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STONE DUALITY FOR VARIETIES GENERATED BY QUASI PRIMAL ALGEBRAS

by

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One of the most famous representation theorems is that of M.H. Stone (1937) which says that for every Boolean algebra B there is a Boolean space X such that B is isomorphic to the Boolean algebra of closed-and-open sets of X. This representation theorem leads to a duality between the category of Boolean algebras and the category of Boolean spaces. In 1969 Tah Kai Hu proved that this representation and duality works for any variety generated by a primal algebra. We want to extend this representation and duality to varieties generated by a <u>weakly independent set of quasi primal algebras</u>, i.e. a finite set $\mathfrak{A}$ of finite algebras having a ternary polynomial $d(x, y, z)$ which satisfies $d(x, y, z) = \begin{cases} x & \text{if } y = z \\ z & \text{if } y \neq z \end{cases}$ on every algebra $A \in \mathfrak{A}$. From now on $\mathfrak{A}$ will always denote a weakly independent set of quasi primal algebras. Examples are any finite set of finite fields

or any finite set of finite chains, regarded as lattices with relative pseudocomplementation and dual relative pseudocomplementation. The existence of $d(x, y, z)$ shows that any algebra in the variety $V\mathfrak{A}$ generated by $\mathfrak{A}$ has permutable and distributive congruences and therefor every $R \in V\mathfrak{A}$ is a subdirect product of subalgebras of algebras in $\mathfrak{A}$. For $R \in V\mathfrak{A}$ let $\mathrm{Hom}(R, \mathfrak{A}) := \{\varphi\colon R \to A \mid \varphi \text{ homomorphism}, A \in \mathfrak{A}\}$ and $\mathrm{Spec}(R) := \{\mathrm{Ker}\,\varphi \mid \varphi \in \mathrm{Hom}(R, \mathfrak{A})\}$. The <u>"equalizer topology"</u> on $\mathrm{Spec}(R)$ is the topology with the basis $\{E(r, s), D(r, s) \mid r, s \in R\}$, where $D(r, s) = \{\theta \mid (r, s) \notin \theta\}$, $E(r, s) = \{\theta \mid (r, s) \in \theta\}$. Let S be the (finite) discrete space $S = \bigcup\mathfrak{A}$.

THM 1: (1) Spec(R) with the equalizer topology is a Boolean space,

(2) Hom(R, $\mathfrak{A}$) with the topology induced by Hom(R, $\mathfrak{A}$) $\subseteq S^R$ is a Boolean space.

Our first representation and duality will be by means of sheaves, therefor we used the standard construction of sheaves from a subdirect representation of a algebra. Let be $R \subseteq \prod_{x \in X} B_x$ a subdirect product of algebras and let τ be a topology on X such that for r, s $\in$ R the set $\{x \in X \mid r(x) = s(x)\}$ is open, then there is a sheaf $\mathcal{G}$ with base space X and stalks B_x $(x \in X)$ such that $R \subseteq \Gamma \mathcal{G}$, $\Gamma \mathcal{G}$ is the algebra of all global sections of $\mathcal{G}$. This standard construction gives us a sheaf $\mathcal{G}(R)$ for every algebra $R \in V\mathfrak{A}$ with base space Spec(R) and subalgebras of algebras in $\mathfrak{A}$ as stalks.

THM 2: (1st representation theorem) For any $R \in V\mathfrak{A}$ $R \cong \Gamma\mathcal{G}(R)$ holds

THM 3: (1st duality theorem) Let $\mathfrak{G}$ be the category of sheaves $\mathcal{G}$ satisfying

(1) the base space X is Boolean, (2) Every stalk is a subalgebra of some $A \in \mathfrak{A}$, (3) If some $A \in \mathfrak{A}$ contains a 1-element subalgebra, then $\mathcal{G}$ has exactly one one-element stalk.

Then Γ: $\mathfrak{G} \to V\mathfrak{A}$ and $\mathcal{G}$: $V\mathfrak{A} \to \mathfrak{G}$ established a duality. We also can give a representation by continuous functions. Let H be the set of all isomorphisms between subalgebras of algebras in $\mathfrak{A}$. Then H with the relational product is an inverse semigroup and acts as an inverse semigroup of partial homeomorphisms on each Hom(R, $\mathfrak{A}$) and on S. If X, Y are Boolean H-spaces (i.e. Boolean spaces with H acting on them) then we denote by $C_H(X, Y)$ the set of all continuous H-preserving maps φ: $X \to Y$.

THM 4: (2nd representation theorem) For any $R \in V\mathfrak{A}$ $R \cong C_H(\mathrm{Hom}\ (R, \mathfrak{A}), S)$ holds.

THM 5: (2^{nd} duality theorem) Let $\mathcal{B}_H$ denote the category of Boolean H-spaces.

Then $C_H(\_\_\_, S)$: $\mathcal{B}_H \to V\mathfrak{A}$ and $\mathrm{Hom}(\_\_\_, \mathfrak{A})$: $V\mathfrak{A} \to \mathcal{B}_H$ establish a duality

COR. 4.1.: (AHRENS-KAPLANSKI): Let K be a field of characteristic p . For every ring $R \in VK$ there is a Boolean space X together with

(i) a closed subset $\tilde{L}$ of X for every subring L of K

(ii) a homeomorphism $\tilde{\alpha}$: $X \to X$ leaving all $\tilde{L}$ invariant

such that R is isomorphic to the ring of all continuous functions f: $X \to K$ satisfying (1) $f(x) \in L$ for all $x \in \tilde{L}$

(2) $f(\tilde{\alpha} x) = f(x)^p$ for all $x \in X$

COR. 4.2.: (AHRENS-KAPLANSKI): The category of all p-rings is dual to the category of pointed Boolean spaces.

COR. 4.3.: (HU): The variety generated by a primal algebra is dual to the category of Boolean spaces.

COR. 4.4.: (STONE): The category of Boolean algebras is dual to the category of Boolean spaces.