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TIGHT RESIDUATED MAPPINGS

by Erik A. Schreiner

1. Introduction. In this note we examine the connection between certain residuated mappings on a complete lattice and the property of complete distributivity. A map $T:L \longrightarrow M$ is residuated with T^+ as its residual if and only if the pair (T,T^+) forms a Galois connection between L and M^*, the dual of M. With this in mind we consider tight residuated mappings which correspond with the tight Galois connections introduced by G. N. Raney [7]. Since Res(L) is a semigroup and a complete lattice, we are able to compose and join tight residuated maps to extend the result of Raney.

In particular, we are able to characterize all complete homomorphisms with completely distributive images. Indeed, for any lattice, we present a method for finding the largest such complete homomorphism.

Tight residuated mappings abound. They are the pointwise join in Res(L,M) of certain basic tight mappings which, by their simplicity, help to illuminate what is occuring. In view of the fact that it is possible to construct a tight residuated map on an atomic Boolean lattice whose image is nonmodular, it is necessary to ask which tight mappings lead to a connection with complete distributivity and which basic tight mappings are the key ones. The answers are, respectively, idempotent tight maps and decreasing basic tight maps.

The image of an idempotent tight residuated map is completely distributive. Conversely, if the image of an arbitrary residuated map is completely distributive then one may find a tight idempotent with the same image. This is used to show that the completely distributive lattices are both injective and projective in the category of complete lattices with residuated mappings. Finally, the consideration of idempotent basic tight maps leads to some simple proofs of certain well-known results concerning atomic Boolean lattices.

2. Basic tight residuated maps. In this paper we consider only complete lattices. A map $T:L \longrightarrow M$ is called residuated if the inverse image of every principal ideal is a principal ideal. With complete lattices, this is equivalent to being a complete join homomorphism. The basic properties and facts concerning residuated maps may be found in the book of Blyth and Janowitz [1]. Let Res(L,M) denote the set of all residuated maps from L to M and Res(L) = Res(L,L). Both are complete lattices under pointwise order while the latter is also a Baer semigroup.

Z. Shmuely [8] has established a one-one correspondence between Res(L,M) and relations $\gamma \subseteq L \times M$ which satisfy:

(1) $(a,b) \in \gamma$, $x \leq a$, $y \geq b$ implies $(x,y) \in \gamma$.

(2) γ is a complete sublattice of $L \times M$.

For such a γ (called a G-relation) the map $T(a) = \wedge\{b \mid (a,b) \in \gamma\}$ is the associated residuated map. For a given $T \in Res(L,M)$, The set $\sigma(T) = \{(a,b) \mid T(a) \leq b\}$ is the corresponding G-relation.

Let $\theta \subseteq L\times M$ and define $\theta^+ = \{(x,y) \mid \text{for each } (a,b) \in \theta,\ x \leq a \text{ or } y \geq b\}$. Then θ^+ is a G-relation. Raney [7] defined his tight Galois connections in terms of relations of the form θ^+. (We have adjusted for the necessary dualization.)

<u>Definition</u> 1. A map $T \in Res(L,M)$ is <u>tight</u> if there exists a $\theta \subseteq L\times M$ such that $\sigma(T) = \theta^+$.

Let $\theta = \{(g,h)\}$. Then the tight map E_h^g obtained from θ^+ is called a <u>basic tight map</u> and is defined by:

$$E_h^g(x) = \begin{cases} 0 & x \leq g \\ h & \text{otherwise} \end{cases}$$

These maps are either nilpotent (if $h \leq g$) or idempotent (if $h \not\leq g$).

<u>Theorem</u> 2 (Shmuely). $T \in Res(L,M)$ <u>is tight if and only if</u> $T = \vee\{E_h^g \mid (g,h) \in \theta\}$ <u>for some</u> $\theta \subseteq L\times M$.

Proof: $\sigma(T) = \cap\{\sigma(E_h^g) \mid (g,h) \in \theta\} = \sigma(\vee\{E_h^g \mid (g,h) \in \theta\})$, where T is defined by θ^+.

Thus we focus our attention on basic tight maps. The set of basic tight maps. The set of basic tight maps and the set of all tight maps both form two sided ideals of the semigroup Res(L). Hence (ii) and (iii) are equivalent in the following theorem [7, Theorem 4].

Theorem 3 (Raney). *The following conditions are mutually equivalent*:

(i) *L is completely distributive*.

(ii) *The identity map* I_L *in* Res(L) *is tight*.

(iii) *All* T *in* Res(L) *are tight*.

It has come to my attention that D. Mowat [4] in his thesis considered basic tight maps (under the name "two point s.p. maps") and derived a similar result.

