Polynomial Approximation of Some Singular Solutions in Weighted Sobolev Spaces

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Abstract

Error estimates for some spectral projection operators in weighted Sobolev spaces of Jacobi type are derived in terms of a new family of weighted spaces, improving standard estimates. Our results are used to improve error estimates for the Jacobi spectral solution of a model problem in a square by taking into account the decomposition of the solution near the corners. This generalizes to the Jacobi framework some results known in the unweighted case.

Key words: weighted Sobolev spaces, Jacoby polynomials, spectral projection operators, singularities in elliptic problems.

AMS subject classifications: 65N35, 41A10, 65N15.

1 Introduction

The analysis of the convergence rate of high order discretizations of elliptic problems over polygonal domains requires us to take into account the structure of the solutions near the corners. An early reference, concerning the p-version of the finite element method (F.E.M.) is [1]. It has been noted therein that the approximation results for the singular part of the solution, as obtained from estimates involving the usual unweighted Sobolev spaces H^s, are not optimal for the p-version. An approximation theory for the p-version in the framework of certain weighted Sobolev spaces is given in [9]. In that paper, estimates for the distance, measured in standard Sobolev norms, between a function u and its high-order polynomial approximation u_p are given in terms of the norm of u in some weighted spaces. This theory has been applied in [10] to the analysis of the p-version of the F.E.M. over polygonal and polyhedral domains. It enables us to recover optimal convergence rates by analyzing separately the singular and the regular part of the solution.

The results for the Legendre spectral discretizations are very close to those for the p-version since in both cases standard Sobolev norms of the error $u - u_p$ are concerned. But for the numerical analysis of Chebyshev spectral methods this error must be measured in terms of weighted Sobolev norms based upon the Chebyshev weight. Moreover, the analysis may require results related to other weights, for instance to the inverse of the Chebyshev weight as in [3]. So, it is useful to consider a wide range of weights, namely the Jacobi weights, as in [4]. The approximation theory for these spaces as developed therein involves high order weighted Sobolev spaces H^s_α. For $\alpha = 0$ these spaces reduce to the usual unweighted ones. So, it can be expected that the application of this theory to the H^1_α approximation of the singular part of the solutions will not yield optimal estimates.

The aim of this paper is to improve the results in [4] for the Jacobi spectral approximation by using a new family of weighted spaces and the decomposition of the solution into a regular and a singular part. So, our results extend some results of [9] and [10] to the Jacobi framework .

Although in this paper we present the analysis of a simple model problem as an application , the techniques and results can be useful in the general context of elliptic problems.

The next section is devoted to introducing the basic notations. In section 3, we define the new family of weighted spaces and state our basic approximation results. Section 4 is devoted to obtaining a characterization of the new spaces in terms of an intrinsic norm. In section 5, we study the approximation of some singular functions related to the solution of some elliptic problems. In section 6, we obtain improved convergence estimates for a simple model

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problem. Finally, we resume our conclusions in section 7.

For the sake of brevity some proofs have been omitted or only sketched but the detailed version of this work will appear in Fdez-Manín [11].

2 Preliminaries and notations

Let $\Lambda = (-1, 1)$ and $\alpha > -1$. We denote

$$L^2_\alpha(\Lambda) = \{v : \Lambda \mapsto \mathcal{C} / \parallel v \parallel_{0,\alpha,\Lambda} = (\int_\Lambda v(x)^2 \rho_\alpha(x) dx)^{\frac{1}{2}} < \infty\}.$$

For any $s > 0$ and $\alpha \in (-1, 1)$ we denote $H^s_\alpha(\Lambda)$ the weighted Sobolev space of order s associated to the weight function $(1 - x^2)^\alpha$. Its norm will be denoted by $\| \cdot \|_{s,\alpha,\Lambda}$. Let $P_N(\Lambda)$ be the space of polynomials with degree $\leq N$ in Λ. Moreover, we shall note:

