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SOME UNSOLVED PROBLEMS BETWEEN LATTICE THEORY AND EQUATIONAL LOGIC[1]

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This is a very modest paper. My aim is to have a look at some problems that arise in the regions where lattice theory and equational logic share common ground. The list of problems is selected from my own mathematical experience, and is not intended to be in any way comprehensive or definitive.

Throughout the paper, $L(\Lambda)$ denotes the lattice of equational theories of lattices. Λ is its least element (the set of identities satisfied by every lattice $\langle L,+,\cdot\rangle$), and Ω is its largest element (the set of all lattice identities), while Δ is its one and only maximal element (the equational theory of distributive lattices). If τ is a similarity type of universal algebras, then $L(\tau)$ denotes the lattice constituted by all equational theories of algebras of type τ. If Θ is a member of $L(\tau)$, then $L(\Theta)$ denotes the lattice composed of all $\Theta' \in L(\tau)$ such that $\Theta \leq \Theta'$.

The problems in the first group are the ones of most recent origin. They concern the congruence identities holding in a variety. For an arbitrary equational theory Θ, we can form a theory $\underset{\sim}{Cg}\,\Theta$ belonging to $L(\Lambda)$, which consists of the identities that hold true in the congruence lattice of every algebra belonging to the variety

[1] This is a greatly revised version of the lecture given by the author at the Conference on Lattice Theory, Houston, 1973.

$\mathbf{Var}\ \Theta$ defined by Θ. Everyone is familiar with the theorems which assert that certain properties of congruences in $\mathbf{Var}\ \Theta$ are equivalent to conditions (of "Mal'cev type"), defined directly by reference to Θ itself. For example, the properties "$\mathbf{Cg}\ \theta \geq \Delta$", "$\mathbf{Cg}\ \theta$ includes the modular law" are equivalent to Mal'cev conditions. We recall that each of these properties has very strong implications for the general algebraic theory of $\mathbf{Var}\ \Theta$; see [10] and [9, Cor. 5.5], for instance. Very recently, in [4], it was shown that if Θ is an equational theory of semigroups and if $\mathbf{Cg}\ \Theta \neq \Lambda$, then $\mathbf{Cg}\ \Theta$ includes the modular law. A related paper, [17], revealed that if Θ is any equational theory such that $\mathbf{Cg}\ \Theta$ intersects a certain set of identities which are weaker than modularity then, again, $\mathbf{Cg}\ \Theta$ includes the modular law. Thus it turns out that the set $\{\mathbf{Cg}\ \Theta\colon \Theta$ is an equational theory$\}$ is a proper subset of $L(\Lambda)$--a dramatic development in an area which seemed thoroughly cultivated and not very promising of new results.

I conjecture, very boldly: (1) every theory $\mathbf{Cg}\ \Theta$, distinct from Λ, includes the modular law. And less boldly: (2) $\{\mathbf{Cg}\ \Theta\colon \Theta$ is an equational theory$\}$ is a sublattice of $L(\Lambda)$.

The class of theories known to have a finite base has expanded greatly in the past decade. Many of the proofs of the positive results in this direction have used, at least implicitly, the satisfaction of congruence identities. If we recall some of these results, it will lead us to two conjectures related to (1) above. (I am using two excellent survey articles by S. Oates MacDonald [12] and A. Tarski [19].)

There are equational theories Θ for which every member of $L(\Theta)$ is finitely based, such as the theory of commutative semigroups [18], or of idempotent semigroups [5]. (Note, however, that there exists a 6-element semigroup whose equational theory has no finite base; see [18].) More commonly, the above result does not hold, but the following one does: if $\mathfrak{A}$ is any finite member of $\underset{\sim}{\mathrm{Var}}\ \Theta$, then the identities satisfied by $\mathfrak{A}$ have a finite base. This is known to be the case for the theory of groups (Oates-Powell [13]), for the theory of rings (Kruse [11]), for the theory of lattices (McKenzie [14]), and generalizing the case of lattices, for any theory Θ of a finite similarity type which satisfies $\underset{\sim}{\mathrm{Cg}}\ \Theta \geq \Delta$.[2] Although the Oates-Powell theorem uses the congruence modularity of the variety of groups, it has not yet been generalized in the way my theorem for lattices was generalized by Baker. In fact, the following conjectures are unresolved (they were printed for the first time in [12]): (3) if Θ is any theory of finite type such that $\underset{\sim}{\mathrm{Cg}}\ \Theta$ includes the modular law, then every finite algebra in $\underset{\sim}{\mathrm{Var}}\ \Theta$ has a finite base for its laws; (4) the same conclusion holds if Θ is a theory of finite type and $\underset{\sim}{\mathrm{Cg}}\ \Theta \neq \Lambda$. The first conjecture was made by S. Oates MacDonald and by K. Baker, and S. Burris originated the second. [Burris is said to have some evidence

