A Nonlinear Galerkin Method: The Two–Level Chebyshev Collocation Case

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Abstract

In this article we study the implementation of the Nonlinear Galerkin method as a multiresolution method when a two-level Chebyshev–collocation discretization is used. A fine grid containing an even number of Gauss–Lobatto points is considered. The grid is decomposed into two coarse grids based on half as many Gauss–Radau points. This splitting suggests a decomposition of the unknowns in low modes and high modes components which is convenient also in the physical space. A nonlinear Galerkin scheme is then applied to a linear parabolic equation in the case of a Chebyshev–Legendre scheme. L^2–norm stability is proved.

Key words: nonlinear Galerkin method, Chebyshev collocation method.

AMS subject classifications: 65N30, 65N35.

1 Introduction

In this article we study the implementation of the Nonlinear Galerkin method in the case of a Chebyshev–collocation discretization.

Following the guidelines of a previous article [3] in which the Fourier space–periodic case was considered, we address here the case of a Chebyshev approximation.

The basic idea in the Nonlinear Galerkin method (and in the theory of inertial manifolds) is the decomposition of the unknown u into its large scale and small scale components, y and z:

$$u = y + z. \tag{1}$$

In the case of a Fourier expansion it is clear that y corresponds to the low modes and z to the high modes. When a Chebyshev expansion is considered:

$$u(x) = \sum_{k=0}^{M} u_k^M T_k(x), \tag{2}$$

we show that the *low modes* are the coefficients of the low degree Chebyshev polynomials ($k \leq N$) and the *high modes* are the coefficients corresponding to the high degree Chebyshev polynomials ($N+1 \leq k \leq M$).

When a collocation method is used (as opposed to a full spectral method) we need to find a decomposition of the kind (1) which is suitable in the physical space. Such a decomposition is accomplished via the splitting of the *fine* grid into two *coarse* grids, based on half the points. A Nonlinear Galerkin method is applied to a linear parabolic equation in the case of a Chebyshev–Legendre approximation. A slightly modified version of the original method proposed in [4] is considered and the L^2–norm stability is proved.

The article is organized as follows: Section 2 describes the choice of collocation points. The fine grid consists of an even number of Gauss–Lobatto points ($M+1 = 2N+2$), the two coarse grids are based on $N+1$ Gauss–Radau points, each one containing one boundary point.

A decomposition of type (1) is proposed in Section 3. Here y contains only low degree coefficients and is based on one coarse grid and z contains only high degree coefficients and is based on the other coarse grid.

Based on a decomposition of type (1), a nonlinear Galerkin method for a linear parabolic equation is discussed in Section 4. A new version of the Chebyshev–Legendre method originally presented in [4] is considered and L^2–norm stability is proved. In a future article we

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will extend the nonlinear Galerkin method based on the Chebyshev–Legendre approach to a nonlinear parabolic equation.

2 Preliminaries

In this article we will use two polynomial spaces; we define, for $M = 2N+1$, the spaces:

$$(3)\qquad \begin{array}{rcl} \mathbf{P}_M &=& \{polynomials\ of\ degree\ \leq M\}, \\ \mathbf{P}_N &=& \{polynomials\ of\ degree\ \leq N\}, \end{array}$$

of dimensions $M+1 = (2N+2)$ and $N+1$ respectively.

In the following we use many standard results in the Chebyshev approximation; for an overview of those results the reader is referred to [5, 7, 6] and [2].

A generic function f, defined on $[-1,1]$ can be projected onto the space $\mathbf{P}_M$ by interpolation on the following set of points:

$$(4)\qquad x_j^M = \cos\frac{\pi j}{M}, \qquad 0 \leq j \leq M,$$

that we will refer to as the *fine grid.* We recall that the Gauss–Lobatto points x_j^M are the zeros of $(1-x^2)T'_M(x)$, where $T_M(x)$ is the Chebyshev polynomial of degree M.

