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On Subalgebras of Partial Universal Algebras

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For a partial universal algebra, the projection of a closed subalgebra of its n^{th} direct power onto its k^{th} diredt power $(1 \leq k < n)$ is not always a closed subalgebra. Partial universal algebras for which this projection of a closed subalgebra is always a closed subalgebra are called here (n, k) correct. A similar notion of $(2, 1)$ correctness was introduced in [2]. Given any set A, the subalgebra systems of $\langle A;F\rangle^n$ for any set F of partial operations on A was described in [4]. In this note we describe the subalgebra systems for (n, k) correct partial universal algebras. Some of the results were announced in [3].

By algebras we shall mean partial universal algebras. By a subalgebra will always be meant a closed subalgebra.

Let k, n be integers such that $1 \leq k < n$. An algebra $A = \langle A;F\rangle$ is said to be (n, k) correct if for any $f \in F$, $a_1, \ldots, a_m \in A^n$ such that $f(a_{1i}, \ldots, a_{mi})$ is defined for all $1 \leq i \leq k$ (a_{si} is the i^{th} component of a_s), there is an m-place polynomial p in F such that $p(a_1, \ldots, a_m)$ is defined, and $p(a_{1i}, \ldots, a_{mi}) = f(a_{1i}, \ldots, a_{mi})$ for all $1 \leq i \leq k$. An algebra will be called n-correct if it is (n, k) correct for all $1 \leq k < n$.

It is clear that full algebras are n-correct for all n. Every full [1] homomorphic image of an (n, k) correct algebra is also (n, k) correct. The same is true for all quotient algebras. Given an (n, k) correct algebra $\langle A;F\rangle$, if p is an r-place polynomial in F and $a_1, \ldots, a_r \in A^n$ such that $p(a_{1i}, \ldots, a_{ri})$ is defined for all $1 \leq i \leq k$, then there is an r-place polynomial q in F such that $q(a_1, \ldots, a_r)$ is defined, and moreover $q(a_{1i}, \ldots, a_{ri}) = p(a_{1i}, \ldots, a_{ri})$ for all $1 \leq i \leq k$. An algebra is (n, k) correct iff the projection of any subalgebra of its n^{th} direct power is again a subalgebra; it is also sufficient to consider only finitely generated subalgebras. It is also evident that any subalgebra of an (n, k) correct algebra is also (n, k) correct. An (n, k) correct algebra is also (n, m) correct for all $k < m < n$.

If $1 \leq k < m < n$, then every (n, m) correct algebra is (n, k) correct. In fact if B is a subalgebra of $\langle A;F \rangle^n$ and C is the projection of B onto A^k (first k components) and D is the projection of B onto A^{k+1} (first k + 1 components), then $A \times C$ is the projection of $A \times D \times A^{(n-k-2)}$ onto A^{k+i}; $A \times C$ is a subalgebra of $\underline{A}^{k+1}$ iff C is a subalgebra of $\underline{A}^k$.

For every $k > 0$ there is an algebra which is (n, k) correct for all $n > k$ and not (n, k+1) correct for any $n > k + 1$.

Let A be the set of all positive integers. Set $F = \{f, g_1, \ldots, g_{k+1}\}$, where all elements of F are unary and all $g_1, \ldots, g_{k+1}$ are full and the domain of definition is $\{1, \ldots, k+1\} = K$.

$$f(j) = j \quad \text{if} \quad 1 \leq j \leq k$$

$$f(k+1) = k+2.$$

Denote by K_i the complement of $\{i\}$ in K.

$$g_i(m) = \begin{cases} f(m) & \text{if } m \in K_i \\ f(\min K_i) & \text{if } m \notin K_i, \end{cases} \qquad 1 \le i \le k+1.$$

It is obvious that $\langle A;F \rangle$ is (n,k) correct for all $n > k$. If $\underline{A}$ were $(k+2, k+1)$ correct then there would exist a polynomial $p = h_1 \cdots h_s$ $(h_1, \ldots, h_s \in F)$ with $p((1,2,\ldots,k,k+1,k+2)) = (1,2,\ldots,k,k+2,x)$, (since $f((1,2,\ldots,k,k+1))$ is defined). Since f is not defined at $k+2$ then $h_s = g_i$ for some i. So $(1,2,\ldots,k,k+2,x) = h_1(h_2(\cdots h_{s-1}(g_i((1,2,\ldots,k,k+1,k+2)))\cdots))$. But g_i identifies two of the first $k+1$ components of the tuple and the application of $h_{s-1}, \ldots, h_1$ will leave these two components equal. Yet $1, \ldots, k, k+2$ are all distinct. So there is no such p and $\underline{A}$ is not $(k+2, k+1)$ correct.

