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Free products and reduced free products of lattices

by

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1. The purpose of this lecture ** is to direct your attention to a series of papers dealing with the structure of free products of lattices and its applications. Some of the basic ideas go back to P.M. Whitman [17] and R.P. Dilworth [2]. The structure theorem is due to G. Grätzer, H. Lakser, and C.R. Platt [10] and it was to some extent extended by B. Jónsson [14]. Some applications use reduced free products which again go back to R.P. Dilworth [2], and were developed in C.C. Chen and G. Grätzer [1], G. Grätzer [7] and further applied in G. Grätzer and J. Sichler [11] and [12].

In view of the fact that a full proof of the structure theorem has never been given I will state and prove the structure theorem in full detail in §2. Some applications are given without proof in §3. A new approach to reduced free products is given in §4 again with full proofs in view of the fact that the result presented is more general than the one in G. Grätzer [7]. Mostly without proofs, applications are given in §5.

2. For this whole section, let L_i, $i \in I$, be a fixed family of lattices; we assume that L_i and L_j are disjoint for $i, j \in I$, $i \neq j$. We set $Q = \bigcup(L_i, i \in I)$ and we consider Q a poset under the following partial ordering:

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for $a, b \in Q$ let $a \leq b$ iff $a, b \in L_i$ for some $i \in I$ and $a \leq b$ in L_i.

A free product L of the L_i, $i \in I$, is a free lattice generated by Q, $F(Q)$ $(= F_{\underset{\sim}{L}}(Q))$ (in the sense of Definition 5.2 of [5]).

Or, equivalently,

Definition 1. The lattice L is a free product of the lattices L_i, $i \in I$, iff the following conditions are satisfied:

(i) each L_i is a sublattice of L and for $i, j \in I$, $i \neq j$, L_i and L_j are disjoint.

(ii) L is generated by $\bigcup(L_i, i \in I)$.

(iii) for any lattice A, for any family of homomorphisms $\varphi_i: L_i \to A$, there exists a homomorphism $\varphi: L \to A$ such that φ on L_i agrees with φ_i for all $i \in I$.

The next definition is a slight adaptation of Definition 4.1 of [5].

Definition 2. Let X be an arbitrary set. The set $\underset{\sim}{P}(X)$ of polynomials in X is the smallest set satisfying (i) and (ii):

(i) $X \subseteq \underset{\sim}{P}(X)$.

(ii) If $p, q \in \underset{\sim}{P}(X)$, then $(p \wedge q), (p \vee q) \in \underset{\sim}{P}(X)$.

The reader should keep in mind that a polynomial is a sequence of symbols and equality means formal equality. As before, parentheses will be dropped whenever there is no danger of confusion.

In what follows, we shall deal with polynomials in $Q = \bigcup(L_i, i \in I)$. Let $a, b, c \in L_i$, $a \vee b = c$. Observe, that as polynomials in Q, $a \vee b$ (which stands for $(a \vee b)$) and c are distinct.

For a lattice A, we define $A^b = A \cup \{0^b, 1^b\}$, where $0^b, 1^b \notin A$; we order A by the rules:

$0^b < x < 1^b$ for all $x \in A$.

$x \leq y$ in A^b iff $x \leq y$ in A, for $x, y \in A$. Thus A^b is a bounded lattice (§6 of [5]). Note, however, that $A^b \neq A$ even if A was itself bounded. It is important to observe that 0^b is meet-irreducible and 1^b is join-irreducible. Thus if $a \wedge b = 0^b$ then either a or b is 0^b, and dually. This will be quite important in subsequent computations.

<u>Definition 3.</u> Let $p \in \underset{\sim}{P}(Q)$ and $i \in I$. The <u>upper</u> i-<u>cover</u> of p, in notation, $p^{(i)}$, is an element of $(L_i)^b$ defined as follows:

(i) for $a \in Q$ we have $a \in L_j$ for exactly one j; if $j = i$, then $a^{(i)} = a$; if $j \neq i$, then $a^{(i)} = 1^b$.

(ii) $(p \wedge q)^{(i)} = p^{(i)} \wedge q^{(i)}$ and $(p \vee q)^{(i)} = p^{(i)} \vee q^{(i)}$ where $\wedge$ and $\vee$ on the right hand side of these equations is to be taken in $(L_i)^b$.

The definition of the <u>lower</u> i-<u>cover</u> of p, in notation, $p_{(i)}$, is analogous, with 0^b replacing 1^b in (i).

An upper cover or a lower cover is <u>proper</u> if it is not 0^b or 1^b. Observe that, however, no upper cover is 0^b and no lower cover is 1^b.

<u>Corollary 4.</u> For any $p \in \underset{\sim}{P}(Q)$ and $i \in I$ we have that

$$p_{(i)} \leq p^{(i)},$$

and if $p_{(i)}$ and $p^{(j)}$ are proper and $p_{(i)} \leq p^{(j)}$, then $i = j$.