To project a function f on the spaces $\mathbf{P}_N$ we will alternatively use the two following *coarse grids*

$$(5)\qquad \xi_j^N = \cos\frac{2\pi j}{M} = \cos\frac{2\pi j}{2N+1}, \qquad 0 \leq j \leq N,$$

$$(6)\qquad \eta_j^N = \cos\frac{(2j+1)\pi}{M} = \cos\frac{(2j+1)\pi}{2N+1}, \quad 0 \leq j \leq N.$$

The Gauss–Radau points ξ_j^N are the zeros of $T_{N+1} - T_N$, while the η_j^N are the zeros of $T_{N+1} + T_N$ (see *e.g.* [2], Chapter II]).

Remark 2.1 *The fine grid, which contains even number of points, is composed of the union of the two coarse grids, which both contain odd number of points:*

$$\{\xi_j^N\}_{j=0}^N \bigcup \{\eta_j^N\}_{j=0}^N = \{x_j^M\} \qquad M = 2N+1.$$

Indeed, we have:

$$(7)\qquad x_{2j}^M = \xi_j^N, \qquad x_{2j+1}^M = \eta_j^N, \quad 0 \leq j \leq 2N.$$

In the following we give a general formula for the Lagrange polynomial interpolating at a given set of points.

Lemma 2.1 *Given $S+1$ points p_j, $j = 0, \cdots, S$, zeros of a polynomial $Q(x)$, the Lagrange polynomial $\mathcal{H}_Q^s$ that interpolates at those points is defined by (see e.g. [5], Section I.11]):*

$$(8)\qquad \mathcal{H}_Q^s(x, p_j) = \frac{Q(x)}{x - p_j}\frac{1}{Q'(p_j)}.$$

Note that, since the p_j are the zeros of Q,

$$\mathcal{H}_Q^s(p_k, p_j) = \delta_{kj}.$$

The Lagrange polynomial $\mathcal{H}_+^N(x, \xi_j^N)$ corresponding to the Gauss–Radau points ξ_j^N is given by (8), with:

$$\begin{array}{rcl} Q(x) &=& (T_{N+1} - T_N)(x), \\ Q'(\xi_j^N) &=& (-1)^{j+1}\dfrac{2N+1}{2\cos\frac{\pi j}{M}}. \end{array}$$

Definition 2.1 Let f be a function defined on $[-1,1]$; the interpolation polynomial $Q_N f \in \mathbf{P}_N$ is defined by:

$$(9)\qquad Q_N f(x) = \sum_{j=0}^N f(\xi_j^N)\mathcal{H}_+^N(x, \xi_j^N).$$

□

An alternative way of representing $Q_N f$ is to use the identity (see *e.g.* [9], Chapter I]):

$$(10)\qquad \mathcal{H}_+^N(x, \xi_j^N) = \frac{4}{(2N+1)\beta_j^N}\sum_{k=0}^N \frac{T_k(\xi_j^N)T_k(x)}{\beta_k^N},$$

where, $\beta_0^N = 2$, $\beta_j^N = 1$ $(1 \leq j \leq N)$. Substituting (10) into (9) we get:

$$Q_N f(x) = \sum_{k=0}^N \hat{f}_k^N T_k(x),$$

where

$$(11)\qquad F_k^N = \frac{4}{\beta_k^N(2N+1)}\sum_{j=0}^N \frac{f(\xi_j^N)T_k(\xi_j^N)}{\beta_j^N}.$$

It can be seen that:

Lemma 2.2 *The polynomial $Q_N f$ interpolates the function f at the collocation points ξ_j^N, i.e. :*

$$(Q_N f)(\xi_j^N) = f(\xi_j^N), \qquad 0 \leq j \leq N.$$

Alternatively, the projection on the space $\mathbf{P}_N$ can be accomplished via collocation at the other coarse grid points η_j^N; in this case the Lagrange polynomial $\mathcal{H}_-^N(x, \eta_j^N)$ is given by (8) substituting:

$$(12)\qquad \begin{array}{rcl} Q(x) &=& (T_{N+1} + T_N)(x), \\ Q'(\eta_j^N) &=& (-1)^{j+1}\dfrac{2N+1}{2\sin\frac{(2j+1)\pi}{2M}}. \end{array}$$