To say that $\langle A;F \rangle^2$ is (n,k) correct is the same as to say that $\langle A;F \rangle$ is $(2n, 2k)$ correct. Hence direct products of (n,k) correct algebras are not always (n,k) correct.

A^k can be injected into A^n (for $n > k$) in such a way that the image of A^k is always a subalgebra of $\underline{A}^n$ (e.g., $(a_1, \ldots, a_k) \to (a_1, \ldots, a_k, a_k, \ldots, a_k)$ is such an injection). Under such an injection every subalgebra of $\underline{A}^k$ appears as a subalgebra of $\underline{A}^n$. Thus in an (n,k) correct algebra the injection of the projection (onto A^k) of a subalgebra of $\underline{A}^n$ is again a subalgebra of $\underline{A}^n$. Such a condition turns out to be sufficient. This is more precisely made in the following definition and lemma.

If α is a nonvoid subset of $\{1, \ldots, n\}$ and $i = \min \alpha$ define [4]:

$$B\alpha = \{a : a \in A^n,\ a_j = b_j \text{ if } j \notin \alpha,\ a_j = b_i \text{ if } j \in \alpha, \text{ for some } b \in B\} \qquad B \subseteq A^n.$$

If $C \subseteq A^n$ denote by $[C]$ the subalgebra of $\langle A;F\rangle^n$ generated by C.

Lemma: $\langle A;F\rangle$ is (n,k) correct iff $[C]\alpha = [C\alpha]$ for all finite nonvoid $C \subseteq A^n$ and for all $\alpha \subseteq \{1,\ldots,n\}$ with cardinality $n-k+1$.

If s is a permutation on $\{1,\ldots,n\}$ define [4]:

$$Bs = \{a : a \varepsilon A^n,\ a_i = b_{s^{-1}(i)},\ 1 \leq i \leq n \text{ for some } b \varepsilon B\}, \quad B \subseteq A^n.$$

If $\underline{A}$ is an algebra $S(\underline{A})$ is the family of all subalgebras of $\underline{A}$.

Theorem: Let $S \subseteq P(A^n)$. $S = S(\langle A;F\rangle^n)$ for some (n,k) correct algebra $\langle A;F\rangle$ iff:

(a) S is an algebraic closure system on A^n

(b) if $B \varepsilon S$, $1 \leq i < j \leq n$ then $B(ij) \varepsilon S$

(d) $[C]_S\{1,2\} \subseteq [C\{1,2\}]_S$ for all finite nonvoid $C \subseteq A^n$

(e) if $\phi \varepsilon S$ then $\phi = \bigcap \{B : B \varepsilon S,\ B \neq \phi\}$

(f) $[C]_S\{1,\ldots,n-k+1\} = [C\{1,\ldots,n-k+1\}]_S$ for all nonvoid finite $C \subseteq A^n$,

where $[C]_S$ is the intersection of all elements of S containing C.

In [4] it was shown that $S = S(\langle A;F\rangle^n)$ for some partial algebra $\langle A;F\rangle$ iff S satisfies conditions (a), (b), (c), (d), (e), where (c) is $\Delta_2 \times A^{n-2} \varepsilon S$ (Δ_2 is the diagonal in A^2). So the necessity of (a), (b), (d), (e) and (f) follows from this result and the lemma. The sufficiency will be established once we show

Claim: Let r be an integer such that $1 < r \leq n$. If S satisfies (a), (b), (d) and (f') $$[C]_S\{1, \ldots, r\} = [C\{1, \ldots, r\}]_S$$

for all finite nonvoid $C \subseteq A^n$, then $\Delta_2 \times A^{n-2} \varepsilon S$.