- Π^α_N the orthogonal projection operator from $L^2_\alpha(\Lambda)$ onto $P_N(\Lambda)$.
- $\{\mathcal{J}^\alpha_n\}$ the family of Jacobi polynomials associated to the weight $(1 - x^2)^\alpha$ normalized in the following way: the degree of $\mathcal{J}^\alpha_n$ is n and it satisfies

$$(1) \qquad \mathcal{J}^\alpha_n(\pm 1) = (\pm 1)^n \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)}$$

where Γ stands for the classical Euler's gamma function. Their norm is given by

$$(2) \qquad \|\mathcal{J}^\alpha_n\|^2_{0,\alpha,\Lambda} = \frac{2^{2\alpha+1}(\Gamma(n + \alpha + 1))^2}{(2n + 2\alpha + 1)\Gamma(n + 1)\,\Gamma(n + 2\alpha + 1)}$$

(3)

We also use the integral relation

$$(4) \qquad \int \mathcal{J}^\alpha_n(t)dt = \frac{1}{2n + 2\alpha + 1}\left(\frac{n + 2\alpha + 1}{n + \alpha + 1}\mathcal{J}^\alpha_{n+1}(x) - \frac{n + \alpha}{n + 2\alpha}\mathcal{J}^\alpha_{n-1}(x)\right)$$

when $\int \mathcal{J}^\alpha_n(t)dt$ is such that

$$\int_{-1}^1 \mathcal{J}^\alpha_n(t)\rho_\alpha(t)dt = 0.$$

A survey on the properties of the $\mathcal{J}^\alpha_n$ can be found in [4] and [12] .

We shall also consider the function spaces defined over $\Omega = \Lambda \times \Lambda$. For any $-1 < \alpha < 1$ and $s > 0$, $H^s_\alpha(\Omega)$ will stand for the Sobolev space of order s related to the weight $(1 - x^2)^\alpha(1 - y^2)^\alpha$ (see [4] for a precise definition). The norm in this space will be denoted $\| \cdot \|_{s,\alpha,\Omega}$. We shall note

- $P_N(\Omega)$ the space of polynomials with degree $\leq N$ with respect to each variable.
- Π^α_N the orthogonal projection operator from $L^2_\alpha(\Omega)$ onto $P_N(\Omega)$.

It is standard to note that $\Pi^\alpha_N = \Pi^{\alpha,(x)}_N \circ$

3 Approximation properties of the $Z^s_\alpha(\Lambda)$ spaces

For each integer $m \geq 0$, we define

$$(5) \quad Z^m_\alpha(\Lambda) = \{v \in L^2_\alpha(\Lambda) / \frac{d^j v}{dx^j} \in L^2_{\alpha+j}(\Lambda), 1 \leq j \leq m\}$$

equipped with the norm

$$(6) \qquad \parallel v \parallel_{Z^m_\alpha(\Lambda)} = (\sum_{j=0}^m \left\| \frac{d^j v}{dx^j} \right\|^2_{L^2_{\alpha+j}(\Lambda)})^{\frac{1}{2}}$$

and for $s > 0$ non integer, $s = m + \sigma$, with $0 < \sigma < 1$, we define $Z^s_\alpha(\Lambda)$ by

$$Z^s_\alpha(\Lambda) = [Z^m_\alpha(\Lambda), Z^{m+1}_\alpha(\Lambda)]_{\sigma,2}$$

where $[.]_{\theta,2}$ stands for the K-interpolation method (see [2]).

The next three lemmas show the behavior of the derivation and integration operators over $Z^s_\alpha(\Lambda)$ and a property which will be useful in the following section.

Lemma 3.1 *For $\alpha > -1$ and $u \in L^2_{\alpha+1}(\Lambda)$ we define (Pu) by $(Pu)(x) = \int_0^x u(t)dt$. Then, for all $m \geq 1$ integer, the mapping P^m is continuous from $L^2_{\alpha+m}(\Lambda)$ into $Z^m_\alpha(\Lambda))$.*

