \wedge g(a^{2i+2}))]\} \\ & \vee \{[\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(c^{2i+1}))] \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))]\} \\ & \vee \{[\underset{\sim}{c}^{2i-1} \vee (\underset{\sim}{d}^{2i-1} \wedge g(a^{2i}))] \wedge [\underset{\sim}{d}^{2i-1} \vee (\underset{\sim}{c}^{2i-1} \wedge g(a^{2i}))]\}\end{aligned}$$

Observe that

$$\begin{aligned} & [\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(c^{2i+1}))] \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))] \\ = {} & [\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge g(d^{2i+1}))] \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))] \\ = {} & [\underset{\sim}{b}^{2i} \vee (\underset{\sim}{a}^{2i} \wedge (\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))))] \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))] \\ = {} & [\underset{\sim}{a}^{2i} \wedge g(d^{2i+1})] \vee (\underset{\sim}{b}^{2i} \wedge [\underset{\sim}{d}^{2i+1} \vee (\underset{\sim}{c}^{2i+1} \wedge g(a^{2i+2}))]).\end{aligned}$$

With the use of this identity, the modular law and the fact that $\underset{\sim}{c}^{2i-1} \wedge \underset{\sim}{d}^{2i-1} \le \underset{\sim}{a}^{2i-1}$ it is easy to show that $\alpha_1(a^{2i}) = g(a^{2i})$. Similar argument show that $\alpha_1 = g$. If we extend g to all of S_{2n+1} by letting $g(x \wedge y) = g(x) \wedge g(y)$ then g is well-defined and is the maximum inverse image function. Since S_{2n+1} is isomorphic to its dual we can calculate the minimal inverse limit function h as well. Then since $a^1 \succ a$ in S_{2n+1} we have the following covering in FM(4).

$$g(a) \vee h(c \wedge d) = g(a) \vee h(a) \vee h(c \wedge d) = g(a) \vee h(a^1) \succ g(a)$$

Letting $\underset{\sim}{a}_i$ and $\underset{\sim}{b}_i$ be the elements dual to $\underset{\sim}{a}^i$ and $\underset{\sim}{b}^i$

in FM(4), we have $h(c \wedge d) = \underset{\sim}{c} \wedge \underset{\sim}{d} \wedge (\underset{\sim}{a}_{2n-1} \vee \underset{\sim}{b}_{2n-1})$. Also, $g(a) = \underset{\sim}{a} \vee (\underset{\sim}{b} \wedge (\underset{\sim}{c}^1 \vee (\underset{\sim}{d}^1 \wedge (\underset{\sim}{a}^2 \vee (\underset{\sim}{b}^2 \wedge \ldots$ $(\underset{\sim}{c}^{2n-1} \vee (\underset{\sim}{d}^{2n-1} \wedge (\underset{\sim}{a} \vee \underset{\sim}{b})\ldots)$. Thus we have proved the following theorem.

Theorem 4: <u>For</u> $n = 1, 2, \ldots$ <u>we have the following coverings in</u> FM(4).

$$[\underset{\sim}{c}\wedge\underset{\sim}{d}\wedge(\underset{\sim}{a}_{2n-1}\vee\underset{\sim}{b}_{2n-1})]\vee\underset{\sim}{a}\vee(\underset{\sim}{b}\wedge(\underset{\sim}{c}^1\vee(\underset{\sim}{d}^1\ldots(\underset{\sim}{c}^{2n-1}\vee(\underset{\sim}{d}^{2n-1}\wedge(\underset{\sim}{a}\vee\underset{\sim}{b})\ldots)$$
$$\underset{\sim}{a}\vee(\underset{\sim}{b}\wedge(\underset{\sim}{c}^1\vee(\underset{\sim}{d}^1\ldots\wedge(\underset{\sim}{c}^{2n-1}\vee(\underset{\sim}{d}^{2n-1}\wedge(\underset{\sim}{a}\vee\underset{\sim}{b})\ldots)$$

<u>Furthermore, if</u> ψ_n <u>is the unique maximal congruence separating this covering then</u> $FM(4)/\psi_n \cong S_{2n+1}$.

Similarly one obtains coverings in FM(4) corresponding to each of the S_{2n}'s.