By (a) S satisfies (f') for all $C \subseteq A^n$. Thus

$$[\Delta_2 \times A^{n-2}]_S\{2, 3, \ldots, r\} \subseteq [(\Delta_2 \times A^{n-2})\{2, \ldots, r\}]_S = [\Delta_r \times A^{n-r}]_S .$$

But

$$[\Delta_r \times A^{n-r}]_S\{1, \ldots, r\} = [(\Delta_r \times A^{n-r})\{1, \ldots, r\}]_S = [\Delta_r \times A^{n-r}]_S$$

by (f'). Also

$$[\Delta_r \times A^{n-r}]_S\{1, \ldots, r\} \subseteq A^n\{1, \ldots, r\} = \Delta_r \times A^{n-r}.$$

Hence

$$\Delta_r \times A^{n-r} \subseteq [\Delta_r \times A^{n-r}]_S \subseteq \Delta_r \times A^{n-2},$$

i.e., $\Delta_r \times A^{n-r} \varepsilon S$. So

$$[\Delta_2 \times A^{n-2}]_S\{2, \ldots, r\} \subseteq [\Delta_r \times A^{n-r}]_S = \Delta_r \times A^{n-r}.$$

From which we deduce $[\Delta_2 \times A^{n-2}]_S \subseteq \Delta_2 \times A^{n-2}$, i.e., $\Delta_2 \times A^{n-2} \varepsilon S$.

<u>Corollary</u>: Let $S \subseteq P(A^n)$. $S = S(\langle A;F\rangle^n)$ for some n-correct algebra $\langle A;F\rangle$ iff S satisfies (a), (b), (e) and

$$[C]_S\{1, 2\} = [C\{1, 2\}]_S \quad \text{for all nonvoid finite } C \subseteq A^n.$$

This follows from the theorem since n-correctness is equivalent to $(n, n-1)$ correctness and in this case condition (f) implies condition (d).

There are 2-correct partial algebras $\langle A;F\rangle$, for which $S(\langle A;F\rangle^2) \neq S(\langle A;G\rangle^2)$ for any set of full operations G on A. Thus problem 19 of [1] for full universal algebras remains open.

If $S \subseteq P(A^n)$ satisfies (a), (b) and (c), then S satisfies (d) iff S satisfies

(g) if $B \varepsilon S$, $B \subseteq \Delta_2 \times A^{n-2}$, then $A \times pr_{2\ldots n}B \varepsilon S$,

where $pr_{2\ldots n}B$ is the projection of B onto the last $n-1$ components.

In other words, conditions (a), (b), (c), (e) and (g) give another characterization for $S(\langle A;F\rangle^n)$.

Let S satisfy (d). By (a), $[C]_S\{1,2\} \subseteq [C\{1,2\}]_S$ for all $C \subseteq A^n$. Let now $B \varepsilon S$, $B \subseteq \Delta_2 \times A^{n-2}$.

$$[(A \times pr_{2\ldots n}B)(12)]_S\{1,2\} \subseteq [((A \times pr_{2\ldots n}B)(12))\{1,2\}]_S = [B]_S = B.$$

Hence

$$A \times pr_{2\ldots n}B \subseteq [A \times pr_{2\ldots n}B]_S \subseteq A \times pr_{2\ldots n}B$$

i.e.,

$$A \times pr_{2\ldots n}B = [A \times pr_{2\ldots n}B]_S \ \varepsilon\ S.$$

Conversely, let S satisfy (g) and $\phi \neq C \subseteq A^n$. Then

$$[C]_S\{1,2\} \subseteq \Delta_2 \times A^{n-2} \ \varepsilon\ S.$$

Hence

$$B = [C\{1,2\}]_S \subseteq \Delta_2 \times A^{n-2}, \quad E = A \times pr_{2\cdots n} B \in S \quad \text{and} \quad G = E(12) \in S.$$

But $C \subseteq G \in S$. Hence $[C]_S \subseteq G$. Thus

$$[C]_S\{1,2\} \subseteq G\{1,2\} = ((A \times pr_{2\cdots n}[C\{1,2\}]_S)(12))\{1,2\} = [C\{1,2\}]_S .$$

A homomorphism h of a join semilattice $\underline{L}$ onto a join semilattice $\underline{L}'$ is said to be correct [2] if for any $a \in L$, $b' \in L'$, $b' < ah$, there is $b \in L$ such that $b \leq a$ and $bh = b'$. h is correct iff h maps ideals of $\underline{L}$ onto ideals of $\underline{L}'$. The mapping $B \to [pr_{1\cdots k}B]$ is a complete semilattice homomorphism of the join semilattices of all subalgebras of $\langle A;F\rangle^n$ onto that of $\langle A;F\rangle^k$. If $\underline{A}$ is (n,k) correct this homomorphism is correct; restricted to finitely generated subalgebras, this mapping remains correct.

References

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