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Orthomodular Logic

by

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In this paper I develop an "orthomodular (OM) logic", a propositional logic, the models of which are orthomodular lattices. Since Boolean algebras are special orthomodular lattices, this locic contains the classical propositional calculus as a maximal extension. The OM-logic is incomparable with the other generalizations of classical logic like intuitionistic, Lukasiewicz- and Post-type logics. In fact, the only common extension of each of these with the OM-logic is the classical logic.

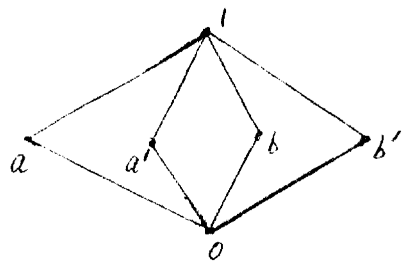
A main concern in this paper is to give a definition of implication which on the one hand has the property, that the modus ponens formulated with it gives a finite axiomatization of the OM-logic and which, on the other hand satisfies the natural requirement that $\alpha \to \beta$ is a tautology iff $v(\alpha) \leq v(\beta)$ holds for every valuation v in an OM-lattice. Among the five definitions of an implication satisfying the second of these requirements only one of them has the property that it gives the completeness theorem. It is shown that the intermediate OM-logics correspond to the equational classes of OM-lattices and that the deduction theorem does not hold in the OM-logic. This answers a question posed by W. Felscher.

1. We determine all two-variable polynomials p with the property

$$(*) \quad p(a,b) = 1 \text{ iff } a \leq b$$

holds in every OML.

Let MO2 be the OML



Every element of the free OML $2^4 \times MO2$ (see G. Bruns and G. Kalmbach in the same Proceedings) on two generators gives rise in the obvious and well-known fashion to a polynomial in two variables. We introduce a special notation "$\to_i$" ($i = 1,\ldots,5$) for some of these polynomials and we write "$a \to_i b$" for the value of $\to_i$ at (a,b). The polynomials $\to_i$ are defined by:

$$a \to_1 b = (a' \wedge b) \vee (a' \wedge b') \vee (a \wedge (a' \vee b))$$

$$a \to_2 b = (a' \wedge b) \vee (a \wedge b) \vee ((a' \vee b) \wedge b')$$

$$a \to_3 b = a' \vee (a \wedge b)$$

$$a \to_4 b = b \vee (a' \wedge b')$$

$$a \to_5 b = (a' \wedge b) \vee (a \wedge b) \vee (a' \wedge b')$$

Theorem: The polynomials p in two variables satisfying (*) are exactly the polynomials $\to_i$ ($i = 1,\ldots,5$). The polynomials $\to_i$ have the additional property that $(1 \to_i b) = 1$ implies $b = 1$ in every OM-lattice.

2. We define the algebra $\mathfrak{F} = (F;\vee,\wedge,\neg)$ of formulae of our orthomodular (propositional) logic as an algebra with two binary operations $\vee,\wedge$ and

a unary operation $\neg$ which is absolutely freely generated by a countable infinite set V, the propositional variables.

A valuation is a map v of F into some OM-lattice L satisfying for all

$$\alpha, \beta \in F : v(\alpha \vee \beta) = v(\alpha) \vee v(\beta);$$

$$v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$$

and

$$v(\neg\alpha) = (v(\alpha))'.$$

A formula $\alpha \in F$ is valid in L iff for every valuation $v: F \to L$ it is $v(\alpha) = 1$. A tautology is a formula $\alpha \in F$ which is valid in every OML. Let T be the set of tautologies.

For notational convenience we introduce a binary operation R in F by

$$\alpha R \beta = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta).$$

It is

$$\alpha R \beta \in T \quad \text{iff} \quad v(\alpha) = v(\beta)$$

for every valuation v.

We also consider the rules of inference

$$R_0: \frac{\alpha \quad \neg\alpha \vee \beta}{\beta} \qquad \text{and} \qquad R_i: \frac{\alpha \quad \alpha \to_i \beta}{\beta} \qquad (1 \leq i \leq 5)$$

These are "correct" rules of inference in the sense that if α and $\neg\alpha \vee \beta$ ($\alpha \to_i \beta$) are valid in an OML M then β is valid in M.

For all formulae α, β, γ the following formulae A1 to A13 are tautologies.