Lemma 3.2 *For $\alpha > -1$, $0 < \theta < 1$ and any integer $m \geq 1$, it holds:*

$$(7) \qquad [Z^m_\alpha(\Lambda), Z^{m+1}_\alpha(\Lambda)]_{\theta,2} = \left\{v \in Z^m_\alpha(\Lambda) / \frac{d^m v}{dx^m} \in [Z^0_{\alpha+m}(\Lambda), Z^1_{\alpha+m}(\Lambda)]_{\theta,2}\right\}$$

Lemma 3.3 *For each non negative integer m and for $u \in Z^m_\alpha(\Lambda)$: $u = \sum_{n=0}^\infty \hat{u}_n \mathcal{J}^\alpha_n(x)$, being $\hat{u}_n$ the corresponding Fourier coefficients , we have:*

$$(8) \qquad \frac{d^j u}{dx^j} = \sum_{n=j}^\infty \hat{u}_n \frac{d^j \mathcal{J}^\alpha_n}{dx^j} \ \ in \ \ L^2_{\alpha+j}(\Lambda) \ , \ 1 \leq j \leq m$$

The next lemma gives a characterization of the space $Z^s_\alpha(\Lambda)$ together with an equivalent norm in this space which is useful in proving the approximation result in norm $\|\cdot|_{0,\alpha,\Lambda}$.

Lemma 3.4 *Let be $s > 0$. For $u \in L^2_\alpha(\Lambda)$, $u = \sum_{n=0}^{\infty} \hat{u}_n \mathcal{J}^\alpha_n(x)$ we define*

$$(9) \qquad \| u \|^*_{Z^s_\alpha(\Lambda)} = \left(\sum_{n=0}^{\infty} |\hat{u}_n|^2 (1+n^2)^s \| \mathcal{J}^\alpha_n \|^2_{0,\alpha,\Lambda}\right)^{\frac{1}{2}}$$

then

$$Z^s_\alpha(\Lambda) = \{u \in L^2_\alpha(\Lambda) / \| u \|^*_{Z^s_\alpha(\Lambda)} < \infty\}$$

and the norm $\| \cdot \|^*_{Z^s_\alpha(\Lambda)}$ is a norm in $Z^s_\alpha(\Lambda)$ equivalent to $\| \cdot \|_{Z^m_\alpha(\Lambda)}$.

Proof When s is an integer, the result easily follows by using lemmas (3.1), (3.3) and properties of the polynomials $\mathcal{J}^\alpha_n$. For the non-integer case we use a standard interpolation argument. □

Theorem 3.1 *If $u \in Z^s_\alpha(\Lambda)$ for $0 \le s' \le s$, the following estimation is satisfied*

$$\| u - \Pi^\alpha_N u \|_{Z^{s'}_\alpha(\Lambda)} \le C\, N^{s'-s} \| u \|_{Z^s_\alpha(\Lambda)} .$$

Proof The result follows from (9) using the classical techniques for these kinds of estimations. □

Theorem 3.2 *Let be $0 \le s' < s$ and $u \in Z^{s'}_\alpha(\Lambda)$. If*

$$\forall N \in I\!N ,\ \exists u_N \in P_N(\Lambda) / \| u \|_{Z^{s'}_\alpha(\Lambda)} \le A N^{s'-s}$$

with $A = A(u)$ independent of N, Then $u \in Z^{s-\epsilon}_\alpha(\Lambda)$ $\forall \epsilon > 0$. Moreover

$$\|u - u_N\|_{Z^{s-\epsilon}_\alpha(\Lambda)} \le C(s, s', \alpha, \epsilon) \left[\| u \|_{Z^{s'}_\alpha(\Lambda)} + A\right]$$

In order to obtain the approximation theorem in norm H^1_α we prove a technical result that gives the expression of the Fourier coefficients of u' with respect to $\{\mathcal{J}^\alpha_n\}$ in terms of those of u with respect to the same basis. This result generalizes well-known expressions for the case $\alpha = 0$ and $\alpha = -1/2$. In the next step we obtain that $\Pi^\alpha_N(u') - (\Pi^\alpha_N u)'$ belongs to a bidimensional subspace of polynomials and finally we conclude the approximation theorem.