[2] This result is due to K. Baker [1]; his proof is unpublished. A different proof is given by M. Makkai in a paper soon to appear in Algebra Universalis.

for (4); namely, that for each of the known finite groupoids $\mathcal{G}$ having no finite base of identities, $\underset{\sim}{\mathrm{Cg}}\,\underset{\sim}{\Theta}\,\mathcal{G} = \Lambda$. Apparently this follows indirectly from the results of [4], since each of these groupoids contains a 2-element subsemigroup that is not a group.]

The second group of problems concerns the free lattice $FL(\omega)$ generated by a denumerable set of freely unrelated elements $\{x_0, x_1, x_2, \ldots\}$. We denote by $FL(n)$, for $1 \leq n < \omega$, the sublattice of $FL(\omega)$ generated by $\{x_0, \ldots, x_{n-1}\}$. The study of these structures shows some analogies, and many differences, to the study of free groups. It is well known for instance that the $FL(\kappa)$ ($\kappa \leq \omega$), like the free groups, are computable algebras: there is an algorithm (discovered by P. M. Whitman) which tells whether two formal words in the generators define the same element of a free lattice.

Let us denote by $\mathcal{F}$ (respectively, by $\mathcal{F}_0$) the class of all finitely generated (respectively, finite) lattices that are isomorphic to a sublattice of $FL(\omega)$. Then in contrast to the situation for groups, where every subgroup of a free group is itself free, the classes $\mathcal{F}$ and $\mathcal{F}_0$ are highly nontrivial, and there are long-standing open questions about them. Some known facts (from [16]) are the following: $\mathcal{F}$ is precisely the class of finitely generated lattices projective in the category of all lattices, where maps are all the homomorphisms (a result due to A. Kostinsky but proved in [16]); $\mathcal{F}_0$ is a computable class (there is an algorithm for determining whether a finite lattice belongs to $\mathcal{F}_0$); every member of $\mathcal{F}$ has a finite presentation by means of generators and relations, relative to the class of lattices (this was not proved in [16], but is easily demonstrated using the "bounded homomorphisms" discussed there).

B. Jónsson once remarked that every sublattice of a free lattice satisfies three simple conditions (which we formulate below), and over a period of years ([6], [7], and [8]) he has obtained deep results tending to confirm the following conjecture: (5) $\mathcal{F}_0$ is characterized, as a subclass of the class of finite lattices, by the conditions (i)-(iii) below. This conjecture is yet unproved. It appears very plausible that $\mathcal{F}$ is characterized by the same conditions. Jónsson's conditions (for a given lattice L): Let $u_0, u_1, v_0, v_1 \in L$. Then (i) $u_0 \cdot u_1 \leq v_0 + v_1$ implies that either $u_i \leq v_0 + v_1$, or else $u_0 \cdot u_1 \leq v_i$, for some $i = 0,1$; (ii) $u_0 + v_0 = u_0 + v_1$ implies $u_0 + v_0 = u_0 + v_0 \cdot v_1$; (iii) same as (ii) with +, $\cdot$ interchanged.