Following McKenzie, call a modular lattice L a <u>splitting modular lattice</u> if there exists an equation ε such that for any variety V of modular lattices either all members of V satisfy ε or $L \in V$. By the above, S_n, $n = 0, 1, 2, \ldots$ is a splitting modular lattice.

COROLLARY: L <u>is a breadth two, four-generated splitting modular lattice if and only if</u> L <u>is isomorphic to</u> S_n <u>for some</u> n, $1 \le n < \infty$.

<u>PROOF</u>: It was shown in [3] that M_6, S_∞, and S_∞^d are not splitting modular lattices. The corollary follows from the

fact that a splitting modular lattice must be subdirectly irreducible.

Now let V be the variety of modular lattices generated by all breadth two modular lattices and let $FL(V, 4)$ be the free V-lattice on four generators. A lattice L is called _weakly atomic_ if for $x > y$ in L there exists $u, v \in L$ such that $x \geq u \succ v \geq y$.

COROLLARY: $FL(V, 4)$ _is a unique irredundant subdirect product of_ 14 _copies of_ S_1, 14 _copies of_ S_2, _and_ 6 _copies of_ S_n, $n = 3, 4, \ldots$. _Moreover_, $FL(V, 4)$ _is weakly atomic_.

PROOF: In [3] it is shown that V is generated by $\{S_n \mid 1 \leq n < \infty\}$. Hence $FL(V, 4)$ is a subdirect product of S_n, $n = 1, 2, \ldots$. It is easy to check that there are 14 distinct congruence relations ψ on $FL(V, 4)$ such that $FL(V, 4)/\psi \cong S_1$, 14 congruences giving S_2, and 6 congruences giving S_n, $n = 3, 4, \ldots$. With the aid of Lemma 2 and Theorem 4 it can be shown that none of these lattices can be removed from a subdirect representation of $FL(V, 4)$.

If $x > y$ in $FL(V, 4)$ then by the above there exists a homomorphism f from $FL(V, 4)$ onto S_n, for some $n < \infty$, such that $f(x) > f(y)$. Since f is bounded there exists $u, v \in FL(V, 4)$ with $u \succ v$ and $f(x) \geq f(u) > f(v) \geq f(y)$. Now it is easy to see that u/v is projective to a subquotient u'/v' of x/y in two or less steps. By modularity $x \geq u' \succ v' \geq y$, proving the corollary.

The above corollary implies that the word problem for $FL(V, 4)$ is solvable. However, by Jónsson's theorem [5] the four-generated subdirectly irreducible members of V are precisely the lattices listed in Theorem 3 (see also [1]). Hence we have the following corollary.

COROLLARY: *If L is the V-lattice freely generated by four generators subject to finitely many relations, then the word problem for L is solvable.*

With the aid of the results of this paper, C. Herrmann has been able to list all subdirectly irreducible four-generated modular lattices in the class C of all lattices embeddable in a complemented modular lattice. From this it follows that the word problem for four-generated lattices in C is solvable. This contrasts the result of G. Hutchinson that the word problem for nine-generated lattices in C is not solvable. An easy modification of Hutchinson's argument yields that the word problem for seven-generated lattices in C is not solvable.