<u>Proof.</u> If $p \in X$, then $p = p_{(i)} = p^{(i)}$ so the first statement is true. If the first statement holds for p and q, then

$$(p \wedge q)_{(i)} = p_{(i)} \wedge q_{(i)} \leq p^{(i)} \wedge q^{(i)} = (p \wedge q)^{(i)},$$

and so the first statement holds for $p \wedge q$ and similarly for $p \vee q$. To prove the second statement it is sufficient to verify that if $p_{(i)}$ is proper, then $p^{(j)}$ is not proper for any $j \neq i$. This is obvious for $p \in Q$ by 3(i). If $p = q \wedge r$, and $p_{(i)}$ is proper, then both $q_{(i)}$ and $r_{(i)}$ are proper, hence $q^{(j)} = r^{(j)} = 1^b$, and so $p^{(j)} = 1^b$. Finally, if $p = q \vee r$ and $p_{(i)}$ is proper, then $q_{(i)}$ or $r_{(i)}$ is proper, hence $q^{(j)} = 1^b$ or $r^{(j)} = 1^b$, ensuring $p^{(j)} = q^{(j)} \vee r^{(j)} = 1^b$, completing the proof.

Finally, we introduce a quasi-ordering of $\underset{\sim}{P}(Q)$.

<u>Definition 5.</u> For $p, q \in \underset{\sim}{P}(Q)$, set $p \subseteq q$ iff it follows from rules (i) - (vi) below:

(i) $p = q$.

(ii) For some $i \in I$, $p^{(i)} \leq q_{(i)}$.

(iii) $p = p_0 \wedge p_1$ where $p_0 \subseteq q$ or $p_1 \subseteq q$.

(iv) $p = p_0 \vee p_1$ where $p_0 \subseteq q$ and $p_1 \subseteq q$.

(v) $q = q_0 \wedge q_1$ where $p \subseteq q_0$ and $p \subseteq q_1$.

(vi) $q = q_0 \vee q_1$ where $p \subseteq q_0$ or $p \subseteq q_1$.

Definition 5 gives essentially the algorithm we have been looking for. For $p, q \in \underset{\sim}{P}(Q)$, it will be shown that p and q represent the same element of the free product iff $p \subseteq q$ and $q \subseteq p$. We shall show this by actually exhibiting the free product as the set of equivalence classes of $\underset{\sim}{P}(Q)$ under this relation. To be able to do this we have to establish a number of properties of the relation $\subseteq$. All the proofs are by induction and will use the rank of a $p \in \underset{\sim}{P}(Q)$ (see §4 of [5]): for $p \in Q$, $r(p) = 1$; $r(p \wedge q) = r(p \vee q) = r(p) + r(q)$.

Lemma 6. Let $p, q, r \in \underset{\sim}{P}(Q)$ and $i \in I$.

(i) $p \subseteq q$ implies that $p_{(i)} \leq q_{(i)}$ and $p^{(i)} \leq q^{(i)}$.

(ii) $p \subseteq q$ and $q \subseteq r$ implies that $p \subseteq r$.

Proof. Let $p \subseteq q$; we shall prove $p_{(i)} \leq q_{(i)}$ by induction on $r(p) + r(q)$. If $r(p) + r(q) = 2$, then $p, q \in Q$ and so only 5(i) or 5(ii) is applicable to $p \subseteq q$. Hence either $p = q$, in which case $p_{(i)} = q_{(i)}$ or $p^{(j)} \leq q_{(j)}$ for some $j \in I$. This implies that $p^{(j)}$ and $q_{(j)}$ are proper, hence $p = p^{(j)}$, $q = q_{(j)}$, and $p \leq q$. Therefore, $p_{(i)} = p \leq q = q^{(i)}$ if $i = j$, and $p_{(i)} = 0^b \leq 1^b = q^{(i)}$ if $i \neq j$.

Now assume that the implication has been proved for all $p' \subseteq q'$ with $r(p') + r(q') < r(p) + r(q)$.

If $p \subseteq q$ follows from 5(i), then $p = q$, and so $p_{(i)} = q_{(i)}$.

If $p \subseteq q$ follows from 5(ii), then $p^{(j)} \leq q_{(j)}$ for some $j \in I$. If $j = i$, then by Corollary 4 $p_{(i)} \leq p^{(i)} \leq q_{(i)}$, which was to be proved. If $j \neq i$, then by Corollary 4 $p_{(i)} = 0^b$, hence $p_{(i)} \leq q_{(i)}$ is obvious.

If $p \subseteq q$ follows from 5(iii), then $p = p_0 \wedge p_1$ where $p_0 \subseteq q$ or $p_1 \subseteq q$, say $p_0 \subseteq q$. Thus $(p_0)_{(i)} \leq q_{(i)}$ and so $p_{(i)} = (p_0)_{(i)} \wedge (p_1)_{(i)} \leq (p_0)_{(i)} \leq q_{(i)}$.

If $p \subseteq q$ follows from 5(iv), then $p = p_0 \vee p_1$ where $p_0 \subseteq q$ and $p_1 \subseteq q$. Hence $(p_0)_{(i)} \leq q_{(i)}$ and $(p_1)_{(i)} \leq q_{(i)}$ and so $(p)_{(i)} = (p_0)_{(i)} \wedge (p_1)_{(i)} \leq q_{(i)}$.

If 5(v) or 5(vi) is applicable to $p \subseteq q$, the proof is analogous to the last two cases.