3. Decreasing basic tight maps. A map $T \in \mathrm{Res}(L)$ is *decreasing* if $T(x) \le x$ for all $x \in L$. A basic tight map E_b^a is decreasing if and only if the ordered pair (a, b) satisfies the condition: $x \not\le a$ implies $x \ge b$. Call such a pair (a,b) a *decreasing pair*. Pairs of the form (1,b) and (a,0) are always trivial decreasing pairs. The map $E_b^a = 0$ iff (a,b) is trivial. Let $\beta_2 = \beta_2(L) = \{(a,b) \mid (a,b) \text{ a nontrivial decreasing pair on } L.\}$ A central role in our study is played by maps of the form:

$$F = \vee\{E_b^a \mid (a,b) \in \beta_2\} .$$

As usual, if β_2 is empty F is the zero map. Note that F is a tight decreasing map in Res(L).

In terms of our basic tight decreasing maps we have the following critical result of Raney [7, Theorem 5].

<u>Lemma</u> 4. L <u>is completely distributive if and only if</u> $x \not\leq y$ <u>implies there exists a decreasing</u> E_b^a <u>in</u> Res(L) <u>with</u> $E_b^a(x) = b$, $E_b^a(y) = 0$ <u>and</u> $b \not\leq y$.

<u>Theorem</u> 5. L <u>is completely distributive if and only if the map</u> $F = \vee\left\{E_b^a \mid (a,b) \in \beta_2\right\} = I_L$ <u>in</u> Res(L).

Proof: Sufficiency follows from Theorem 3. Necessity is established using Lemma 4.

As we are interested in maps of the form of F above, the relation β_2 may contain surplus pairs. If $\left\{(a_i,b) \mid i \in I\right\} \subseteq \beta_2$ and $a = \wedge a_i$, then $(a,b) \in \beta_2$ and E_b^a dominates the other associated basic maps. Similarly, if $\left\{(a,b_i) \mid i \in I\right\} \subseteq \beta_2$ and $b = \vee b_i$, then $(a,b) \in \beta_2$ with E_b^a again an upper bound. The resulting pair (a,b) in either case may be called a <u>minimax</u> pair. Let $\beta_1 = \beta_1(L) = \left\{(a,b) \in \beta_2 \mid (a,b) \text{ is minimax}\right\}$. Note that $\vee\left\{E_b^a \mid (a,b) \in \beta_1\right\} = \vee\left\{E_b^a \mid (a,b) \in \beta_2\right\}$. The mappings eliminated in our transition from β_2 to β_1 were all nilpotent maps.

We are primarily interested in maps of the form $\vee\left\{E_b^a \mid (a,b) \in \theta \subseteq \beta_1\right\}$ which are idempotent. This will always be the case if all the E_b^a are idempotent. However, the elimination of all nilpotents is too drastic a step in the quest for an idempotent join. For if L is the closed unit interval [0,1] of the real numbers under the usual order, then L is completely distributive. The minimax decreasing pairs are all of the form (b,b) where $b \in (0,1)$. Thus all maps E_b^b are nilpotent yet $\vee\left\{E_b^b \mid b \in (0,1)\right\}$ is the identity map on L.

In order to eliminate the problems presented by isolated nilpotents in our study of idempotent tight maps we make one final adjustment to our relation. Let $F = \vee\left\{E_b^a \mid (a,b) \in \beta_1\right\}$. Define $\beta = \beta(L) = \left\{(a,b) \in \beta_1 \mid F(b) = b\right\}$. Note that if there are no nilpotent decreasing maps E_b^a, then $\beta = \beta_1$ but not conversely. For the remainder of the paper we shall use the notation:

$$E = \vee\left\{ E_b^a \mid (a,b) \in \beta\right\} .$$

We may then restate Theorem 5 as:

L <u>is completely distributive iff</u> E <u>is the identity in</u> Res(L).

4. Idempotent tight residuated maps.

Lemma 6. *For any complete lattice* L, E *is an idempotent decreasing tight map in* Res(L).

For an arbitrary map $T \in$ Res(L), the image T(L) = M has the following properties:

(1) $0 \in M$.

(2) $(M,\leq)$ is a complete lattice.

(3) The join in $(M,\leq)$ is the same as the join in L.

In general M is not a sublattice of L due to differing meet operations. Given a subset M of L satisfying the above properties, whether one can always find a $T \in \text{Res}(L)$ with $T(L) = M$ is an open question. The following theorem presents a partial answer. (See also Mowat [4, Theorem 17, page 42] for a related result.)