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FIGURE 1

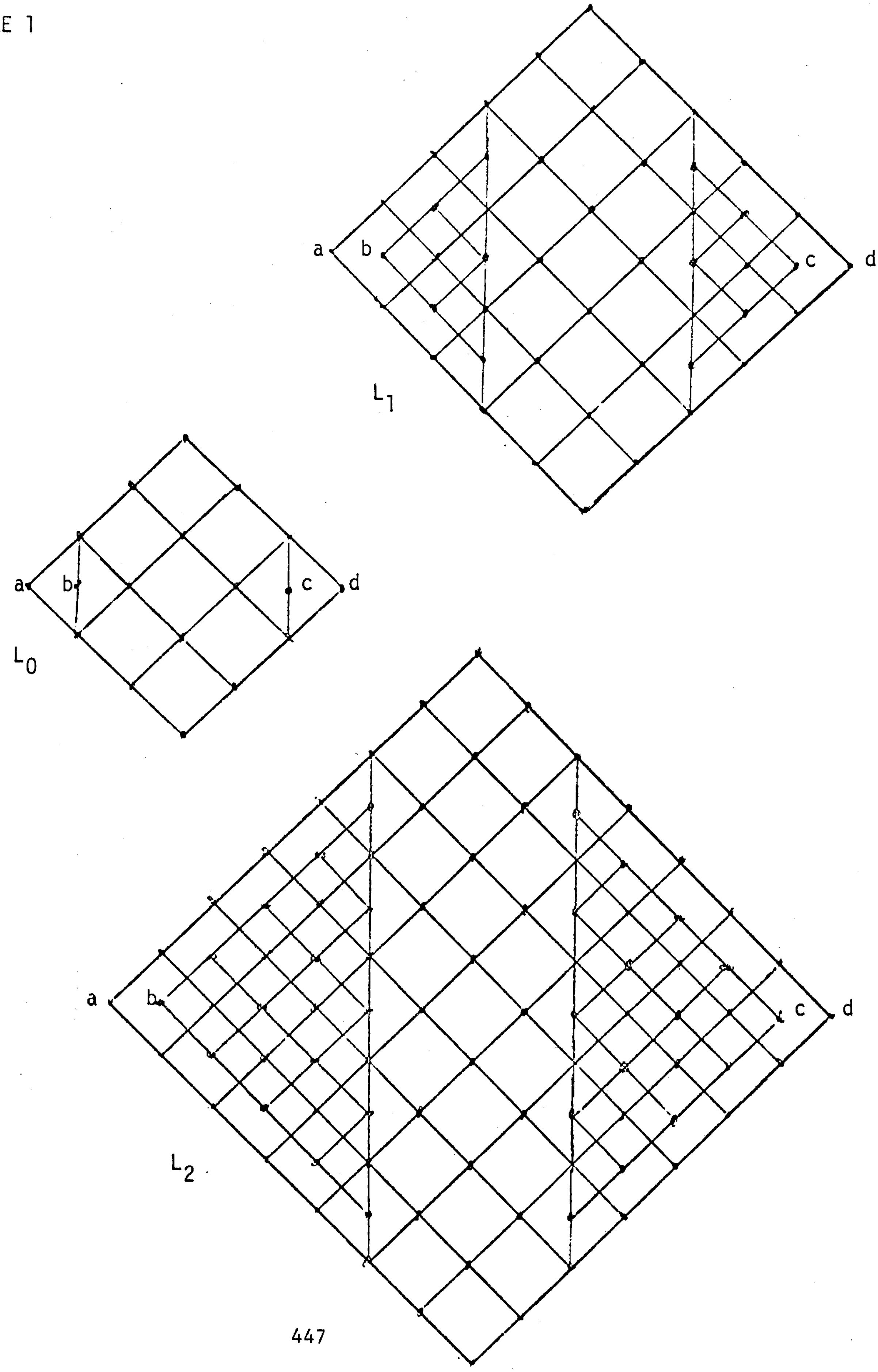