The proof of $p^{(i)} \leq q^{(i)}$ follows by duality.

To prove (ii), let $p \subseteq q$ and $q \subseteq r$. We shall proceed by induction on $a = r(p) + r(q) + r(r)$. If $a = 3$, then $p, q, r \in Q$. If $p = q$ or $q = r$, then $p \subseteq r$ is obvious; otherwise, $p, q, r \in L_i$ for some $i \in I$ and $p \leq r$, so $p \subseteq r$ follows from 5(ii).

Now assume the statement true for sums smaller than a. We can further assume that $p \neq q$ and $q \neq r$.

If $p \subseteq q$ follows from 5(ii), then $p^{(i)} \leq q_{(i)}$ for some $i \in I$. Since $q \subseteq r$, by Corollary 4, $q_{(i)} \leq r_{(i)}$, hence $p^{(i)} \leq r_{(i)}$. Thus $p \subseteq r$, by 5(ii).

If $p \subseteq q$ follows from 5(iii), then $p = p_0 \wedge p_1$ where $p_0 \subseteq q$ or $p_1 \subseteq q$. Thus, by the induction hypotheses, $p_0 \subseteq r$ or $p_1 \subseteq r$, and so by 5(iii), $p_0 \vee p_1 = p \subseteq r$.

If $p \subseteq q$ follows from 5(iv), then $p = p_0 \vee p_1$, $p_0 \subseteq q$ and $p_1 \subseteq q$, and so again $p_0 \subseteq r$ and $p_1 \subseteq r$, implying $p_0 \vee p_1 = p \subseteq r$ by 5(iv).

If $q \subseteq r$ follows from 5(v) or 5(vi) we can proceed dually (that is, by interchanging $\wedge$ and $\vee$). Only two cases remain; since the second is the dual of the first, we shall state only one:

$q = q_0 \wedge q_1$, 5(v) applies to $p \subseteq q$, and 5(iii) is applicable to $q \subseteq r$ (observe that 5(iv) is not applicable). In this case, 5(v) yields $p \subseteq q_0$ and $p \subseteq q_1$ and 5(iii) yields $q_0 \subseteq r$ or $q_1 \subseteq r$. Hence $p \subseteq q_i \subseteq r$ for $i = 0$ or 1, hence by the induction hypotheses, $p \subseteq r$.

Since by 5(i), $p \subseteq p$ for any $p \in \underset{\sim}{P}(Q)$, the relation $\subseteq$ is a quasi-ordering and so (see Exercise 2.28 of [5]) we can define

$$p \equiv q \text{ iff } p \subseteq q \text{ and } q \subseteq p \quad (p, q \in \underset{\sim}{P}(Q)).$$

$$R(p) = \{q \mid q \in \underset{\sim}{P}(Q) \text{ and } p \equiv q\} \quad (p \in \underset{\sim}{P}(Q)).$$

$$R(Q) = \{R(p) \mid p \in \underset{\sim}{P}(Q)\}.$$

$$R(p) \leq R(q) \text{ if } p \subseteq q.$$

In other words, we split $\underset{\sim}{P}(Q)$ into blocks under the equivalence relation $p \equiv q$; $R(Q)$ is the set of blocks which we partially order under $\leq$.

Lemma 7. $R(Q)$ is a lattice, in fact,

$$R(p) \wedge R(q) = R(p \wedge q) \quad \text{and} \quad R(p \vee q) = R(p) \vee R(q).$$

Furthermore, if $a, b, c, d \in L_i$, $i \in I$, and $a \wedge b = c$, $a \vee b = d$ in L_i, then

$$R(a) \wedge R(b) = R(c) \quad \text{and} \quad R(a) \vee R(b) = R(d).$$

Proof. $p \wedge q \subseteq p$ and $p \wedge q \subseteq q$ by 5(iii). If $r \subseteq p$ and $r \subseteq q$, then $r \subseteq p \wedge q$ by 5(v); this argument and its dual give the first statement.

$c \subseteq a$ and $c \subseteq b$ is obvious by 5(ii), hence $R(c) \leq R(a)$ and $R(c) \leq R(b)$ Now let $R(p) \leq R(a)$ and $R(p) \leq R(b)$ for some $p \in \underset{\sim}{P}(Q)$. Then $p \subseteq a$ and $p \subseteq b$, and so by Lemma 6 $p^{(i)} \leq a^{(i)} = a$ and $p^{(i)} \leq b^{(i)} = b$. Therefore $p^{(i)} \leq c = c_{(i)}$ and thus $p \subseteq c$ by 5(ii). The second part follows by duality.

Let $p, q \in L_i$ $i \in I$ and $R(p) = R(q)$. Then $p \subseteq q$ and $q \subseteq p$. Since only 5(i) and 5(ii) can be applied to these, we easily conclude that $p \leq q$ and $q \leq p$, hence $p = q$. Thus by Lemma 7

$$p \rightarrow R(p) \quad p \in L_i$$

is an embedding of L_i into $R(Q)$. Therefore, by identifying $p \in L_i$ with $R(p)$ we get each L_i as a sublattice of $R(Q)$ and hence $Q \subseteq R(Q)$. It is also obvious that the partial ordering induced by $R(Q)$ on Q agrees with the original partial ordering.