Definition 2.2 Let $f = f(x)$ be a function defined on $[-1,1]$; the interpolation polynomial $\tilde{Q}_N f \in \mathbf{P}_N$ is defined by:

$$\tilde{Q}_N f(x) = \sum_{j=0}^{N} f(\eta_j^N)\mathcal{H}_{-}^{N}(x, \eta_j^N). \tag{13}$$

□

An alternative representation of $\tilde{Q}_N f$ is:

$$\tilde{Q}_N f(x) = \sum_{k=0}^{N} \tilde{F}_k^N T_k(x), \tag{14}$$

where,

$$\tilde{F}_k^N = \frac{4}{\beta_k^N(2N+1)} \sum_{j=0}^{N} \frac{f(\eta_j^N)T_k(\eta_j^N)}{\sigma_j^N}, \tag{15}$$

where, $\sigma_N^N = 2$, $\sigma_j^N = 1$ $(0 \le j \le N-1)$.

We can summarize the above by:

Lemma 2.3 *The polynomial $\tilde{Q}_N$ interpolates f at the collocation points η_j^N,i.e. :*

$$(\tilde{Q}_N f)(\eta_j^N) = f(\eta_j^N), \qquad 0 \le j \le N. \tag{16}$$

The projection on the space $\mathbf{P}_M$ is accomplished via collocation at the Gauss–Lobatto points x_j^M. The corresponding Lagrange polynomial $\mathcal{H}_{\mathrm{GL}}^{\mathrm{M}}(x, x_j^M)$, is given by (8), substituting

$$\begin{aligned} Q(x) &= (1-x^2)T'_M(x), \\ Q'(x_j^M) &= (-1)^{j+1}M^2\alpha_j^M, \end{aligned} \tag{17}$$

where $\alpha_0^M = \alpha_M^M = 2$, $\alpha_j^M = 1$ $(1 \le j \le M-1)$.

Definition 2.3 Let f be a function defined on $[-1,1]$; the interpolation polynomial $I_M f \in \mathbf{P}_M$ is defined by:

$$I_M f(x) = \sum_{j=0}^{M} f(x_j^M)\mathcal{H}_{\mathrm{GL}}^{\mathrm{M}}(x, x_j^M), \tag{18}$$

□

Using the following equality, similar to (10),

$$\mathcal{H}_{\mathrm{GL}}^{\mathrm{M}}(x, x_j^M) = \frac{2}{\alpha_j^M M} \sum_{k=0}^{M} \frac{T_k(x_j^M)T_k(x)}{\alpha_k^M}, \tag{19}$$

we can give an alternative representation of $I_M f$:

$$I_M f(x) = \sum_{j=0}^{M} \hat{f}_k^M T_k(x), \tag{20}$$

where

$$\hat{f}_k^M = \frac{2}{M\alpha_k^M} \sum_{j=0}^{M} \frac{f(x_j^M)T_k(x_j^M)}{\alpha_j^M}. \tag{21}$$

In the following we define the scalar product with which we endow the spaces $\mathbf{P}_N$.

Definition 2.4 Suppose that u and v are given at the points ξ_j^N, then the scalar product $(u,v)_N$ in $\mathbf{P}_N$ is defined as follows:

$$(u,v)_N = \frac{\pi}{N} \sum_{j=0}^{N} \frac{u(\xi_j^N)v(\xi_j^N)}{\beta_j^N}. \tag{22}$$

□

3 Fine/coarse grids *vs* high/low modes

The main goal of this section is to find a decomposition of the fine grid into two coarse grids such that the difference between the projection operators at the different meshes turns out to be *small* both in the physical and the polynomial space. In other words we want to construct two collocation operators, $\mathcal{J}_N$ on $\mathbf{P}_N$ and $\mathcal{G}_M$ on $\mathbf{P}_M$ that decompose a generic function into its low modes and high modes components.