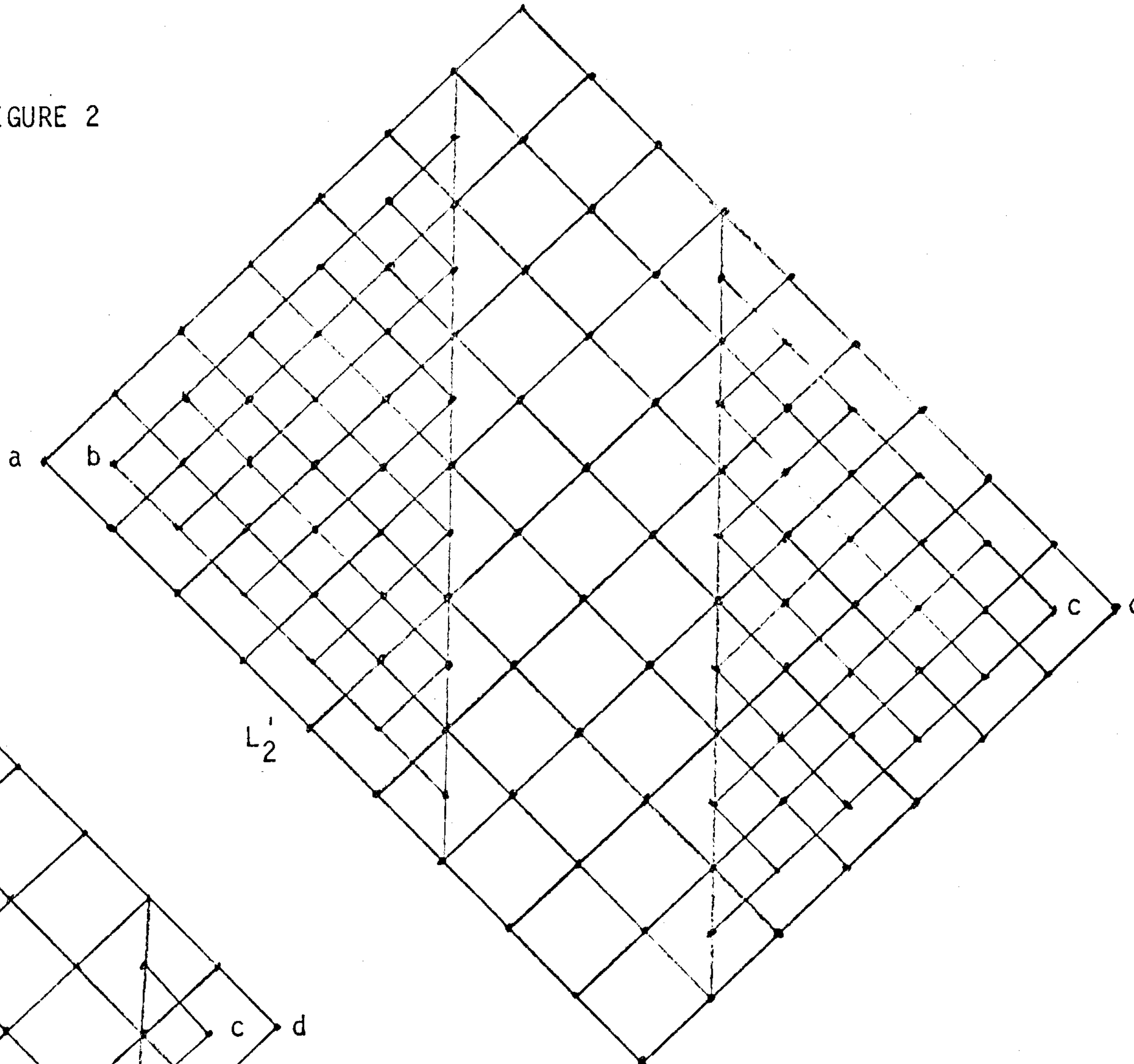
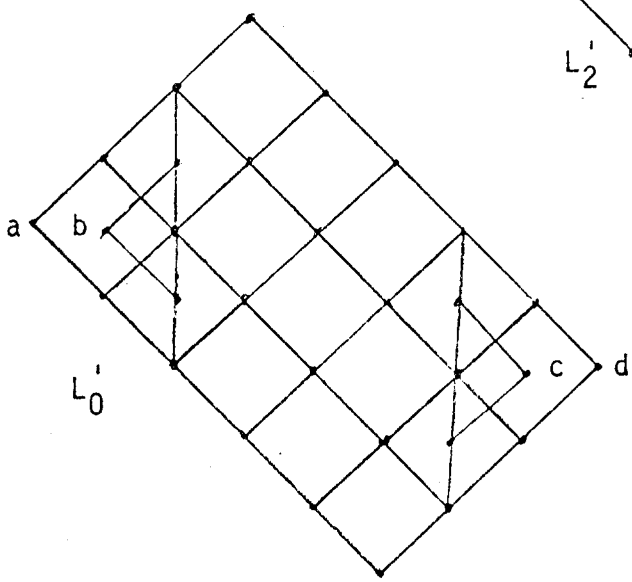


