Proc. Univ. of Houston
Lattice Theory Conf..Houston 1973

Equational classes of Foulis semigroups and orthomodular lattices

Donald H. Adams

I. Introduction. This paper follows on from our work exhibiting Baer semigroups as an equational class and investigating the connection between equational classes of lattices and equational classes of Baer semigroups [1]. Here we consider the particular case of Foulis semigroups and orthomodular lattices and, as is usual when one particularizes, we are rewarded with more specific results. This paper does not depend on [1] and may be read separately. However, a fair knowledge of Foulis semigroups and orthomodular lattices is assumed and we refer to Blyth and Janowitz [2] for the basic theory and further references.

In this paper we show that Foulis semigroups from an equational class when they are regarded as algebras of type $<2,1,1,0>$ where the two unary operations are the involution and the focal map. We show that the class of Foulis semigroups coordinatizing the members of an equational class of orthomodular lattices is equational. Conversely, the class of orthomodular lattices coordinatized by the members of an equational class of Foulis semigroups is also equational. We exhibit a homomorphism from the lattice of equational classes of Foulis semigroups onto the lattice of equational classes of orthomodular lattices.

We generally follow the conventions of Grätzer [4], except that we rarely bother to distinguish between an algebra and its base set. We are ambidextrous in the way we write maps: homomorphisms are on the left and residuated maps on the right. We feel that this tends to clarify rather than confuse the situation. We skip over foundational difficulties, especially when dealing with the lattice of equational classes, because all these can be handled by standard tricks - see Grätzer [4] for details.

2. Foulis semigroups. We follow Blyth and Janowitz [2] in using the term *Foulis semigroup* for what was originally called a *Baer *-semigroup* by Foulis [3]. We refer the reader to these two sources for the proofs of any assertions that we leave unproven.

Definition 1. A *Foulis semigroup* is an algebra $\langle F;\cdot,*,',0\rangle$ of type $\langle 2,1,1,0\rangle$ such that

(i) $\langle F;\cdot,0\rangle$ is a semigroup with zero;

(ii) $*$ is an *involution*, i.e. for any $x,y \in F$, $x^{**} = x$ and $(xy)^* = y^*x^*$. If $x = x^*$ then x is *self-adjoint*.

(iii) for each $x \in F$ the element x' is a self-adjoint idempotent or *projection*.

(iv) for each $x \in F$

$$r(x) = \{y \mid xy = 0\} = x'F.$$

In other words, the right annihilator of x is a principal right ideal generated by a projection.

The elements of F of the form x' are called *closed projections*. The map defined by $x \to x'$ is called the *focal map*, $1 = 0'$ is an identity in F and $1' = 0$.

Proposition 2. Condition (iv) of Definition 1 is equivalent to

$$\text{(iv)}' \quad x'y(xy)' = y(xy)' \text{ \underline{for all} } x,y \in F.$$

Proof. Since x' is idempotent, (iv) is equivalent to:

$$xy = 0 \text{ if and only if } x'y = y.$$

Suppose that (iv) holds. Then

$$\begin{aligned}(xy)'F &= \{t \mid xyt = 0\} \\ &= \{t \mid x'yt = yt\}\end{aligned}$$

and since $(xy)' \in (xy)'F$, we get $x'y(xy)' = y(xy)'$.

If, on the other hand, (iv)' holds, then substituting $x = 1$ in the formula gives us the identity $yy' = 0$. Hence, if $x'y = y$, then $xy = xx'y = 0$. If $xy = 0$, then substitution in the formula gives us $x'y = y$. We have shown that $xy = 0$ if and only if $x'y = y$ and so (iv) holds.

Notice that (iv) was the only one of the defining properties of a Foulis semigroup that could not readily be expressed as an identity. Proposition 2 shows us that it can be so expressed and we have the following result.

Corollary 3. The class of all Foulis semigroups is equational.

Foulis semigroups are of interest mainly because, if F is a Foulis semigroup, then the set of closed projections in F, $P'(F)$, is an orthomodular lattice where the operations are given by the formulas:

(1) $\qquad e \wedge f = (f'e)'e;$

(2) $\qquad e^{\perp} = e' \qquad ;$

(3) $\qquad 0 = 0 \qquad .$

We denote the equational class of Foulis semigroups by $\mathcal{B}$ and the equational class of orthomodular lattices by $\mathcal{L}$. From the formulas (1) - (3) the following is clear.