In the case of a Fourier expansion it is clear that the large scale component corresponds to the low modes, (*i.e.* to $e^{iN\pi x}$ for small N) and the small scale component corresponds to the high modes, ($e^{iN\pi x}$ for large N). In the case of a Chebyshev expansion we need to understand better the concept of small and large scales. Consider the Fourier expansion of the function $(1-x^2)^{-\frac{1}{2}}T_N$:

$$\frac{T_N}{\sqrt{1-x^2}} = \sum_{k=-\infty}^{\infty} b_k^N e^{ik\pi x}. \tag{23}$$

The coefficients b_k^N can be expressed in terms of the Bessel function $B_N(x)$(see [[1], Chapter 10] and [[8], Lemma 2.6]):

$$b_k^N = \frac{i^N\pi}{2}B_N(\pi k), \tag{24}$$

and they verify the following estimate:

$$|b_k^N| \le A \min\left(1, (\frac{e\pi k}{2N})^N\right), \tag{25}$$

where A is a constant independent of k or N. Estimate (25) shows that, for large N, the *lower* terms in (23) decay *exponentially* with N. We conclude that, for large N, basically only the terms that have mode number larger than N appear in the Fourier expansion (23). Therefore, T_N, for large N, can be considered a small scale function.

This justifies the following terminology of "low and high modes" in the case of a Chebyshev expansion.

Consider a function f, expanded in the Chebyshev basis:

$$f(x) = \sum_{k=0}^{M} \hat{f}_k^M T_k(x); \tag{26}$$

we refer to the first N coefficients as the *low modes* of f and to the coefficients corresponding to the Chebyshev polynomials of degree from $N+1$ to M as the *high modes.*

The results of this section are based on the following:

Theorem 3.1
Let $M = 2N+1$ and let Q_N, $\tilde{Q}_N$ and I_M be defined as in (9), (13) and (18). We set:

$$\mathcal{J}_N = \frac{Q_N + \tilde{Q}_N}{2}, \tag{27}$$

$$\mathcal{G}_M = I_M - \mathcal{J}_N. \tag{28}$$

Then, any function $f \in \mathbf{P}_M$ can be written as:

$$f = \mathcal{J}_N f + \mathcal{G}_M f, \tag{29}$$

where $\mathcal{J}_N f \in \mathbf{P}_N$ and $G_M f \in \mathbf{P}_N^M$ (the orthogonal complement of $\mathbf{P}_N$ in $\mathbf{P}_M$). Hence, $\mathcal{J}_N f$ has only low modes and $\mathcal{G}_M f$ has only high modes, i.e. :

$$\mathcal{G}_M f(x) = \sum_{k=1}^{N+1} \hat{f}_{N+k}^M T_{N+k}(x). \tag{30}$$

A similar decomposition holds for $I_M f$, for any function f:

$$I_M f = \mathcal{J}_N f + \mathcal{G}_M f. \tag{31}$$

Proof We show that the coefficients of f and $\mathcal{J}_N f$, corresponding to $0 \le k \le N$, agree. Recalling the definition of $\hat{f}_k^M$ and using (7) we have:

$$\begin{aligned} \hat{f}_k^M &= \frac{2}{M\alpha_k^M}\left(\sum_{j=0}^{N}\frac{f(x_{2j}^M)T_k(x_{2j}^M)}{\alpha_{2j}^M} + \sum_{j=0}^{N}\frac{f(x_{2j+1}^M)T_k(x_{2j+1}^M)}{\alpha_{2j+1}^M}\right) \\ &= \frac{2}{M\beta_k^N}\left(\sum_{j=0}^{N}\frac{f(\xi_j^N)T_k(\xi_j^N)}{\beta_j^N} + \sum_{j=0}^{N}\frac{f(\eta_j^N)T_k(\eta_j^N)}{\sigma_j^N}\right) \\ &= \frac{1}{2}\left(F_k^N + \tilde{F}_k^N\right). \end{aligned} \tag{32}$$

□

The following Lemma provides an explicit inversion formula expressing the Chebyshev coefficients of a function $f \in \mathbf{P}_N^M$ in terms of its values at the points ξ_j^N.