Theorem 7. *Let M be a subset of L satisfying (1), (2) and (3). Then there exists an idempotent tight map $T \in \text{Res}(L)$ with $T(L) = M$ if and only if $(M,\leq)$ is completely distributive.*

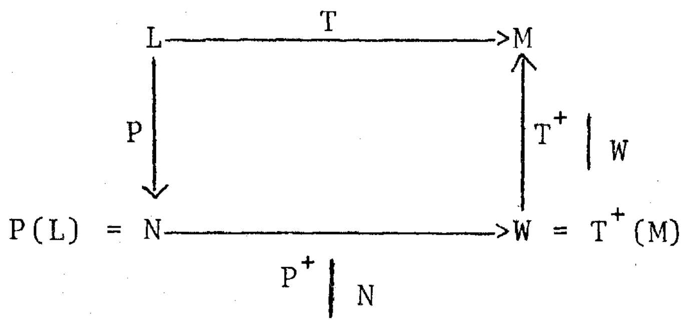
Proof: Given a tight idempotent T, the restriction of T to M is the identity map and may be shown to be tight as a map in Res(M). Thus $(M,\leq)$ is completely distributive by Theorem 3. Conversely, if $(M,\leq)$ is completely distributive, the identity map on M is $E_M = \vee \left\{ E_b^a \mid (a,b) \in \beta(M) \right\}$. Since each E_b^a may trivially be extended to a map in Res(L), the desired idempotent in Res(L) is $T = \vee \left\{ E_b^a \mid (a,b) \in \beta(M) \right\}$.

To see that the idempotency of T is essential, consider the Boolean lattice $\underline{2}^3$ with atoms a,b,c and respective complements $d = a'$, $e = b'$, $f = c'$. The image of the tight map $T = E_c^f \vee E_d^e \vee E_f^d$ is nonmodular.

In the next theorem we combine the fact that the residual T^+ sets up an isomorphism between the image of T and the image of T^+, appropriate duality and the extension of the identity used in Theorem 7.

Theorem 8. *Let* T $\in$ Res(L,M) *be an onto map. If* M *is completely distributive, then* $T = S \cdot P$ *where* P *is a tight idempotent in* Res(L) *and* S *is an isomorphism.*

Proof: The following commutative diagram may be obtained:



The restricted maps $P^+|_N$ and $T^+|_W$ are isomorphisms.

Corollary 9 (Crown [2]). *A completely distributive lattice is both injective and projective in the category of complete lattices with residuated maps.*

Proof: It is enough to show that if M is completely distributive, for every monomorphism P $\in$ Res(M,L) there is a T $\in$ Res (L,M) such that $T \cdot P = I_M$. This follows easily from

Theorem 7. Thus M is injective. Dually, or by use of Theorem 8, M is projective.

Theorem 10. <u>The map E is a complete homomorphism of L onto a completely distributive image. Moreover, E is the largest complete homomorphism with a completely distributive image.</u>

Proof: E is a decreasing idempotent, thus a complete homomorphism by [3, Theorem 3.6]. Since E is also tight, E(L) is completely distributive.

For any complete homomorphism onto a completely distributive image, consider the complete congruence Θ generated and the associated lattice $L/_\Theta$. The identity map on $L/_\Theta$ is generated from pairs $(\bar{a},\bar{b})$ in $\beta(L/_\Theta)$. These may be pulled back to obtain pairs (a,b) in $\beta(L)$. The given homomorphism thus was of the form $\vee\left\{ E^a_b \,\middle|\, (a,b) \in \theta \subseteq \beta(L) \right\}$ which is less than E in Res(L).

Thus for each $\theta \subseteq \beta$ such that $\vee\left\{ E^a_b \,\middle|\, (a,b) \in \theta \right\}$ is idempotent we obtain a complete homomorphism with completely distributive image, and all such homomorphisms are of this form. If distinct subrelations of β give rise to distinct maps, as will be the case if there are no nilpotents associated with β, we have a means of enumerating such homomorphisms.

We now turn our attention to idempotent basic decreasing maps. An element b is the image of an idempotent decreasing E_b^a if and only if [b,1] is a completely prime dual ideal, that is, in Raney's terms, b is a completely join-irreducible element. His result [5, Theorem 2] may thus be stated in terms of decreasing maps.

Theorem 11. *L is isomorphic to a complete ring of sets if and only if* $I_L = E$ *and* $\beta(L)$ *contains no nilpotent pairs*.

If L is (dual) semicomplemented, all decreasing E_b^a are idempotents. They may be characterized by the conditions: b is an atom, a is a dual atom, a and b are complements, and a is a distributive element. Combining this with the properties of the map E provides simple proofs of the following well-known results.

Theorem 12. *A (dual) semicomplemented completely distributive lattice is an atomic Boolean lattice*.

Theorem 13. *Any two of the following conditions on a complete lattice imply the third*.

(i) L *is atomistic*.

(ii) L *is completely distributive*.

(iii) L *is a Boolean lattice*.

Finally, we combine these observations with the remark following Theorem 10.

Theorem 14. Let L be a complete (dual) semicomplemented lattice. Then every completely distributive complete homomorphic image of L is an atomic Boolean lattice.

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Western Michigan University