Proposition 4. (i) *If $h: F_1 \to F_2$ is a Foulis semigroup homomorphism, then $h': P'(F_1) \to P'(F_2)$, the restriction of h to the closed projections, is an orthomodular lattice homomorphism. If h is onto, then h' is also onto.*

(ii) *If F_1 is a subalgebra of F_2 where F_1 and F_2 are Foulis semigroups, then $P'(F_1)$ is a subalgebra of $P'(F_2)$ as an orthomodular lattice.*

We denote the direct product of a family of algebras $(A_i | i \in I)$ by $\Pi(A_i | i \in I)$. It is straightforward to prove the following result.

Proposition 5. *Let $(F_i | i \in I)$ be a family of Foulis semigroups. Then $\Pi(P'(F_i) | i \in I)$ is isomorphic as an orthomodular lattice to $P'(\Pi(F_i | i \in I))$.*

These two propositions will give us an immediate proof of our first main result. But first we observe that the mapping given by $h \to h'$ in Proposition 4 (i) is a functor from the category of Foulis semigroups to the category of orthomodular lattices. By Proposition 5 it is product-preserving and in this situation one always gets a result of the following type.

<u>Theorem 6</u>. <u>Let</u> $\mathcal{L}_1$ <u>be an equational class of orthomodular lattices</u>. <u>Then</u>

$$b(\mathcal{L}_1) = \{F \mid F \in \mathcal{B},\ P'(F) \in \mathcal{L}_1\}$$

<u>is an equational class of Foulis semigroups</u>.

<u>Proof</u>. It follows immediately from Propositions 4 and 5 that $b(\mathcal{L}_1)$ is closed under the taking of homomorphic images, subalgebras and direct products.

Actually, formulas (1)-(3) give us an immediate technique for translating orthomodular lattice identities into Foulis semigroup identities. This means that, given an equational base for the equational class $\mathcal{L}_1$, we can in principle calculate an equational base for the corresponding equational class of Foulis semigroups $b(\mathcal{L}_1)$. If $\mathcal{L}_1$ is the equational class of Boolean lattices, then $b(\mathcal{L}_1)$ is the class of all Foulis semigroups satisfying the identity $x'y' = y'x'$. This follows from the standard result [2, p. 201] that $P'(F)$ is Boolean if and only if $ef = fe$ for all $e,f \in P'(F)$.

Foulis [3] showed that any orthomodular lattice is isomorphic to the lattice of closed projections of the Foulis semigroup $S(L)$ of residuated maps on L (see also [2]). The involution is given by $x\phi^* = (x^{\perp}\phi^{+})^{\perp}$ where ϕ^{+} denotes the residual of ϕ and $x \in L$. The closed projections are precisely the Sasaki projections defined by $x\phi_y = (x \vee y^{\perp}) \wedge y$ for $x,y \in L$ and the focal map is given by $\phi' = \phi_g$ where $g = (1\phi)^{\perp}$. We say that a Foulis semigroup F <u>coordinatizes</u> L if L is isomorphic to $P'(F)$.

<u>Proposition 7</u>. <u>Let</u> F <u>be a Foulis semigroup and let</u> $L = P'(F)$. <u>The map</u> $h: F \to S(L)$ <u>defined by</u> $h(x) = \phi_x$ <u>where</u> $\phi_x: P'(F) \to P'(F)$ <u>is defined by</u> $e\phi_x = (ex)''$ <u>for</u> $e \in L$ <u>is a Foulis semigroup homomorphism of</u> F <u>into</u> $S(L)$. <u>Moreover, if</u> $e,f \in L$, <u>then</u> $e\phi_f = (e \vee f^{\perp}) \wedge f$.

<u>Proof</u>. Foulis [3] proved all this except the fact that h preserves the local map. Observe that $(\phi_x)' = \phi_g$ where $g = (1\phi_x)' = (1x)''' = x'$ to get $h(x') = [h(x)]'$.

<u>3. Small Foulis semigroups</u>. If F is a Foulis semigroup and if $x \in F$ is the product of closed projections i.e. $x = e_1e_2\ldots e_n$ where $e_i \in P'(F)$ for $i = 1,2,\ldots n$, then $x^* = e_ne_{n-1}\ldots e_1$ is also a product of closed projections and x' is a closed projection. Therefore F_o, the subsemigroup of F generated by $P'(F)$, is a subalgebra of F and is a Foulis semigroup coordinatizing $P'(F)$. We call a Foulis semigroup small when it is generated by its

closed projections. Clearly any Foulis semigroup F has a unique small subalgebra F_0 coordinatizing $P'(F)$. If L is an orthomodular lattice we denote by $S_0(L)$ the small Foulis semigroup of products of Sasaki projections on L.