Theorem 3.3 *For $u \in C^\infty(\bar{\Lambda})$, $u = \sum_{n=0}^{\infty} \hat{u}_n \mathcal{J}^\alpha_n$ and $u' = \sum_{n=0}^{\infty} \hat{u}^{(1)}_n \mathcal{J}^\alpha_n$, then*

$$(10) \qquad \hat{u}^{(1)}_k = \frac{(2k+2\alpha+1)\Gamma(k+2\alpha+1)}{\Gamma(k+\alpha+1)} \sum_{\substack{n=k+1 \\ n+k\ odd}}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+2\alpha+1)} \hat{u}_n$$

(in the exceptional case $k = 0$ and $\alpha = -1/2$ we set $(2k + 2\alpha + 1)\Gamma(k + 2\alpha + 1) = \Gamma(2\alpha + 2)$)

Proof Firstly by using (4) we obtain the equality

$$(11) \quad \hat{u}_n = \frac{(n+2\alpha)}{(n+\alpha)(2n+2\alpha-1)} \hat{u}^{(1)}_{n-1} - \frac{(n+\alpha+1)}{(n+2\alpha+1)(2n+2\alpha+3)} \hat{u}^{(1)}_{n+1}$$

secondly we solve the homogeneous difference equation

$$(12) \qquad \frac{(n+2\alpha)}{(n+\alpha)(2n+2\alpha-1)} \mathcal{X}_{n-1} - \frac{(n+\alpha+1)}{(n+2\alpha+1)(2n+2\alpha+3)} \mathcal{X}_{n+1} = 0$$

$$\mathcal{X}_1 = \mathcal{X}_2 = 1$$

to obtain the general term for $k \ge 1$

$$\mathcal{X}_k = \frac{(2k+2\alpha+1)}{(2\alpha-2l+5)} \frac{\Gamma(k+2\alpha+1)}{\Gamma(k+\alpha+1)} \frac{\Gamma(\alpha-l+3)}{\Gamma(2\alpha-l+3)}$$

with l=0 if k even, l=1 if k odd. For $k \ge 1$ let $\hat{v}^{(1)}_k$ be defined by

$$\hat{u}^{(1)}_k = \mathcal{X}_k \hat{v}^{(1)}_k .$$

Then, equation (11) leads to

$$\hat{v}^{(1)}_{n-1} - \hat{v}^{(1)}_{n+1} = \frac{(n+\alpha)(2n+2\alpha-1)}{n+2\alpha} \frac{1}{\mathcal{X}_{n-1}} \hat{u}_n$$

using that $u \in C^\infty(\bar{\Lambda})$ we deduce that $\lim_{k\to\infty} \hat{v}^{(1)}_k = 0$. Therefore

$$\hat{v}^{(1)}_k = \sum_{\substack{n=k+1 \\ n+k odd}} \frac{(n+\alpha)(2n+2\alpha-1)}{n+2\alpha} \frac{1}{\mathcal{X}_{n-1}} \hat{u}_n$$

and we conclude the result for $k \ge 1$. The identity (11) with $n = 1$ solves the case $\hat{u}^{(1)}_0$.

□

Lemma 3.5 *For* $u \in H^1_\alpha(\Lambda)$ *and* $u' = \sum_{n=0}^{\infty} \hat{u}^{(1)}_n \mathcal{J}^\alpha_n$ *we have the following property:*

$$\Pi^\alpha_N(u') - (\Pi^\alpha_N u)' = \begin{cases} \lambda^\alpha(N)\phi^N_0 + \lambda^\alpha(N+1)\phi^N_1 & \text{for } N \text{ even} \\ \lambda^\alpha(N+1)\phi^N_0 + \lambda^\alpha(N)\phi^N_1 & \text{for } N \text{ odd} \end{cases} \tag{13}$$

where

$$\phi^N_0 = \sum_{\substack{n=0 \\ n \text{ even}}}^{N} \frac{(2n+2\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)} \mathcal{J}^\alpha_n$$

$$\phi^N_1 = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{(2n+2\alpha+1)\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)} \mathcal{J}^\alpha_n$$

$$\lambda^\alpha(m) = \frac{\Gamma(m+\alpha+1)}{(2m+2\alpha+1)\Gamma(m+2\alpha+1)} \hat{u}^{(1)}_m.$$

The previous lemma allows us to state the following theorem.