Lemma 3.1 *Consider a function $f \in \mathbf{P}_N^M$ of the form*

$$f(x) = \sum_{k=1}^{N+1} \hat{f}_{N+k}^M T_{N+k}(x), \tag{33}$$

where $\hat{f}_{N+k}^M$ is given in (21). For the sake of simplicity, we will denote

$$F_k = \hat{f}_{2N+1-k}^M, \qquad k = 0, \cdots, N. \tag{34}$$

Let ξ_j^N be defined as in (5); then, for $k = 0, \cdots, N$,

$$F_k = \frac{4}{\beta_k^N(2N+1)} \sum_{j=0}^{N} \frac{f(\xi_j^N)T_k(\xi_j^N)}{\beta_j^N}. \tag{35}$$

Proof From (33), evaluating $f(\xi_j^N)$, we obtain

$$\begin{aligned} f(\xi_j^N) &= \sum_{k=1}^{N+1} \hat{f}_{N+k}^M \cos\left[\frac{(2N+1+k-(N+1))2\pi j}{2N+1}\right] \\ &= \sum_{k=1}^{N+1} \hat{f}_{N+k}^M T_{N+1-k}(\xi_j^N) = \sum_{k=0}^{N} F_k T_k(\xi_j^N). \end{aligned} \tag{36}$$

Thus, using (22),

$$\begin{aligned} \sum_{j=0}^{N} \frac{f(\xi_j^N)T_p(\xi_j^N)}{\beta_j^N} &= \sum_{k=0}^{N} \hat{f}_{N+k}^M \sum_{j=0}^{N} \frac{T_k(\xi_j^N)T_p(\xi_j^N)}{\beta_j^N} \\ &= \frac{N}{\pi}\sum_{k=0}^{N} \hat{f}_{N+k}^M (T_k, T_p)_N. \end{aligned} \tag{37}$$

The result follows then from the fact that

$$(38)\qquad \begin{cases} (T_k, T_p)_N = 0, & \text{for } k \neq p, \\ (T_p, T_p)_N = \frac{\pi}{2}\beta_p^N. \end{cases}$$

Given the inversion formula (35), we can express a function $f \in \mathbf{P}_N^M$ in terms of its values at the points ξ_j^N; the corresponding Lagrange interpolation operator turns out to be a linear combination of the Lagrange Kernels $\mathcal{H}_+^{\text{N}}$ and $\mathcal{H}_{\text{GL}}^{\text{M}}$. □

Theorem 3.2
Consider a function f in $\mathbf{P}_N^M$, of the form

$$(39)\qquad f(x) = \sum_{k=1}^{N+1} \hat{f}_{N+k}^M T_{N+k}(x);$$

let ξ_j^N be defined as in (5). Then we have:

$$(40)\qquad f(x) = \sum_{k=1}^{N+1} f(\xi_j^N) g_j(x),$$

where,

$$(41)\qquad g_j(x) = 2\mathcal{H}_{\text{GL}}^{\text{M}}(x, \xi_j^N) - \mathcal{H}_+^{\text{N}}(x, \xi_j^N).$$

Proof Substituting in (39) the Chebyshev coefficients of $f(x)$ expressed as in (35), we find:

$$(42)\qquad \begin{aligned} f(x) = & \sum_{k=0}^{N} F_k T_{2N+1-k}(x) \\ = & \sum_{k=0}^{N} \frac{4}{(2N+1)\beta_k^N} \times \\ \times & \sum_{j=0}^{N} \frac{f(\xi_j^N) T_k(\xi_j^N)}{\beta_j^N} T_{2N+1-k}(x), \end{aligned}$$

thus, since $T_k(\xi_j^N) = T_{2N+1-k}(\xi_j^N)$:

$$(43)\qquad \begin{aligned} f(x) = & \sum_{j=0}^{N} f(\xi_j^N) \frac{4}{(2N+1)\beta_j^N} \times \\ \times & \sum_{k=0}^{N} \frac{T_{2N+1-k}(\xi_j^N) T_{2N+1-k}(x)}{\beta_k^N}. \end{aligned}$$