Proposition 8. $S_0(L)$ is a homomorphic image of any small Foulis semigroup coordinatizing L.

Proof. The homomorphism in Proposition 7 carries closed projections onto closed projections.

We can now prove some partial converses to Proposition 4.

Proposition 9. Let $h\colon L_1 \to L_2$ be an orthomodular lattice homomorphism onto L_2. There exists a unique Foulis semigroup homomorphism $k\colon S_0(L_1) \to S_0(L_2)$ onto $S_0(L_2)$ such that the restriction of k to the closed projections in $S_0(L_1)$ coincides with h.

Proof. Let $\phi_1(x)$ denote the Sasaki projection on L_1 generated by $x \in L_1$ and let $\phi_2(y)$ denote the Sasaki projection on L_2 generated by $y \in L_2$. A typical element of $S_0(L_1)$ is of the form $\Pi_{i=1}^{n} \phi_1(x_i)$ and we define the map k by

$$k(\Pi_{i=1}^{n} \phi_1(x_i)) = \Pi_{i=1}^{n} \phi_2(h(x_i)).$$

Suppose that $\Pi_{i=1}^{n} \phi_1(x_i) = \Pi_{j=1}^{m} \phi_1(y_j)$ as maps on L_1. If $u \in L_2$ there exists $u \in L_1$ such that $h(v) = u$ and

$$\begin{aligned} &u\, \Pi_{i=1}^{n} \phi_2(h(x_i)) \\ &= h(v)\, \Pi_{i=1}^{n} \phi_2(h(x_i)) \\ &= h[v\, \Pi_{i=1}^{n} \phi_1(x_i)] \end{aligned}$$

since this expression is just an orthomodular lattice polynomial. This calculation shows that $\Pi_{i=1}^{n}\ \phi_2(h(x_i)) = \Pi_{j=1}^{m}\ \phi_2(h(y_j))$ and hence the map k is well-defined.

The map k is clearly a semigroup homomorphism preserving the zero and the involution. To prove that k preserves the focal map remember that $[\Pi_{i=1}^{n}\ \phi_1(x_i)]' = \phi_1(y)$ where $y = [1\ \Pi_{i=1}^{n}\ \phi_1(x_i)]^{\perp}$. Since this expression is an orthomodular lattice polynomial it follows that $h(y) = [1\ \Pi_{i=1}^{n}\ \phi_2(h(x_i))]^{\perp}$ and so we get

$$\begin{aligned} k([\Pi_{i=1}^{n}\ \phi_1(x_i)]') &= k(\phi_1(y)) \\ &= \phi_2(h(y)) \\ &= [k(\Pi_{i=1}^{n}\ \phi_1(x_i))]' \end{aligned}$$

This completes the proof.

Proposition 10. *Let L_1 be a subalgebra of L_2 as orthomodular lattices. There is a small Foulis semigroup F_1 coordinatizing L_1 and F_1 is a subalgebra of $S_0(L_2)$.*

Proof. If $x \in L_2$ let $\phi(x)$ denote the Sasaki projection on L_2 generated by x. Form the set

$$F_1 = \{\Pi_{i=1}^{n}\ \phi(x_i) \mid x_i \in L_1,\ n \geq 1\},$$

i.e. F_1 is the set of all products of Sasaki projections on L_2 generated by elements of L_1. F_1 is clearly a subsemigroup of $S_0(L_2)$ closed under involution and containing the zero. If $x_i \in L_1$ for $i = 1,2,\ldots,n$, then

$y = 1\ \Pi_{i=1}^{n}\ \phi(x_i) \in L_1$ since it is just an orthomodular lattice polynomial. Since $[\Pi_{i=1}^{n}\ \phi(x_i)]' = \phi(y^{\perp})$ we get that F_1 is closed under the focal map and is hence a subalgebra of $S_0(L)$. Since $[\Pi_{i=1}^{n}\ \phi(x_i)]'' = \phi(y)$, it follows that the closed projections of F_1 are exactly those generated by elements of L_1 and L_1 is isomorphic to $P'(F_1)$ since the lattice operations can be expressed in terms of Foulis semigroup operations.

