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ORDERING UNIFORM COMPLETIONS OF PARTIALLY ORDERED SETS

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We give here a discussion and summary of results whose proofs will appear elsewhere [6].

The familiar construction of the real numbers from the rational numbers via Cauchy sequences was put into its final form by Cantor [2]. In the 1930's, Weil [8] - an alternate view is Tukey's [7] - showed that the essence of this process was a certain similarity between neighbourhood systems of different points. In particular, a Hausdorff uniform space in the sense of [1] can be "completed" by a process which mimics the Cauchy sequence construction mentioned above, of the reals from the rationals.

Both the real numbers and the rational numbers are (additive) groups, and, in fact, the rational numbers are usually considered as a subgroup of the real numbers. It is well known (see, for example, [4]) that this extension of the group operation can be done in the more general case of Hausdorff uniform spaces, provided that the uniform structure and the group structure are sufficiently intertwined. Specifically, on any Hausdorff group with continuous group operations, there are several natural uniformities, at least one of whose completions can always be endowed with a group multiplication which extends the original operation.

Now the real numbers and the rational numbers are not only groups, but <u>totally ordered</u> groups as well. It seems reasonable, therefore, to ask for conditions whereby the order structure of a partially ordered set with

Hausdorff uniformity may be extended to the uniform completion of the set. As in the case of groups, it seems intuitively clear that some sort of connection between the order structure and the uniform structure must be assumed to ensure a satisfactory extension. That connection is provided by the idea of a uniform ordered space. This concept was introduced by Nachbin [5] who used it to investigate the ramifications of complete regularity on ordered sets with uniformities.

To define uniform ordered spaces, we need a little notation and a definition: Let X be a set; as usual, let

$$\Delta(X) = \{(x,x) \in X \times X \mid x \in X\}$$

be the <u>diagonal</u> of X. A <u>semi-uniform structure</u> on X is a filter $\mathcal{J}$ on $X \times X$ such that

(i) for all $V \in \mathcal{J}$, $\Delta(X) \subseteq V$;

(ii) for all $V \in \mathcal{J}$, there exists $U \in \mathcal{J}$ such that $U \circ U \subseteq V$.

Thus a semi-uniform structure is almost a uniformity: it lacks only a symmetric base. This requirement may be added by considering

$$\mathcal{J}^* = \{U \cap V^{-1} \mid U, V \in \mathcal{J}\} .$$

It is easy to see that $\mathcal{J}^*$ is a uniformity, which we call the <u>uniformity generated by</u> $\mathcal{J}$.

Let $(P, \leq)$ be a partially ordered set. Let

$$G(\leq) = \{(x,y) \in P \times P \mid x \leq y\}$$

be the <u>graph</u> of $\leq$. A <u>nearly uniform ordered space</u> is a partially ordered set $(P, \leq)$ with Hausdorff uniformity $\mathcal{U}$ such that there exists a semi-uniform structure $\mathcal{J}$ on P satisfying $\mathcal{J}^* = \mathcal{U}$ and $\bigcap \mathcal{J} \supseteq G(\leq)$. A uniform ordered space [5] is a nearly uniform ordered space with a semi-uniform structure $\mathcal{J}$ satisfying the conditions for a nearly uniform ordered space $(\mathcal{J}^* = \mathcal{U}, \bigcap \mathcal{J} \supseteq G(\leq))$ and additionally $\bigcap \mathcal{J} \subseteq G(\leq)$.

Every nearly uniform ordered space is locally convex, and there exist nearly uniform ordered spaces which are not uniform ordered spaces. Thus the concepts of partially ordered set with Hausdorff uniformity, nearly uniform ordered space, and uniform ordered space are distinct. In the totally ordered case, however, nearly uniform ordered spaces and uniform ordered spaces are the same.

Since nets are usually easier than filters to use where relations on sets are concerned, we will express our ideas and results in terms of nets rather than filters. This convention will be simplified by restricting ourselves to a single domain as follows: Let $(Y, \mathcal{U})$ be a Hausdorff uniform space. Let $(\tilde{Y}, \tilde{\mathcal{U}})$ be the completion of $(Y, \mathcal{U})$ at $\mathcal{U}$. Let $\tilde{y} \in \tilde{Y}$ and let $\{y_\delta \mid \delta \in \Delta\} \subseteq Y$ be a Cauchy net converging to $\tilde{y}$. If $\mathcal{U}^s$ is the set of symmetric entourages of $\mathcal{U}$ directed downwards, then there exists a Cauchy net $\{x_U \mid U \in \mathcal{U}^s\} \subseteq Y$, with domain $\mathcal{U}^s$, such that $\{x_U\}$ converges to $\tilde{y}$ and such that, as subsets of Y, $\{x_U\} \subseteq \{y_\delta\}$.