Rearranging the terms in (43), we eventually get:

$$(44)\qquad \begin{aligned} f(x) = & \textstyle\sum_{j=0}^{N} f(\xi_j^N) \frac{4}{(2N+1)\beta_j^N} \times \\ \times & \sum_{p=1}^{N+1} \frac{T_{N+p}(\xi_j^N) T_{N+p}(x)}{\sigma_p^{N+1}}, \end{aligned}$$

where $\sigma_{N+1}^{N+1} = 2$, and $\sigma_p^{N+1} = 1$ $(p = 1, \cdots, N)$. The Theorem is proved, providing that:

$$(45)\qquad \begin{aligned} 2\mathcal{H}_{\text{GL}}^{\text{M}}(x, \xi_j^N) & - \mathcal{H}_+^{\text{N}}(x, \xi_j^N) = \\ & = \frac{4}{(2N+1)\beta_j^N} \sum_{p=1}^{N+1} \frac{T_{N+p}(\xi_j^N) T_{N+p}(x)}{\sigma_p^{N+1}}; \end{aligned}$$

this is a consequence of the alternative representation of the Lagrange polynomials $\mathcal{H}_{\text{GL}}^{\text{M}}$ and $\mathcal{H}_+^{\text{N}}$, (19) and (10). □

4 The linear case

We consider in this section the instructive case of the linear parabolic equation:

$$(46)\qquad \begin{cases} u_t - \nu u_{xx} = 0, & x \in (-1, 1), \quad t > 0, \\ u(-1, t) = u(1, t) = 0, & t > 0, \\ u(x, 0) = u_0, & x \in (-1, 1), \end{cases}$$

where ν is a positive constant. The nonlinear Burgers equation will be considered in a future work.

The choice of homogeneous boundary conditions is just for the sake of simplicity; all the results we present extend easily to the non homogeneous case.

In the following we propose a Nonlinear Galerkin Method for problem (46) and compare it with a slightly modified version of the Chebyshev–Legendre method presented in [4].

4.1 The Chebyshev–Legendre collocation method

We describe hereafter the Chebyshev–Legendre Collocation Method for (46) on the fine grid x_j^M. We recall that in the Chebyshev–Legendre Collocation method the boundary conditions are imposed via a penalty method in such a way that the method is stable in the usual L^2–norm (rather than the weighted L^2–norm). We present here a slightly modified version of the method, in which a penalty term is still present, but the boundary conditions are satisfied exactly. In order to prove the L^2 stability, we need to introduce the Legendre collocation points ζ_j^M defined as the roots of the polynomial $(1-x^2)P_M'$, for $M = 2N+1$. Note that we do not need to use the Legendre points in the actual computations but they are "ghost points" introduced only for the sake of the proof. The discrete scalar product corresponding to the points ζ_j^M is defined as follows:

$$(47)\qquad \langle f, g \rangle_M = \sum_{j=0}^{M} f(\zeta_j^M) g(\zeta_j^M) \omega_j,$$

where the ω_j are the usual Legendre weights (see *e.g.* [5], Chapter II]).

For $M = 2N+1$, let $\mathbf{P}_M$ be defined as in (3); we denote

$$(48)\quad B_M^+ = \frac{(1+x)P_M'(x)}{2P_M'(1)} \qquad B_M^- = \frac{(1-x)P_M'(x)}{2P_M'(-1)}.$$

In the classical Chebyshev–Legendre method we seek the polynomial $u_M \in \mathbf{P}_M$ that satisfies:

$$(49)\quad \frac{d}{dt}u_M(x) = \nu\frac{\partial^2 u_M(x)}{\partial x^2} - \tau_0 u(1,t)B_M^+(x) - \tau_M u(-1,t)B_M^-(x),$$

at the points $x = x_j^M$, the Chebyshev collocation points defined in (4); τ_0 and τ_M are positive parameters determined in the stability proof.