Note that it is, in general, not true that $S_0(L_1)$ is isomorphic to a subalgebra of $S_0(L_2)$. This is because products of Sasaki projections generated by elements of L_1 may be equal as mappings on L_1 but not when they are regarded as mappings on L_2. However, $S_0(L_1)$ is a homomorphic image of F_1.

4. Equational classes. We are now ready to prove our second main result, the converse to theorem 6.

Theorem 11. *If $\mathcal{B}_1$ is an equational class of Foulis semigroups, then*

$$\ell(\mathcal{B}_1) = \{L \mid L \in \mathcal{L},\ S_0(L) \in \mathcal{B}_1\}$$

is an equational class of orthomodular lattices.

Proof. Proposition 9 implies that $\ell(\mathcal{B}_1)$ is closed under the taking of homomorphic images and Propositions 8 and 10 imply that $\ell(\mathcal{B}_1)$ is closed under the formation of subalgebras. If $(L_i \mid i \in I)$ is a family of members of $\ell(\mathcal{B}_1)$

then Proposition 8 implies that $S_0(\Pi(L_i|i\varepsilon I))$ is a homomorphic image of the unique small subalgebra of $\Pi(S_0(L_i)|i\varepsilon I)$ and therefore $\Pi(L_i|i\varepsilon I)$ is in $\ell(\mathcal{B}_1)$. This proves that $\ell(\mathcal{B}_1)$ is an equational class.

In this case it is not as easy to see how identities may be carried over, but it is again possible in principle. The general idea is to take the Foulis semigroup identity and write it as an orthomodular lattice identity in terms of products of Sasaki projections. For example, the Foulis semigroup identity $xy = yx$ goes over to the orthomodular lattice identity $e\phi_f\phi_g = e\phi_g\phi_f$ (it is equivalent to assume that only two Sasaki projections commute). It is not immediately transparent that this determines the equational class of Boolean lattices.

Note that if $\mathcal{B}_1$ is an equational class of Foulis semigroups, then $S_0(L) \varepsilon \mathcal{B}_1$ if and only if L is coordinatized by F for some $F \varepsilon \mathcal{B}_1$. Therefore

$$\ell(\mathcal{B}_1) = \{L|L \varepsilon \mathcal{L}, \text{ L is isomorphic to } P'(F) \text{ for some } F \varepsilon \mathcal{B}_1\}.$$

Let $\underline{B}$, $\underline{L}$ denote the lattices of equational classes of Foulis semigroups and orthomodular lattices respectively. Then $\ell\colon \underline{B} \to \underline{L}$ and $b\colon \underline{L} \to \underline{B}$ are monotone maps and we can readily prove the following.

<u>Proposition 12</u>. (i) $\ell(b(\mathcal{L}_1)) = \mathcal{L}_1$ for any $\mathcal{L}_1 \varepsilon \underline{L}$;
(ii) $b(\ell(\mathcal{B}_1)) \supseteq \mathcal{B}_1$ for any $\mathcal{B}_1 \varepsilon \underline{B}$.

The inclusion in part (ii) may be strict, i.e. one equational class of orthomodular lattices may be coordinatized by many different equational classes of Foulis semigroups. As an example of this observe that the class of Boolean lattices is coordinatized by each of the following equational classes of Foulis semigroups:

(i) Boolean lattices themselves;

(ii) pseudo-complemented semilattices;

(iii) commutative Foulis semigroups.

Corollary 13. *The map ℓ is residuated and b is its residual.*

It follows from this that ℓ preserves joins and it is straightforward to check that it also preserves meets since the meet in the lattice of equational classes is intersection (there are some foundational difficulties here).

Theorem 14. *The map ℓ: $\underline{B} \to \underline{L}$ is a complete lattice homomorphism.*

This result illustrates the main point of this paper: that a study of $\underline{B}$ should give information about the structure of $\underline{L}$.

REFERENCES

[1] D. H. Adams, Baer semigroups as an equational class, submitted.

[2] T. S. Blyth and M. F. Janowitz, Residuation Theory, Pergamon Press, 1972.

[3] D. J. Foulis, Baer *-semigroups, Proc. Amer. Math. Soc. 11 (1960) 648-654.

[4] George Grätzer, Universal Algebra, Van Nostrand, 1968.

University of Massachusetts, Amherst.