Theorem 3.4 *For* $s > 1-\alpha$ *and* $u \in Z^s_\alpha(\Lambda) \bigcap H^1_\alpha(\Lambda)$ *we have*

$$\| \Pi^\alpha_N(u') - (\Pi^\alpha_N u)' \|_{0,\alpha,\Lambda} \leq C\, N^{2-s}\ \| u \|_{Z^s_\alpha(\Lambda)}\ . \tag{14}$$

We now focus on the two-dimensional case. First, for $s \geq 0$ we define the space $Z^s_\alpha(\Omega)$ by

$$Z^s_\alpha(\Omega) = L^2_\alpha(\Lambda; Z^s_\alpha(\Lambda)) \bigcap Z^s_\alpha(\Lambda; L^2_\alpha(\Lambda)). \tag{15}$$

Using the standard tensorization argument and theorem 3.1 we obtain the following result concerning the approximation in the $\| \ \|_{0,\alpha,\Lambda}$ norm.

Theorem 3.5 *For* $s > 0$ *and* $u \in Z^s_\alpha(\Omega)$ *we have:*

$$\| u - \Pi^\alpha_N u \|_{0,\alpha,\Omega} \leq C\, N^{-s}\ \| u \|_{Z^s_\alpha(\Omega)}$$

For the approximation in $\| \ \|_{1,\alpha,\Lambda}$, now we state:

Theorem 3.6 *For all* $r > 0, s > 1-\alpha$ *and* $u \in Z^s_\alpha(\Omega)$ *such that* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in Z^r_\alpha(\Omega)$ *we have:*

$$\| u - \Pi^\alpha_N u \|_{1,\alpha,\Omega} \leq C\, \Big[N^{2-s}\ \| u \|_{Z^s_\alpha(\Omega)} +$$

$$N^{-r} \left(\| \frac{\partial u}{\partial x} \|_{Z^r_\alpha(\Omega)} + \| \frac{\partial u}{\partial y} \|_{Z^r_\alpha(\Omega)} \right) \Big] \tag{16}$$

Proof We use the following decomposition

$$\frac{\partial u}{\partial x} - \left(\frac{\partial}{\partial x}(\Pi^\alpha_N u)\right) = \left[\frac{\partial u}{\partial x} - \Pi^{\alpha,(y)}_N \frac{\partial u}{\partial x}\right] + \Pi^{\alpha,(y)}_N \left(\frac{\partial u}{\partial x} - \Pi^{\alpha,(x)}_N \frac{\partial u}{\partial x}\right) + \Pi^{\alpha,(y)}_N \left(\Pi^{\alpha,(x)}_N \frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(\Pi^{\alpha,(x)}_N u)\right)$$

and we apply the monodimensional result to the first and second terms. The third term is handled by using theorem 3.3

□

4 Intrinsic norms of the spaces $Z^s_\alpha(\Lambda)$

In this section we use the notations appearing in the works Bernardi-Dauge-Maday [7] and Triebel [13], as well as some results therein.

For m non negative integer we denote the space

$$V^{m,2}_\alpha(\Lambda) = \{v \in D'(\Lambda) / \| v \|_{V^{m,2}_\alpha} < \infty\}$$

with the norm defined by

$$\| v \|_{V^{m,2}_\alpha} = \left(\sum_{j=0}^{m} \left\| \frac{d^j v}{dx^j} \right\|^2_{0,\alpha+2(j-m),\Lambda} \right)^{\frac{1}{2}}. \tag{17}$$