Theorem 8. $R(Q)$ is a free product of the L_i, $i \in I$.

Proof. 1(i) and 1(ii) have already been observed. Let $Q_i: L_i \to A$ be given for all $i \in I$. We define inductively a map

$$\psi: \underset{\sim}{P}(Q) \longrightarrow A$$

as follows: for $p \in Q$ there is exactly one $i \in I$ with $p \in L_i$; set $p\psi = p\varphi_i$; if $p = p_0 \wedge p_1$ or $p = p_0 \vee p_1$, $p_0\psi$ and $p_1\psi$ have already been defined, thus set $p\psi = p_0\psi \wedge p_1\psi$ and $p\psi = p_0\psi \vee p_1\psi$, respectively. Now we prove:

(i) If $p_{(i)}$ is proper, then $p_{(i)}\psi \le p\psi$.

Lemma 9. For $p \in \underset{\sim}{P}(Q)$ and $i \in I$.

(ii) If $p^{(i)}$ is proper, then $p\psi \le p^{(i)}\psi$ for $p \in P(Q)$ and $i \in$

(iii) $p \subseteq q$ implies that $p\psi \le q\psi$ for $p, q \in \underset{\sim}{P}(Q)$.

Proof. (i) If $p \in Q$ and $p_{(i)}$ is proper, then $p \in L_i$, hence $p = p_{(i)}$ and so $p_{(i)}\psi \le p\psi$ is obvious. The induction step is obvious by 3(ii).

(ii) This follows by duality from (i).

(iii) If $p, q \in Q$, then $p, q \in L_i$ for some $i \in I$ and $p \le q$. Therefore, $p\varphi_i \le q\varphi_i$, and so $p\psi \le q\psi$.

If $p \subseteq q$ follows from 5(i), then $p\psi = q\psi$.

If $p \subseteq q$ follows from 5(ii), then, for some $i \in I$, $p^{(i)} \le q_{(i)}$. Thus $p^{(i)}$ and $q_{(i)}$ are proper. Therefore, $p\psi \le p^{(i)}\psi$ by (ii), $p^{(i)}\psi \le q_{(i)}\psi$ because $p^{(i)}$ and $q_{(i)} \in Q$, and $q_{(i)}\psi \le q\psi$ by (i), implying $p\psi \le q\psi$.

If $p \subseteq q$ follows from 5(iii), then $p = p_0 \wedge p_1$ where $p_0 \subseteq q$ or $p_1 \subseteq q$. Hence $p_0\psi \leq q\psi$ or $p_1\psi \leq q\psi$, therefore $p\psi = p_0\psi \wedge p_1\psi \leq q\psi$.

If $p \subseteq q$ follows from 5(iv) - 5(vi), the proof is analogous to the last one.

Now take a $p \in \underset{\sim}{P}(Q)$ and define

$$R(p)\varphi = p\psi.$$

φ is well-defined since if $R(p) = R(q)$ $(p, q \in \underset{\sim}{P}(Q))$, then $p \subseteq q$ and $q \subseteq p$. Hence by Lemma 9 $p\psi \leq q\psi$ and $q\psi \leq p\psi$, and so $p\psi = q\psi$. Since

$$(R(p) \wedge R(q))\varphi = R(p \wedge q)\varphi = (p \wedge q)\psi = p\psi \wedge q\psi = R(p)\varphi \wedge R(q)\varphi$$

and similarly for $\vee$, we conclude that φ is a homomorphism. Finally, for $p \in L_i$, $i \in I$,

$$R(p)\varphi = p\psi = p\varphi_i$$

by the definition of ψ, hence φ restricted to L_i agrees with φ_i.

Lemma 6(i) implies that if $p \equiv q$ $(p, q \in \underset{\sim}{P}(Q))$, then, for all $i \in I$, $p_{(i)} = q_{(i)}$ and $p^{(i)} = q^{(i)}$. Hence we can define

$$(R(p))_{(i)} = p_{(i)} \quad \text{and} \quad (R(p))^{(i)} = p^{(i)}.$$

All our results will now be summarized. The Structure Theorem of Free Products (G. Grätzer, H. Lakser, and C.R. Platt [10]):

<u>Theorem 10.</u> Let L_i, $i \in I$, be lattices and let L be a free product of the L_i, $i \in I$. Then for every $a \in L$ and $i \in I$ if some element of L_i is contained in a, then there is a largest one with this property, $a_{(i)}$.

If $a = p(a_0, \cdots, a_{n-1})$, where p is a n-ary polynomial and $a_0, \cdots, a_{n-1} \in \bigcup(L_j, j \in I)$, then $a_{(i)}$ can be computed by the algorithm given in Definition 3. Dually, $a^{(i)}$ can be computed. For $a, b \in L$, $a = p(a_0, \cdots, a_{n-1})$, $b = q(b_0, \cdots, b_{m-1})$, $a_0, \cdots, a_{n-1}, b_0, \cdots, b_{m-1} \in \bigcup(L_i, i \in I)$, we can decide whether $a \leq b$ using the algorithm of Definition 5.