Let $(P, \mathcal{U})$ be a nearly uniform ordered space, with completion $(\tilde{P}, \tilde{\mathcal{U}})$ at $\mathcal{U}$. We might reasonably expect the following definition to extend the order on P to $\tilde{P}$: $\tilde{x} \preccurlyeq \tilde{y}$ if and only if for each Cauchy net $\{x_U\} \subseteq P$ converging to $\tilde{x}$, there exists a Cauchy net $\{y_U\} \subseteq P$ converging to $\tilde{y}$ such that $x_U \leq y_U$ for all $U \in \mathcal{U}^s$. However, this relation, which we call the strong order on $\tilde{P}$, does not necessarily extend the original order: the condition that $x_U \leq y_U$ is too restrictive. Thus we are led to weaken this definition as follows: Let $\mathcal{J}$ be a semi-uniform structure for P such that $\mathcal{J}^* = \mathcal{U}$ and $\bigcap \mathcal{J} \supseteq G(\leq)$. Then $\tilde{x} \leq_{\mathcal{J}} \tilde{y}$ if and only if for each Cauchy net $\{x_U\} \subseteq P$ converging to $\tilde{x}$, and for each $V \in \mathcal{J}$, there exists a Cauchy net $\{y_U\} \subseteq P$ converging to $\tilde{y}$ such that $(x_U, y_U) \in V$ for all $U \in \mathcal{U}^s$. (We call this relation the $\underline{\mathcal{J}\text{-order}}$ on P.) Thus, if $(P, \mathcal{U}, \leq)$ is a uniform ordered space, and if $\mathcal{J}$ is a semi-uniform structure for P such that $\mathcal{J}^* = \mathcal{U}$ and $\bigcap \mathcal{J} = G(\leq)$, then $\tilde{x} \leq_{\mathcal{J}} \tilde{y}$ means that for every net $\{x_U\} \subseteq P$ converging to $\tilde{x}$, we can find nets $\{y_U\} \subseteq P$ converging to $\tilde{y}$ which are as close as we want to being (pointwise) greater than or equal to $\{x_U\}$.

For every nearly uniform ordered space, the strong order and the $\mathcal{J}$-orders are partial orders on $\tilde{P}$. Clearly $G(\preccurlyeq) \subseteq G(\leq_{\mathcal{J}})$. Furthermore, if $(P, \mathcal{U}, \leq)$ is a uniform ordered space, then $G(\leq) = G(\leq_{\mathcal{J}}) \cap (P \times P)$ for any $\mathcal{J}$-order $\leq_{\mathcal{J}}$. Also, any $\mathcal{J}$-order makes the uniform completion of a nearly uniform ordered space into a uniform ordered space.

Now the uniform completion of a Hausdorff uniform space $(X, \mathcal{U})$ is "free" in the sense that it satisfies the following universal mapping property [1]: If $(Y, \mathcal{V})$ is any complete Hausdorff uniform space, and if f is any uniformly continuous function from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$, then there exists a unique uniformly continuous function $\tilde{f}$ from the uniform completion $(\tilde{X}, \tilde{\mathcal{U}})$ of $(X, \mathcal{U})$, to $(Y, \mathcal{V})$ such that the diagram

$$\begin{tikzcd}
(X, \mathcal{U}) \arrow[r, "i"] \arrow[dr, "f"'] & (\tilde{X}, \tilde{\mathcal{U}}) \arrow[d, "\tilde{f}"] \\
& (Y, \mathcal{V})
\end{tikzcd}$$

commutes, where i is the canonical embedding of X into $\tilde{X}$. In other words, the uniform completion provides an adjoint to the functor which embeds the category of complete Hausdorff uniform spaces in the category of Hausdorff uniform spaces.

If we hope to achieve a similar property for the ordered case, we must be able to pick out a particular semi-uniform structure with which to define an $\mathcal{J}$-order. We can do this as follows: Let $\mathcal{U}$ be a Hausdorff uniformity on a partially ordered set $(P, \leq)$. Let

$$\mathcal{J}(\mathcal{U}) = \{V \in \mathcal{U} \mid \text{there exist } V_1, V_2, \ldots \in \mathcal{U} \text{ such that } V = V_1, \text{ and for all } n, V_n \supseteq G(\leq) \text{ and } V_{n+1} \circ V_{n+1} \subseteq V_n\}.$$

Thus $\mathcal{J}(\mathcal{U})$ consists of all those entourages which could conceivably be members of a semi-uniform structure which generates $\mathcal{U}$ and whose intersection contains the graph of $\leq$. Then for every (nearly) uniform ordered space, $\mathcal{J}(\mathcal{U})$ is the unique maximal semi-uniform structure satisfying $(\bigcap \mathcal{J}(\mathcal{U}) \supseteq G(\leq)) \cap \mathcal{J}(\mathcal{U}) = G(\leq)$ and $\mathcal{J}(\mathcal{U})^* = \mathcal{U}$. Furthermore, if e is the embedding functor of the category of complete uniform ordered spaces into the category of uniform ordered spaces, both with uniformly continuous order-preserving functions, then the functor $\hat{e}$, which takes a uniform ordered space $(P, \mathcal{U}, \leq)$ to $(\tilde{P}, \tilde{\mathcal{U}}, \leq_{\mathcal{J}(\mathcal{U})})$ and a uniformly continuous function to its unique uniformly continuous extension, is adjoint to e.