For any number $s > 0$ non integer we put $s = m + \theta$ with $0 < \theta < 1$ and m integer, and denote

$$V^{s,2}_\alpha(\Lambda) = \{v \in D(\Lambda) / \| v \|_{V^{s,2}_\alpha} < \infty\}$$

with the norm

$$\| v \|^2_{V^{s,2}_\alpha} = \sum_{j=0}^{m} \left\| \frac{d^j v}{dx^j} \right\|^2_{0,\alpha+2(j-s),\Lambda} + \int\!\!\int_{\Delta_{\Lambda,a}} \frac{\left| v^{(m)}(x) - v^{(m)}(y) \right|^2}{|x-y|^{1+2\theta}} (1-x^2)^\alpha dx\, dy \tag{18}$$

where for all $a > 1$ the domain $\Delta_{\Lambda,a}$ is defined by:

$$\Delta_{\Lambda,a} = \left\{ \begin{array}{r} (\xi,\eta) \in \Lambda \times \Lambda / \xi < 0 \ \text{ and } \ \frac{1+\xi}{a} < 1+\eta < a(1+\xi) \\ \text{or } \xi > 0 \ \text{ and } \ \frac{1-\xi}{a} < 1-\eta < a(1-\xi) \end{array} \right\}$$

The space denoted by $W^{s,2}_\alpha(\Lambda)$ in [7] coincides with $H^s_\alpha(\Lambda)$. We characterize the space $Z^s_\alpha(\Lambda)$ by the following theorem.

Theorem 4.1 *Let s be a positive real number which is not an integer, $s = m + \theta$ with $0 < \theta < 1$ and $s \neq 1 + \alpha$ when $\alpha \in (-1, 0)$. Then, a norm in $Z^s_\alpha(\Lambda)$ which defines the space is:*

$$(19) \quad \| u \| = \Big[\| u \|^2_{Z^m_\alpha(\Lambda)} + \int\int_{\Delta_{\Lambda,a}} \frac{|u^{(m)}(x) - u^{(m)}(y)|^2}{|x-y|^{1+2\theta}} (1-x^2)^{\alpha+s} dx\, dy \Big]^{\frac{1}{2}}.$$

Proof We consider three cases depending on the values of s and α

First case: $\alpha > 0$, and $0 < s < 1$.

Following the notation in Triebel [13] we have:

$$(20) \qquad Z^0_\alpha(\Lambda) = W^0_2(\Lambda, \rho_\alpha, \rho_\alpha)$$

$$V^{1,2}_{\alpha+1}(\Lambda) = W^1_2(\Lambda, \rho_{(\alpha+1)}, \rho_{(\alpha-1)})$$

and with the notation in Bernardi-Dauge-Maday [7] we can state the inclusions:

$$V^{1,2}_{\alpha+1}(\Lambda) \subset Z^1_\alpha(\Lambda) \subset W^{1,2}_{\alpha+1}(\Lambda)$$

By using theorem (1.b.10) and lemma (1.b.22) in [7] we have that $V^{1,2}_{\alpha+1}(\Lambda) = W^{1,2}_{\alpha+1}(\Lambda)$. Then

$$(21) \quad Z^1_\alpha(\Lambda) = V^{1,2}_{\alpha+1}(\Lambda) = W^1_2(\Lambda, \rho_{(\alpha+1)}, \rho_{(\alpha-1)}).$$

Moreover the identifications (20) and (21) lead to:

$$Z^s_\alpha(\Lambda) = [W^0_2(\Lambda, \rho_\alpha, \rho_\alpha), W^1_2(\Lambda, \rho_{\alpha+1}, \rho_{\alpha-1})]_{s,2}$$

From the results in section 3.4.2 of [13] we have:

$$Z^s_\alpha(\Lambda) = W^s_2(\Lambda, \rho_{(\alpha+s)}, \rho_{(\alpha-s)}).$$

Finally we apply proposition (1.c.2) and remark (1.c.3) in [7] to deduce that for s non integer $W^s_2(\Lambda, \rho_{(\alpha+s)}, \rho_{(\alpha-s)}) = V^{s,2}_{\alpha+s}(\Lambda)$ and therefore

$$Z^s_\alpha(\Lambda) = W^s_2(\Lambda, \rho_{(\alpha+s)}, \rho_{(\alpha-s)}) = V^{s,2}_{\alpha+s}(\Lambda)$$

Using again theorem (1.b.10) of [7] we obtain

$$V^{s,2}_{\alpha+s}(\Lambda) = W^{s,2}_{\alpha+s}(\Lambda)$$

and conclude the theorem in this case.