We propose, instead:

$$(50)\quad \frac{d}{dt}u_M(x) = \nu\frac{\partial^2 u_M(x)}{\partial x^2} - u_{M,xx}(1,t)B_M^+(x) - u_{M,xx}(-1,t)B_M^-(x),$$

at the points $x = x_j^M$. Note that in this case, since $\frac{d}{dt}u_M(\mp 1) = 0$, the boundary conditions are satisfied for any $t > 0$ if they are initially satisfied.

In both cases the penalty term is different from zero for all the Chebyshev points ξ_j^N; this adds some penalty terms to the differentiation matrix; however, since $P_M'(\xi_j^N)$ are given explicitly in [4], Section 3, these additional terms can be evaluated once and for all for any grid size M. In [4] the following stability result is stated.

Theorem 4.1
Let $u = u(x,t) \in \mathbf{P}_M$ be the solution of the Chebyshev–Legendre scheme (49). If τ_0 and τ_M satisfy the following conditions:

$$(51)\quad \tau_0 \geq \frac{\nu}{4\omega_0^2}, \qquad \tau_M \geq \frac{\nu}{4\omega_M^2},$$

then u satisfies:

$$(52)\quad \langle u,u\rangle_M \leq \langle u(0),u(0)\rangle_M - 2\int_0^t \sum_{j=1}^{M-1} u_x^2(\zeta_j^M,t)\,dt.$$

Remark 4.1 *The crucial point in the stability proof for this method relies on the fact that equality (49) is actually verified at every point $x \in (-1,1)$, since both sides of (49) are polynomials of degree M that agree at $M+1$ points. Hence, we can read the equality at the Legendre points ζ_j^M and thus carry on the proof as in the usual Legendre-collocation case. For all the details see [4], Lemma 4.1, Theorem 4.1]. The same remark is valid for the scheme (50) with the simplification that, since the boundary conditions are exactly satisfied, the boundary terms in the integrations by parts are zero.*

4.2 The nonlinear Galerkin method

As suggested by Theorem (3.1), we have a natural decomposition of any function $u \in \mathbf{P}_M$:

$$(53)\quad u = \mathcal{J}_N u + \mathcal{G}_M u = y + z,$$

where $\mathcal{J}_N = (Q_N + \tilde{Q}_N)/2$ and $\mathcal{G}_M = I_M - \mathcal{J}_N$. Projecting equation (49) with $\mathcal{J}_N$ and $\mathcal{G}_M$, respectively, we obtain the following scheme:

$$(54)\quad \begin{cases} y_t(x) - \nu y_{xx}(x) - \nu\mathcal{J}_N z_{xx}(x) \\ \quad = -\mathcal{J}_N[\tau_0 u(1,t)B_M^+(x) + \tau_M u(-1,t)B_M^-(x)], \\ \alpha z_t(x) - \nu(I_M - \mathcal{J}_N)z_{xx}(x) \\ \quad = -(I_M - \mathcal{J}_N)[\tau_0 u(1,t)B_M^+(x) + \tau_M u(-1,t)B_M^-(x)], \end{cases}$$

at the points $x = \xi_j^N$; the coefficient α is equal to 0 or 1 according to the term z_t being removed or not.

Remark 4.2 *Note that the penalty terms $\mathcal{J}_N B_M^+$, $\mathcal{J}_N B_M^-$, $(I_M - \mathcal{J}_N)B_M^+$ and $(I_M - \mathcal{J}_N)B_M^-$ are different from zero at all the points ξ_j^N, but they can be computed once and for all at the beginning of the iteration. This leads to a slight modification of the differentiation matrix.*

As in the previous Section, one can consider a slightly different scheme (corresponding to (50)):

$$(55)\quad \begin{cases} y_t(x) - \nu y_{xx}(x)\nu\mathcal{J}_N z_{xx}(x) \\ \quad = -\mathcal{J}_N[(y+z)_{xx}(1,t)B_M^+(x) + (y+z)_{xx}(-1,t)B_M^-(x)], \\ \alpha z_t(x) - \nu(I_M - \mathcal{J}_N)z_{xx}(x) \\ \quad = -(I_M - \mathcal{J}_N)[(y+z)_{xx}(1,t)B_M^+(x) + (y+z)_{xx}u(-1,t)B_M^-(x)], \end{cases}$$

at the points $x = \xi_j^N$.