3. Using the Structure Theorem of Free Products one can develop a theory which contains most of the known results on free lattices. The normal form theorem of P.M. Whitman [17] stating that the shortest representation of an element of a free lattice is unique up to commutativity and associativity has the following analogue for free products. Let L, L_i, $i \in I$, and Q be as in §2. For $a \in L$ and $p = p(a_0, \cdots, a_{n-1}) \in \underset{\sim}{P}(Q)$ $(a_0, \cdots, a_{n-1} \in Q)$ $a = p$ is a minimal representation of a if $r(p)$ is minimal and we call p a minimal polynomial.

<u>Theorem 1</u> (H. Lakser [15]). Let $p \in \underset{\sim}{P}(Q)$. Then p is a minimal representation iff $p \in Q$, or if $p = p_0 \vee \cdots \vee p_{n-1}$, $n > 1$ where no p_j is a join of more than one polynomial and conditions (i) - (v) below hold, or the dual of the preceding case holds.

(i) Each p_j is minimal, $0 \leq j < n$.

(ii) For each $0 \leq j < n$, $p_j \nleq p_0 \vee \cdots \vee p_{j-1} \vee p_{j+1} \vee \cdots \vee p_{n-1}$.

(iii) If $0 \leq j < r$, $r(p_j) > 1$, $i \in I$, then $(p_j)^{(i)} \nleq p_{(i)}$ in L_i.

(iv) If $p_j = p_j' \wedge p_j''$ $(0 \leq j < n$ and $p_j', p_j'' \in \underset{\sim}{P}(Q))$, then $p_j' \nleq p$.

(v) If $p_j, p_k \in L_i$ $(0 \leq j \leq k < n$ and $i \in I)$, then $j = k$.

Another result of H. Lakser [16] (which is applied in G. Grätzer and J. Sichler [12]) is based on Theorem 1:

<u>Theorem 2.</u> Let M be a sublattice of L, a free product of the L_i, $i \in I$. Assume that $M \cong M_5$ the five-element nondistributive lattice. Then $M \subseteq L_i$ for some i or some L_i has a sublattice isomorphic to $M_5 \times C_2$, where C_2 is the two-element chain.

The most important properties of the free lattice are the following (P.M. Whitman [17] and B. Jónsson [13]):

(W) $x \wedge y \leq u \vee v$ implies that $x \leq u \vee v$ or $y \leq u \vee v$ or $x \wedge y \leq u$ or $x \wedge y \leq v$.

$(SD_\wedge)$ $x \wedge y = x \wedge z = u$ implies that $x \wedge (y \vee z) = u$.

$(SD_\vee)$ is the dual of $(SD_\wedge)$.

The next result is due to G. Grätzer and H. Lakser [9]:

<u>Theorem 3.</u> Let (X) be one of the properties (W), $(SD_\wedge)$, and $(SD_\vee)$. Let A_i be a sublattice of L_i, $i \in I$, and let L be the free product of the L_i, $i \in I$. Let K be a sublattice of L with the property that for all $a \in K$, $a_{(i)}$ and $a^{(i)} \in (A_i)^b$. If all A_i, $i \in I$, satisfy (X), then so does K.

We obtain that the free lattice has (W), $(SD_\wedge)$, and $(SD_\vee)$ by taking $L_i = C_1$ (the one-element chain) and $L = K$.

Naturally, not all results on free lattices have been successfully generalized to free products. As an interesting example I mention the result of F. Galvin and B. Jónsson [4] according to which every chain in a free lattice is countable. A natural generalization of this is the following conjecture:

Let $\mathfrak{m}$ be a regular cardinal and let L_i, $i \in I$, be lattices with the property that any chain in any of the L_i has cardinality less than $\mathfrak{m}$. Then all chains in the free product of the L_i, $i \in I$, have cardinality less than $\mathfrak{m}$.

Of course, $\mathfrak{m} = \aleph_1$ is the most interesting case. The only result relating to the conjecture above is in B. Jónsson [14] in which the general conjecture is reduced to the case $|I| = 2$.

In the same paper, B. Jónsson generalizes some of the results of §2 to K-free products for an arbitrary equational class K of lattices. The problem stated above is completely settled for distributive free product in G. Grätzer and H. Lakser [8].

4. Let L_i, $i \in I$, be bounded lattices and let L be a $\{0, 1\}$-free product of the L_i, $i \in I$. As we shall see, a pair of elements x, y is complementary in L (that is, $x \wedge y = 0$ and $x \vee y = 1$) iff they are complementary in some L_i or if $x_0 \leq x \leq y_0$, $x_1 \leq y \leq y_1$ in L_i and $\{x_0, y_0\}$, $\{x_1, y_1\}$ are complementary in L_i. We need a construction in which there are many more complements however we can still keep track of the complements. We call this construction the reduced free product.

In the discussion below let L_i, $i \in I$, be bounded lattices.

<u>Definition 1.</u> A C-relation $\mathcal{C}$ on L_i $i \in I$, is a symmetric binary relation on $\bigcup(L_i, i \in I)$ with the property that if $\{a, b\} \in \mathcal{C}$, $a \in L_i$, $b \in L_j$, then $i \neq j$.