Second case: $\alpha > -1$, and $m \geq 1$.

If follows easily from the first case together with lemma (3.2).

Third case: $-1 < \alpha < 0$ and $0 < s < 1$.

In this case, we prove first an analogous result for the spaces defined on the interval $\mathcal{I} = (0, 1)$ in a quite similar way. This allows us to handle only one singular point instead of two. We consider the weight x^α and define the natural spaces $L^2_\alpha(\mathcal{I})$, $W^{s,2}_\alpha(\mathcal{I})$, $V^{s,2}_\alpha(\mathcal{I})$ and $Z^s_\alpha(\mathcal{I})$. Then, the mapping

$$\begin{array}{rccl} T: & Z^m_\alpha(\mathcal{I}) & \rightarrow & W^{m,2}_{1+2\alpha}(\mathcal{I}) \\ & v & \rightarrow & \hat{v}(t) = v(t^2) \end{array}$$

is an isomorphism for m=0, 1. Therefore

$$T : [Z^0_\alpha(\mathcal{I}), Z^1_\alpha(\mathcal{I})]_s \rightarrow [W^{0,2}_{1+2\alpha}, W^{1,2}_{1+2\alpha}]_s = W^{s,2}_{1+2\alpha}(\mathcal{I})$$

is also an isomorphism. As we have a characterization of the norm in $W^{s,2}_{1+2\alpha}(\mathcal{I})$ we can obtain a norm in $Z^s_\alpha(\mathcal{I})$. Finally, the result holds for the domain Λ from standard localization techniques . □

5 Approximation of singular functions

We are interested in the approximation by polynomials of the functions $W_\epsilon(x, y)$ defined by:

$$W_\epsilon(x, y) = ((1-x) + i(1-y))^\epsilon$$

for $\epsilon > -1$, because they coincide with the singular part of the solution of the Dirichlet or Neumann problem for the Laplace operator, and also for the biharmonic operator.

Theorem 5.1 *For any real number $\epsilon > -1$ the function $W_\epsilon(x, y)$ belongs to $Z^s_\alpha(\Omega)$ for any positive real number $s < 2(\epsilon + \alpha + 1)$*

Proof The proof must distinguish between two cases, depending on whether s is an integer or not.

If $s = m$ integer, taking into account (15) we only have to check that $W_\epsilon(x, y) \in Z^s_\alpha(\Lambda; L^2_\alpha(\Lambda))$ because the other case is analogous. Let $0 \leq j \leq m$, the strongest singularity in the expression of $\left\| \frac{\partial^j W_\epsilon}{\partial x^j} \right\|_{L^2_{\alpha+j}(\Lambda, L^2_\alpha(\Lambda))}$ arises near the point $(1, 1)$, so:

$$(22) \qquad \left\| \frac{\partial^j W_\epsilon}{\partial x^j} \right\|_{L^2_{\alpha+j}(\Lambda; L^2_\alpha(\Lambda))} \leq C_1 + C_2 \int_0^1 \int_0^1 \left| \frac{\partial^j W_\epsilon}{\partial x^j} \right| (1-x^2)^{\alpha+j} (1-y^2)^\alpha dx\, dy$$

for some positive constants C_1 and C_2. In order to bound the last term in (21), we consider the change of variables:

$$1 - x = r \cos\phi$$

$$1 - y = r \sin\phi$$

then we obtain the integral

$$(23)\ \int_0^1 \int_0^{\frac{\pi}{2}} r^{2(\epsilon-j)+2\alpha+j+1} \cos^{\alpha+j}\phi \sin^{\alpha}\phi \, dr \, d\phi$$

which is finite because $\alpha + j > -1$, $\alpha > -1$ and $2(\epsilon - j) + 2\alpha + j + 1 > -1$ when $m < 2(\epsilon + \alpha + 1)$.
When s is not an integer we must verify