The stability of the methods is a direct consequence of the stability of (49) and (50). In fact, from (54) it is immediately seen that $y + z$ satisfies equation (49) at the points ξ_j^N. Actually, equality (54a) and (54b) are true also at the points η_j^N. Let us consider the first equation: both sides are polynomials of degree N that agree at $N+1$ points, thus they agree at all the points $x \in [-1,1]$, in particular at η_j^N. Regarding the second equation, we observe that both sides are polynomials in $\mathbf{P}_N^M$, of the form $\phi(x) = \sum_{k=1}^N \hat{\phi}_{N+k}^M T_{N+k}(x)$ and, thanks to Theorem (3.2), these polynomials are uniquely determined by their values at the points ξ_j^N. Hence, we deduce that equality (54b) is verified at all points in the interval $[-1,1]$, and thus at the points η_j^N. The following Theorem is, therefore, proved.

Theorem 4.2
Let $y(x,t)$ and $z(x,t)$ be the solutions of the Chebyshev–Legendre scheme (54). If τ_0 and τ_M satisfy (51), then $y+z$ satisfies:

$$\begin{aligned}(56)\qquad & \langle (y+z)(\cdot,t),(y+z)(\cdot,t)\rangle_M \\ & \le \langle u(\cdot,0),u(\cdot,0)\rangle_M - 2\int_0^t \sum_{j=1}^{M-1}(y+z)_x^2(\zeta_j^M,t)\,dt.\end{aligned}$$

In a similar way one can prove that the scheme (55) is stable in norm L^2.

5 Implementation issues

In this section we discuss some aspects of the implementation of the Nonlinear Galerkin method introduced in the previous section. We will show how the splitting of the equation in low and high modes produces a significant gain in terms of the total computational cost, compared to the classical method (49).

We consider in the following the scheme (54), but similar considerations are valid in the case of the scheme (55). We propose to solve both equations in (54) on the *physical space*, via collocation on the coarse grid points $\{\xi_j^N\}$. In fact, the unknown z contains only modes with wave numbers larger than N, thus it belongs to the space U_N^M. The representation formula (40) given in Theorem (3.2) is used to solve the high modes equation in (54) on the coarse grid $\{\xi_j^N\}$. Note also that the penalty terms $\mathcal{J}_N B_M^+$, $\mathcal{J}_N B_M^-$, $(I_M-\mathcal{J}_N)B_M^+$ and $(I_M-\mathcal{J}_N)B_M^-$ can be computed once and for all at the beginning of the iteration.

In the following we will compare the number of operations needed to advance the solution of from time $t=0$ to time $t=1$. For the sake of simplicity, we assume that all the derivatives are computed via vector–matrix multiplications and that the time stepping is done via an explicit method.

Consider the Chebyshev–Legendre scheme (49) based on the fine grid $\{x_j^M\}$, containing $2N+2$ points. In order to compute the spatial derivative we need $(2N)^2$ operations. Since $\Delta t \sim \frac{1}{(2N)^4}$, we have the total number of $64N^6$ operations.

Consider now the Chebyshev–Legendre nonlinear Galerkin scheme (54). Assume that y and z are given at the fine mesh. In order to solve (54a) $3N^2$ operations are needed for the derivatives (note that $\frac{\partial}{\partial x}Q_N$ and $\frac{\partial}{\partial x}\tilde{Q}_N$ should be taken separately). To solve equation (54b), using (40), we can compute the spatial derivative in just N^2 operations. The total count to evaluate the spatial derivatives sums up to $5N^2$. Also in view of the application we have in mind to a nonlinear equation, we suggest to advance the coarse grid equation in time with an explicit method, and therefore consider a time step of order $\Delta t \sim \frac{1}{N^4}$. On the other hand, we suggest to use for the high modes equation, which is linear in z, an implicit method. The total computational cost is then $5N^6$, offering a substantial saving over the non–split scheme.

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