The strong order is clearly easier to work with than any of the $\mathcal{J}$-orders; so it is of interest to enquire when the strong order is equivalent to some $\mathcal{J}$-order (and thus to the $\mathcal{J}(\mathcal{U})$-order). It turns out that if

$(P, \mathcal{U}, \leq)$ is a nearly uniform ordered space which satisfies

$$\text{(M) for all } V \in \mathcal{U}, \text{ there exists } W \in \mathcal{U}$$
$$\text{such that } W \circ G(\leq) \subseteq G(\leq) \circ V,$$

then the strong order is equivalent to the $\mathcal{J}(\mathcal{U})$-order. This condition (M) will in fact be satisfied for certain semilattices, lattices, and ℓ-groups.

We would hope that the extension procedure outlined thus far would take lattices to lattices and ℓ-groups to ℓ-groups. As long as the various operations are sufficiently well-behaved with respect to the uniform structure, this does indeed happen. Specifically, we say that a join-semilattice $(L, \vee)$ with Hausdorff uniformity $\mathcal{U}$ is a j-<u>uniform semilattice</u> in case $\vee$ is uniformly continuous with respect to $\mathcal{U}$. A <u>uniform lattice</u> $(L, \vee, \wedge, \mathcal{U})$ is defined similarly, with both operations uniformly continuous. Then a j-uniform semilattice $(L, \vee, \mathcal{U})$ is a uniform ordered space satisfying condition (M), and furthermore, the strong order on its uniform completion $\tilde{L}$ is the unique partial order on $\tilde{L}$ such that $\tilde{L}$ is a join-semilattice, L is a join-subsemilattice of $\tilde{L}$, and the join on $\tilde{L}$ is uniformly continuous. A similar result holds for uniform lattices.

The case for ℓ-groups answers most of a question raised by Conrad in [3]. (It was this question which led the present author to consider the general problem in the first place.) Consider an abelian ℓ-group with Hausdorff group and lattice topology $\mathfrak{J}$. Is the strong order on the completion $\tilde{B}$ of B at the usual uniformity the minimal lattice order on $\tilde{B}$ such that $\tilde{B}$ is an abelian ℓ-group, the positive cone of $\tilde{B}$ contains the positive cone of B, and the lattice operations on $\tilde{B}$ are continuous? With one assumption, some of our previous results may be used to give an affirmative answer in the following more general situation: Let B be an ℓ-group with Hausdorff group and lattice topology $\mathfrak{J}$. Let $\tilde{B}$ be its completion at one of the usual uniformities associated with $\mathfrak{J}$. If at least one of the lattice operations on B is uniformly continuous, then the strong order on $\tilde{B}$ is the minimal lattice order on $\tilde{B}$ whose graph contains that of $\leq$ and whose join is continuous. Furthermore, if $\tilde{B}$ is a group (at least one of whose lattice operations is uniformly continuous), then $(\tilde{B}, \lhd)$ is an ℓ-group.

The assumption of uniform continuity of at least one of the lattice operations is not at all stringent. In fact, for an ℓ-group B with group and lattice topology $\mathfrak{J}$, the following statements are equivalent:

(i) $\vee$ is uniformly continuous with respect to the right, left or two-sided uniformity;

(ii) $\wedge$ is uniformly continuous with respect to the right, left or two-sided uniformity;

(iii) the topology on B is locally convex.

In the non-locally convex case, it is difficult to see intuitively how to order the completion, and thus a requirement of local convexity does not really restrict the cases that one might expect to consider.

REFERENCES

1. N. Bourbaki, General Topology, Addison-Wesley Pub. Co., Don Mills, Ontario, 1966 (translated from the French, Hermann, Paris).

2. G. Cantor, Gesammelte Abhandlungen, Springer, Berlin, 1932.

3. P. Conrad, "The topological completion and the linearly compact hull of an abelian ℓ-group", (to appear).

4. J. L. Kelley, General Topology, D. Van Nostrand Co., Inc., New York, 1955.

5. L. Nachbin, Topology and Order, D. Van Nostrand Co., Inc., Princeton, 1965.

6. R. H. Redfield, "Ordering uniform completions of partially ordered sets", (to appear).

7. J. W. Tukey, Convergence and Uniformity in Topology, Ann. Math. Studies 2, Princeton University Press, 1940.

8. A. Weil, Sur les Espaces à Structure Uniforme et sur la Topologie Générale, Act. Scient. et Ind., no. 551, Hermann, Paris, 1937.

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