Unlike the situation for groups, it is easily demonstrated that $FL(\kappa)$ and $FL(\lambda)$ (for distinct $\kappa, \lambda \leq \omega$) do not satisfy precisely the same elementary (that is, first-order) sentences. (A famous open problem, due to Tarski, asks whether it is the same with free groups.) For free lattices, as for free groups, the following is open: (6) for each κ ($3 \leq \kappa \leq \omega$), is the elementary theory of $FL(\kappa)$ decidable? Even a very special case of this problem has not been settled. Conjecture: (7) the existential first-order theory of $FL(3)$ is decidable. We should remark that, since $FL(\omega)$ is embeddable into $FL(3)$, all $FL(\kappa)$ for $3 \leq \kappa \leq \omega$ have the same existential theory. The decision problem for this theory is quite different from the so-called "embedding problem for lattices"--the decision problem for the universal theory of lattices--for which Evans [2] gave a positive solution. Also, it is broader in scope than the (solvable) problem of determining membership in $\mathcal{F}_0$.

Here is a concrete elementary sentence which, as shown in [16], has the same truth value in every lattice FL(κ) with $3 \leq \kappa < \omega$, and is certainly false in FL(ω). At present, we have no way of deciding whether it is true or false in FL(3). This would be decided as a particular case by any algorithm giving a positive solution to problem (6).

$$\Phi:\ \forall x,y\ \exists u,v\ \forall z.\quad x < y \rightarrow (x \leq u < v \leq y \wedge \neg\, u < z < v).$$

This sentence has importance, independently, from the study of equational theories of lattices. (See [16, Problem 6].) Given a lattice L and two of its members, x and y, we write x/y for the set $\{z\colon x \leq z \leq y\}$. A <u>nontrivial quotient</u> in L is any set of the form x/y having at least two members; an <u>atomic quotient</u> in L is any nontrivial quotient with exactly two members. (So x/y is atomic iff y <u>covers</u> x.) Clearly, L satisfies Φ iff every nontrivial quotient of L contains an atomic quotient of L, in short, iff L is <u>weakly atomic</u>. It turns out that if we identify any $w_0, w_1 \in$ FL(ω) just in the case that both $w_0/w_0 \cdot w_1$ and $w_1/w_0 \cdot w_1$ contain no quotients u/v which are atomic in any FL(n) ($n < \omega$), then the resulting relation is a fully invariant congruence relation on FL(ω). This relation is the equational theory of the class of all so-called "splitting lattices." Thus, FL(3) is not weakly atomic just in case there exists a nontrivial lattice identity that holds in each and every splitting lattice.

The third group of problems is a small selection from among those mentioned in [16]. There are many open problems about the abstract structure of $L(\Lambda)$, about particular members of $L(\Lambda)$, and about related properties of equational theories and their models. Among them are the following: (8) Has $L(\Lambda)$ any automorphisms aside from the identity map, and the involution that results from the duality of the two basic operations in lattices? (9) If $\Theta \in L(\Lambda)$, and $L(\Theta)$ is finite, must Θ cover only a finite set of elements of $L(\Lambda)$? (10) Is the equational theory of modular lattices decidable? (This is a long-standing open problem. Only recently, a somewhat richer theory, namely, the universal theory of modular lattices, was proved undecidable by G. Hutchinson.[3] See his article in this volume, and the article by C. Herrmann.) (11) Is it true for every $\Theta \in L(\Lambda)$ that the following are equivalent: (a) $L(\Theta)$ is finite; (b) there is a finite lattice L with $\Theta = \underset{\sim}{\Theta} L$? (Compare this with problem (9).)

Finally, I should like to repeat two conjectures about the lattices $L(\tau)$, where τ is an arbitrary similarity type. I proved in [15] basically two results about these lattices: first, that τ is recoverable from the abstract structure of $L(\tau)$; second, that most familiar equational theories--for instance, the theory of groups, of rings, of lattices, or of Boolean algebras--can be singled out

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[3] This important result was probably obtained independently and simultaneously by Leonard Lipschitz, and possibly can be found in his doctoral dissertation which was completed about 1972-73 under the direction of Kochen at Princeton.

abstractly in their type lattice and defined as the unique member satisfying a first-order lattice formula $\varphi(x)$, where φ depends of course on the theory to be defined by it. I conjecture: (12) all automorphisms of $L(\tau)$ are basic ones, generated by exchanging operation symbols that have the same rank, and by permuting the "places" of some operation symbols; (13) every member of $L(\tau)$ that is finitely based as a theory, and is a fixed element under all automorphisms of $L(\tau)$, is a first-order definable member of this lattice.

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