A1 $\quad \neg(\alpha R \beta) \vee (\neg\alpha \vee \beta)$

A2 $(\alpha R\beta)R(\beta R\alpha)$

A3 $\neg(\alpha R\beta) \vee (\neg(\beta R\gamma) \vee (\alpha R\gamma))$

A4 $\alpha R(\neg(\neg\alpha))$

A5 $(\alpha R\beta)R(\neg\alpha R\neg\beta)$

A6 $\neg(\alpha R\beta) \vee ((\alpha \wedge \gamma)R(\beta \wedge \gamma)$

A7 $(\alpha \wedge \beta)R(\beta \wedge \alpha)$

A8 $\neg(\alpha \vee \beta)R(\neg\alpha \wedge \neg\beta)$

A9 $(\alpha \wedge (\alpha \vee \beta))R\alpha$

A10 $(\alpha \wedge (\beta \wedge \gamma))R((\alpha \wedge \beta) \wedge \gamma)$

A11 $(\alpha \vee(\neg\alpha \wedge (\alpha \vee \beta))R(\alpha \vee \beta)$

A12 $(\neg\alpha \wedge \alpha)R((\neg\alpha \wedge \alpha) \wedge\beta)$

A13 $(\neg\alpha \vee \beta) \rightarrow_1 (\alpha \rightarrow_1 (\alpha \rightarrow_1 \beta))$

Let B_0 be the set of tautologies of the form A1 to A13.

Lemma: A set of formulae containing B_0 is closed under R_0 iff it is closed under R_1.

3. Using the well-known technique of Lindenbaum-Tarski algebras we prove that the tautologies A1 to A13 together with the rule of inference R_0 give an axiomatization of our logic.

For an arbitrary set $M \subseteq F$ let $\Gamma_i M$ be the smallest set $S \subseteq F$ containing M and closed under R_i, i.e. satisfying: if $\alpha \in S$ and $\neg\alpha \vee \beta \in S$ resp. $\alpha \rightarrow_i \beta \in S$ then $\beta \in S$.

We define a relation ρ in F by $\alpha\rho\beta$ iff $\alpha R\beta \in \Gamma_0 B_0$.

In the following Lemma α/ρ is the equivalence class of α modulo ρ for $\alpha \in F$.

Lemma: ρ is a congruence relation in $\mathfrak{F}$. The quotient $\mathfrak{F}/\rho$ is an

OML. For every formula $\alpha \in F$ it is

$$\alpha \in \Gamma_0 B_0 \text{ iff } \alpha/\rho = 1.$$

By the Lemma of 2. and the preceeding Lemma we have

Theorem: $$\Gamma_0 B_0 = T = \Gamma_1 B_0$$

4. For a set A of formulae define mod A to be the class of all orthomodular lattices L in which all formulae $\alpha \in A$ are valid. For a class $\mathfrak{K}$ of OML define form $\mathfrak{K}$ to be the set of all formulae α which are valid in all $L \in \mathfrak{K}$. Then the pair (mod, form) is a Galois-correspondence, i.e. the following rules hold:

$$\text{if } A_1 \subseteq A_2 \text{ then mod } A_2 \subseteq \text{mod } A_1$$

$$\text{if } \mathfrak{K}_1 \subseteq \mathfrak{K}_2 \text{ then form } \mathfrak{K}_2 \subseteq \text{form } \mathfrak{K}_1$$

$$A \subseteq \text{form (mod } A)$$

$$\mathfrak{K} \subseteq \text{mod (form } \mathfrak{K})$$

We define an intermediate OM-logic to be a set A of formulae which is closed under this Galois-correspondence, i.e. satisfies A = form(mod A).

Theorem: A set M of formulae is in intermediate logic iff it contains B_0 and is closed under substitution and the rule R_0 or R_1.

Corollary: For every set A of formulae:

$$\Gamma_0(B_0 \cup \overline{A}) = \text{form(mod } A) = \Gamma_1(B_0 \cup \overline{A}).$$

Here $\overline{A}$ is the set of formulae β for which there exists a formula

$\alpha \in A$ and an endomorphism φ of F such that $\beta = \varphi(\alpha)$.

To determine the closed classes $\mathfrak{K}$ of orthomodular lattices under the Galois-correspondence above we note first that all classes $\mathfrak{K}$ of the form $\mathfrak{K} = \text{mod } A$ form some set $A \subset F$ are obviously equational classes. Using standard techniques of universal algebra it is not difficult to prove the converse. Thus our theory of intermediate logics is equivalent with the equational theory of orthomodular lattices.

With respect to the other possible rules of inference we can prove

Theorem: There exists a set of formulae A which is closed under substitution, under $\Gamma_i (2 < i \leq 5)$ and which contains T, but for which $A \neq \text{form}(\text{mod } A)$.

The deduction theorem fails to hold in the OM-logic:

Theorem: There exist formulae α, β in F such that $\beta \in \Gamma_1(T \cup \{\alpha\})$, but $(\alpha \rightarrow_1 \beta) \notin T$ and $(\neg \alpha \vee \beta) \notin T$.

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