<u>Definition 2.</u> Let $\mathcal{C}$ be a C-relation on L_i, $i \in I$. A lattice L is a $\mathcal{C}$-<u>reduced free product</u> of the L_i, $i \in I$ iff the following conditions hold:

(i) Each L_i, $i \in I$, is a $\{0, 1\}$-sublattice of L and $L = [\bigcup(L_i \mid i \in I)]$.

(ii) If $\{a, b\} \in \mathcal{C}$, then a, b is a complementary pair in L.

(iii) If, for $i \in I$, φ_i is a $\{0, 1\}$-homomorphism of L_i into the bounded lattice A, and $\{a, b\} \in \mathcal{C}$ $(a \in L_i, b \in L_j)$ implies that $a\varphi_i$, $b\varphi_j$ are complementary in A, then there is a homomorphism φ of L into A extending all the φ_i, $i \in I$.

It is obvious that a $\mathcal{C}$-reduced product is unique up to isomorphism. The next result shows that it actually exists, and what is more important we can describe the complementary pairs in it (Theorem 5). Let $Q = \bigcup(L_i, i \in I)$ and define a subset S of $\underset{\sim}{P}(Q)$:

Definition 3. For $p \in \underset{\sim}{P}(Q)$, $p \in S$ is defined by induction on $r(p)$:

(i) $r(p) = 1$, that is, $p \in L_i$ $(i \in I)$ and $p \notin \{0_i, 1_i\}$.

(ii) $p = q \wedge r$ where $q, r \in S$ and the following two conditions hold:

(ii_1) $p \subseteq 0_i$ for no $i \in I$.

(ii_2) $q \subseteq x$ and $r \subseteq y$ for no $\{x, y\} \in \mathcal{C}$.

(iii) $p = q \vee r$ where $q, r \in S$ and the following two conditions hold:

(iii_1) $1_i \subseteq p$ for no $i \in I$.

(iii_2) $x \subseteq q$ and $y \subseteq r$ for no $\{x, y\} \in \mathcal{C}$.

Now we set

$$L = \{0, 1\} \cup \{R(p) \mid p \in S\},$$

and partially order L by

$$0 < R(p) < 1 \quad \text{for} \quad p \in S,$$

$$R(p) \leq R(q) \quad \text{iff} \quad p \subseteq q.$$

If we identify $a \in L_i$ with $R(a)$, then we get the setup we need:

Theorem 4. L is a $\mathcal{C}$-reduced free product of the L_i, $i \in I$.

Proof. L is obviously a poset. To show that L is a lattice we have to find the meet of $R(p)$ and $R(q)$ in L $(p, q \in S)$, and dually. We claim that $R(p) \wedge R(q) = R(p \wedge q)$ if $p \wedge q \in S$ and otherwise $R(p) \wedge R(q) = 0$. This is obvious since if $p \wedge q$ fails (ii_1) or (ii_2), then any $r \subseteq p \wedge q$ will fail (ii_1) or (ii_2).

Now it is obvious that $a \longrightarrow R(a)$ is a $\{0, 1\}$-embedding of L_i into L. So after the identification 2(i) becomes obvious. 2(ii) is clear in view of $3(ii_1)$, $3(ii_2)$, and our description of meet and join in L.

Let K be the free product of the L_i, $i \in I$, as constructed in §1. Then $L - \{0, 1\} \subseteq K$. We define a congruence Θ on K:

$$\Theta = \bigvee(\Theta(x, 0_i) \mid i \in I, x \le 0_i) \vee \bigvee(\Theta(x, 1_i) \mid i \in I, x \ge 1_i) \vee$$
$$\vee \bigvee(\Theta(x, u \wedge v) \mid x \le u \wedge v, \{u, v\} \in \mathcal{C}) \vee \bigvee(\Theta(x, u \vee v) \mid x \ge u \vee v, \{u, v\} \in \mathcal{C}).$$

In other words, Θ is the smallest congruence relation under which all 0_i and $u \wedge v$ $(u, v \in \mathcal{C})$ are in the smallest congruence class and dually. We claim that

$$K/\Theta \cong L.$$

To see this, it is sufficient to prove that every congruence class modulo Θ except the two extremal ones contain one and only one element of S.

Let ϵ_i be the identity map as a map of L_i into L. Then there is a map φ extending all ϵ_i, $i \in I$, into a homomorphism of K into L. Let Φ be the congruence induced by φ ($a \equiv b(\Phi)$ iff $a\varphi = b\varphi$). Since L satisfies 2(i) and 2(ii), $\Theta \le \Phi$. Now if $p, q \in S$, and $R(p)\varphi = R(q)\varphi$, then $R(p) = R(q)$. In other words, $R(p) \equiv R(q)(\Phi)$ implies $R(p) = R(q)$. Therefore, the same holds for Φ. This proves that there is at most one $R(p)$ in the non-extremal congruence classes of Θ. To show "at least one" take a $p \in \underset{\sim}{P}(Q)$ such that $R(p) \not\equiv 0_i(\Theta)$ and $R(p) \equiv 1_i(\Theta)$ (for any/all $i \in I$); we prove that there exists a $q \in S$ such that $R(p) \equiv R(q)(\Theta)$.