FIGURE 2





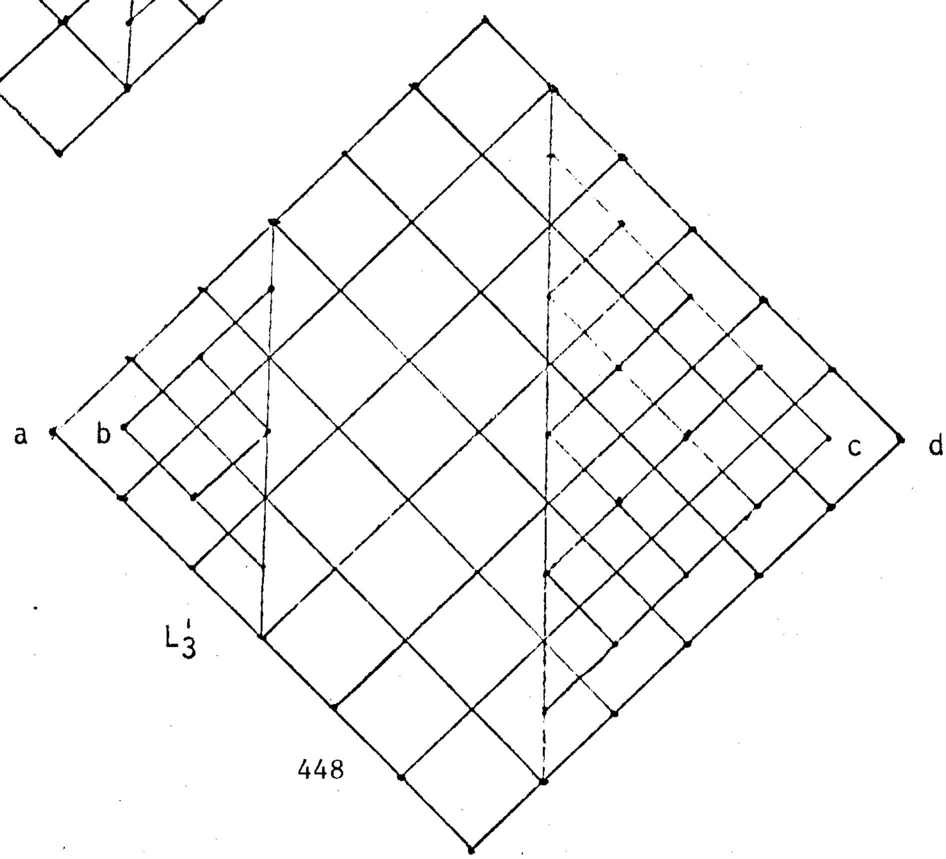


FIGURE 3

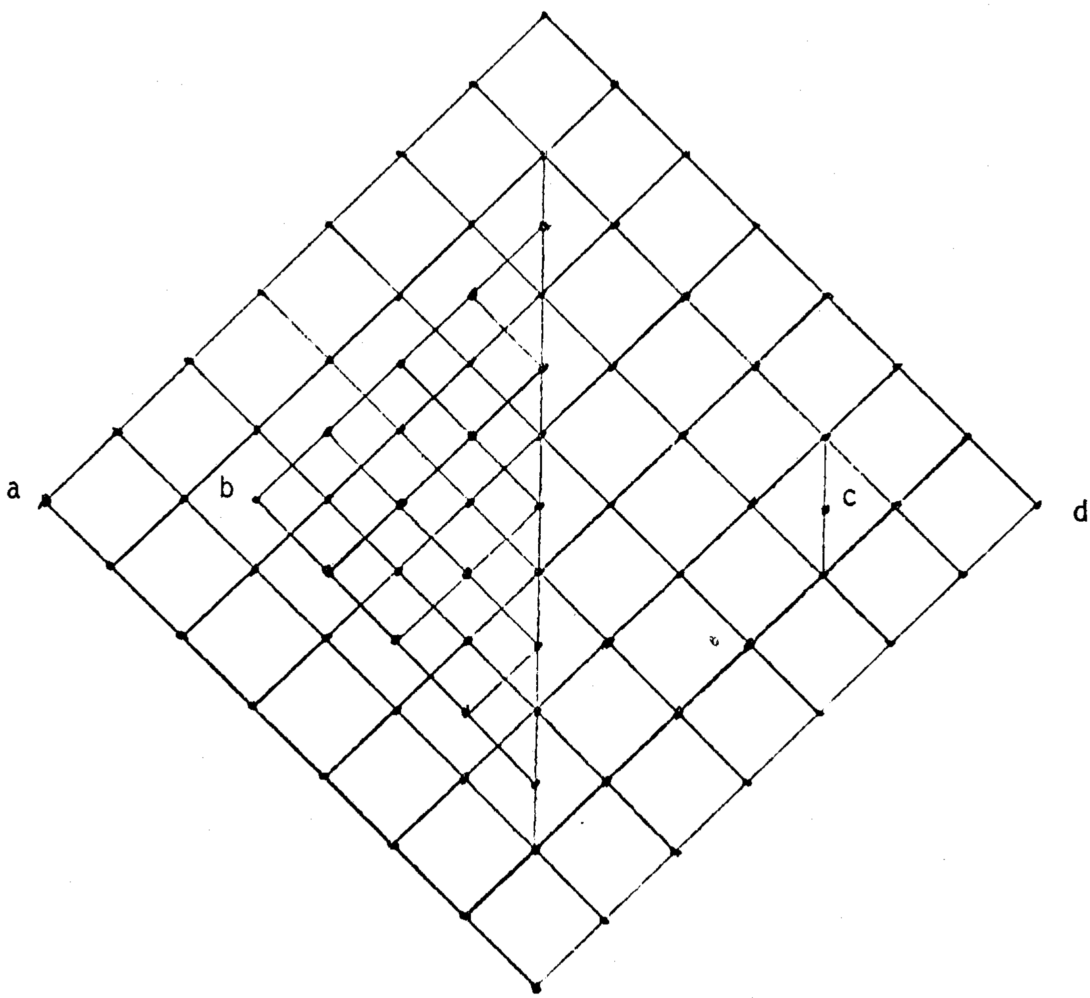
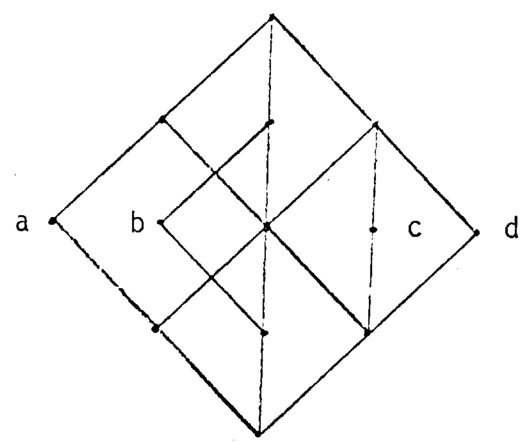