Let $p \in L_i$ for some $i \in I$. Then, by assumption, $p \neq 0_i$ and 1_i; hence we can take $q = p$. Let $q = a$, $p = p_0 \wedge p_1$, $R(p_0) \equiv R(q_0)(\Theta)$, $R(p_1) \equiv R(q_1)(\Theta)$ where $q_0, q_1 \in S$. If $q_0 \wedge q_1 \in S$ take $q = q_0 \wedge q_1$. Otherwise, by 3(ii), $q_0 \wedge q_1 \equiv 0_i(\Theta)$, hence $p \equiv 0_i(\Theta)$, contrary to our assumption. The dual argument completes the proof. Thus we have verified that $K/\Theta \cong L$.

Now we are ready to verify 2(iii). For each $i \in I$, let φ_i be a $\{0, 1\}$-homomorphism of L_i into the bounded lattice A. Since K is the free product of the L_i, $i \in I$, there is a homomorphism ψ of K into A extending all the φ_i, $i \in I$. Let Ψ be the congruence induced by ψ (that is, $a \equiv b(\Psi)$ if $a\psi = b\psi$). It obviously follows from the definition of Θ that $\Theta \leq \Psi$. Therefore, by the Second Isomorphism Theorem (see e.g. Lemma 15.8 in [5])

$$[x]\Theta \longrightarrow x\psi$$

is a homomorphism of K/Θ into A. Combining this with the isomorphism $L \cong K/\Theta$ as described above, we get a $\{0, 1\}$-homomorphism φ of L into A extending all the φ_i, $i \in I$.

Theorem 5. Let a, b be a complementary pair in the $\mathcal{C}$-reduced free product L of the L_i, $i \in I$. Then there exist a_0, b_0 and a_1, b_1 such that

$$a_0 \leq a \leq a_1 \quad \text{and} \quad b_0 \leq b \leq b_1$$

such that either $\{a_0, b_0\}, \{a_1, b_1\} \in \mathcal{C}$ or, for some $i \in I$, a_0, b_0 and a_1, b_1 are complementary pairs in L_i, and conversely.

Proof. The converse is, of course, obvious. In either case, by Definition 2, a_0, b_0 and a_1, b_1 are complementary in L, hence

$$a \wedge b \leq a_1 \wedge b_1 = 0 \ , \ a \vee b \geq a_0 \vee b_0 = 1 \ ,$$

and so a, b is complementary in L.

Now to prove the main part of the theorem, take $p, q \in S$ such that $a = R(p)$ and $b = R(q)$ are complementary in L. Then $p \wedge q$ violates $3(ii_1)$ or $3(ii_2)$ and $p \vee q$ violates $3(iii_1)$ or $3(iii_2)$. The four cases will be handled separately.

Case 1. $p \wedge q$ violates $3(ii_1)$ and $p \vee q$ violates $3(iii_1)$. Hence, for some $i, j \in I$, $p \wedge q \subseteq 0_i$ and $1_j \subseteq p \vee q$. Thus in the free product K of the L_i, $i \in I$, $(p \wedge q)^{(i)} = 0_i$ and $(p \vee q)_{(j)} = 1_j$. Note that $q_{(i)}$ is proper, because otherwise $p^{(i)} = 0_i$, that is, $p \subseteq 0_i$ contradicting $p \in S$. Similarly, $q^{(j)}$ is proper. This is a contradiction unless $i = j$, in which case we can put $a_0 = p_{(i)}$, $b_0 = q_{(i)}$, $a_1 = p^{(i)}$, $b_1 = q^{(i)}$ and these obviously satisfy the requirements of the theorem.

Case 2. $p \wedge q$ violates $3(ii_1)$ and $p \vee q$ violates $3(iii_2)$. Hence there exist $i \in I$ and $\{x, y\} \in \mathcal{C}$ such that

$$p \wedge q \subseteq 0_i, \ x \subseteq p, \text{ and } \ y \subseteq q.$$

Let $x \in L_j$ and $y \in L_k$ ($j, k \in I$ and $j \neq k$). Just as in Case 1 we conclude that in K $p^{(i)}$, $q^{(i)}$ are proper, $p^{(i)} \wedge q^{(i)} = 0_i$ $p_{(j)} \geq x$, and $q_{(k)} \geq y$. Hence $i = j$, $i = k$, from which $j = k$ follows, contradicting $j \neq k$.

Case 3. $p \wedge q$ violates $3(ii_2)$ and $p \vee q$ violates $3(iii_1)$. This leads to a contradiction just as Case 2 does.

Case 4. $p \wedge q$ violates $3(ii_2)$ and $p \vee q$ violates $3(iii_2)$. Then there exist $\{a_0, b_0\} \in C$ and $\{a_1, b_1\} \in C$ such that

$$p \subseteq a_1, \ q \subseteq b_1, \ a_0 \subseteq p, \text{ and } \ b_0 \subseteq q .$$

These obviously satisfy the requirements of the theorem. This completes the proof of Theorem 5.

Theorem 5 is the main result on reduced free products. It is a generalization of the results of G. Grätzer [7], which in turn generalized C.C. Chen and G. Grätzer [1].