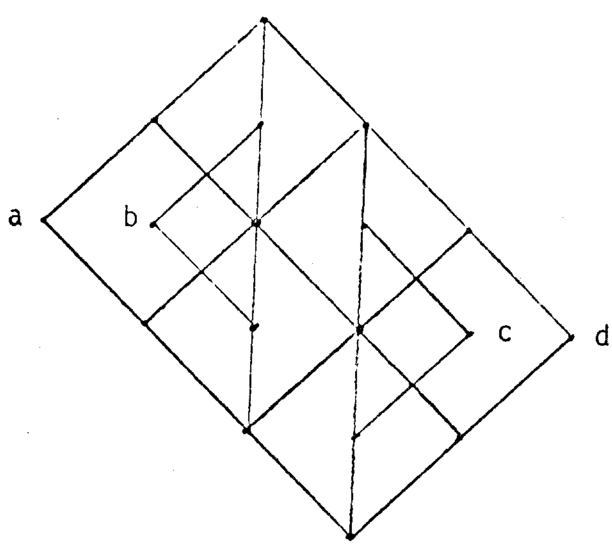


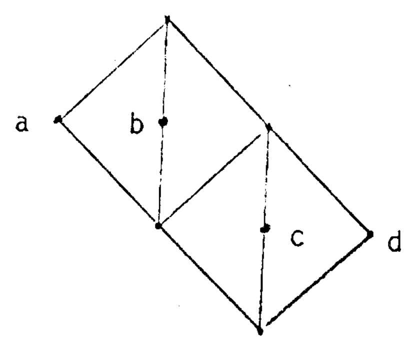
FIGURE 4



S_4



S_5



S_3

FIGURE 5