5. The simplest application of the results of §4 is to uniquely complemented lattices, that is to lattices in which every element has exactly one complement. A longstanding conjecture of lattice theory was disproved by R.P. Dilworth [2] by showing that not every uniquely complemented lattice is distributive. In fact Dilworth proved that every lattice can be embedded in a uniquely complemented lattice. This result is further sharpened by a theorem of C.C. Chen and G. Grätzer [1]:

Theorem 1. Let L be a bounded lattice in which every element has at most one complement. Then L has a 0 and 1 preserving embedding into a uniquely complemented lattice.

Observe that Theorem 1 implies the Dilworth embedding theorem; indeed, if L is an arbitrary lattice, then by adding a 0 and 1 to L we obtain a lattice L_1 in which every element has at most one complement (in fact if $x \in L_1$, $x \neq 0, 1$, then x has no complement). Apply Theorem 1 to L_1 to get a uniquely complemented lattice containing L as a sublattice.

The proof of Theorem 1 is so simple that we reproduce a sketch of the proof.

If L is complemented, then set $K = L$. Otherwise let $L = L_0$. We define by induction the lattice L_n. If L_{n-1} is defined let I_{n-1} be the set of noncomplemented elements of L_{n-1}. For $i \in I_{n-1}$ let $L_i = \{a_i\}^b$. Define the C-relation C_{n-1} on the family $\{L_{n-1}\} \cup (L_i \mid i \in I_0)$ by the rule

$$\{a, b\} \in C_{n-1} \text{ iff } \{a, b\} = \{i, a_i\} \text{ for some } i \in I_{n-1}.$$

Let L_n be the C_{n-1}-reduced free product. Since

$$L = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$$

and all these containments are $\{0, 1\}$-embeddings, we can form $K = \bigcup(L_i \mid i \in I)$.

Now consider for $n > 0$ the property

(P_n) if $a_0, b_0, a_1, b_1 \in L_n$, a_0 is a complement of b_0, a_1 is a complement of b_1, $a_0 \leq b_0$, and $a_1 \leq b_1$, then $a_0 = a_1$ and $b_0 = b_1$; if $\{a_0, b_0\}, \{a_1, b_1\} \in \mathcal{C}_n$ and $a_0 \leq a_1$, $b_0 \leq b_1$, then $a_0 = a_1$ and $b_0 = b_1$.

Obviously, (P_0) holds. An easy induction using Theorem 5 of §4 shows that (P_n) holds for all $n \geq 0$. Again by Theorem 5 a is a complement of b in L_n iff the same holds in L_{n-1} or $\{a, b\} \in \mathcal{C}_n$. Therefore we obtain that the direct limit of the L_n is uniquely complemented.

Many variants of Theorem 1 are considered in C.C. Chen and G. Grätzer [1]: Bi-uniquely complemented lattices, lattices in which complementation is a transitive relation, and so on. All these results are based on Theorem 5 of §4.

Another application is to the endomorphism monoid of a bounded lattice. For a bounded lattice L let $\mathrm{End}_{0,1}(L)$ denote the monoid of 0 and 1 preserving endomorphisms of L.

The following result is due to G. Grätzer and J. Sichler [11]:

<u>Theorem 2.</u> Let M be a monoid. Then there exists a bounded lattice L such that

$$M \cong \mathrm{End}_{0,1}(L) .$$

Let $\langle G; R \rangle$ be a <u>graph</u>, that is, a set G with a symmetric binary relation R such that $\langle a, a \rangle \notin R$ for any $a \in G$. We associate with the graph a family of lattices L_a, $a \in G$, where each L_a is a three-element chain $0_a, a, 1_a$. Set $\mathcal{C} = R$; then $\mathcal{C}$ is a C-relation so we can form the $\mathcal{C}$-reduced free product L. We then prove (using Theorem 5 of §4) that every endomorphism extends to a $\{0, 1\}$-endomorphism, and conversely, provided that every element of G lies on a cycle of odd length. We get from the results of Z. Hedrlin and A. Pultr a graph $\langle G; R \rangle$ with $\mathrm{End}(\langle G; R \rangle) \cong M$ satisfying the cycle condition and so we obtain Theorem 2.

The final application I would like to mention concerns hopfian lattices. A lattice L is called hopfian iff $L \cong L/\Theta$ implies that Θ is the trivial congruence relation ω. Equivalently, L is hopfian iff every onto endomorphism is an automorphism.

T. Evans [3] has proved that every finitely presented lattice is hopfian.

Motivated by H. Neumann's results, the question arose whether the free product of two hopfian lattices is hopfian again.

Theorem 3. There exist two bounded hopfian lattices whose bounded free product is not hopfian.

Theorem 4. There exist two hopfian lattices whose free product is not hopfian.

These results are due to G. Grätzer and J. Sichler [12]. Theorem 3 is based on Theorem 2 which reduces Theorem 3 to a graph construction. Theorem 4 is more complicated and it also uses Theorem 2 of §3.

There are many more results on free products and many more results using free products. I hope, however, that this restricted exposition is sufficient to substantiate my claim that the free product is an important construction in lattice theory with which all experts should be familiar.

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