$$(24)\qquad \| W_\epsilon(x,y) \|_{Z^s_\alpha(\Lambda;L^2_\alpha(\Lambda))} < \infty$$

because the other term is analogous. To do this, we use the intrinsic norm given by theorem 4.1. Since the mapping

$$(x,y) \xrightarrow{f} f(x,y) = ((1-x) + i(1-y))^{\epsilon-m}$$

is C^∞ away from $(1,1)$, it clearly suffices to bound the integral

$$(25)\qquad \int_{\ominus} \frac{|W_{(\epsilon-m)}(x,y) - W_{(\epsilon-m)}(x',y)|^2}{|x-x'|^{1+2\theta}} \cdot (1-x^2)^{\alpha+s}(1-y^2)^{\alpha} dx\, dx'\, dy$$

where

$$\ominus = \{0 < x < 1, 0 < x' < 1, 1 - \frac{1-x}{a} < y < 1 - a(1-x)\}.$$

We make the change of variables

$$1 - x = (1-y)t$$

$$1 - x' = (1-y)tz$$

and we must verify that the integral

$$(26)\qquad I = \int_{\oplus} (1-y)^{2\epsilon+2\alpha-s+1} t^{\alpha+s-2\theta} \cdot \frac{|(t+i)^{\epsilon-m} - (tz+i)^{\epsilon-m}|^2}{|z-1|^{1+2\theta}} dz\, dt\, dy$$

is finite, where

$$\oplus = \{0 < y < 1, 0 < t < \frac{1}{1-y}, \frac{1}{a} < z < a\}$$

For fixed t, we apply Hardy's inequality in $[\frac{1}{a}, 1]$ and in $[1, a]$ to obtain

$$(27)\quad \int_{\frac{1}{a}}^{a} \frac{|(t+i)^{\epsilon-m} - (tz+i)^{\epsilon-m}|^2}{|z-1|^{1+2\theta}} dz \leq C \int_{\frac{1}{a}}^{a} t^2 |z-1|^{1-2\theta}(1+t^2h^2)^{\epsilon-m-1} dz \leq Ct^2(1+t^2h^2)^{\epsilon-m-1}$$

where

$$h = \begin{cases} \frac{1}{a} & for \quad \epsilon - m - 1 < 0 \\ a & for \quad \epsilon - m - 1 \geq 0 \end{cases}$$

From (26) and (27), we obtain that I is bounded by

$$(28)\qquad C \int_0^1 (1-y)^{2\epsilon+2\alpha-s+1} \int_0^{\frac{1}{1-y}} t^{\alpha+s-2\theta} t^2 (1+t^2h^2)^{\epsilon-m-1} dt\, dy.$$

Finally, we bound the integral with respect to t and afterwards with respect to y to obtain the desired result. □

Remark 5.1 *For any real number $\epsilon > -1$, we also introduce the function $\tilde{W}_\epsilon(x,y)$ defined by*

$$\tilde{W}_\epsilon(x,y) = ((1-x) + i(1-y))^{\epsilon} \, log\, ((1-x) + i(1-y))$$

By using the same proof, it can be verified that theorem (5.1) is still valid with W_ϵ replaced by $\tilde{W}_\epsilon$.

6 Application to elliptic problems

We consider the Dirichlet problem on the square Ω

$$(29)\qquad \begin{aligned} -\Delta u &= f \quad in\ \Omega \\ u &= 0 \quad in\ \partial\Omega. \end{aligned}$$

Let (a_j, b_j) , $1 \leq j \leq 4$ denote the vertices of Ω. If the function f belongs to $H^\rho_\alpha(\Omega)$, for $\rho \geq 0$. then a decomposition result of M. Dauge [8], guarantees that the solution u of (29) can be written as

$$u = u_r + \sum_{j=1}^{4} \alpha_j \chi_j \tilde{W}_j.$$

The function u_r belongs to $H^s_\alpha(\Omega)$ for all non negative real number $s < min\{\rho + 2, 5 + \alpha\}$, the χ_j are C^∞ cut-off functions and the functions $\tilde{W}_j$ $1 \leq j \leq 4$ are

$$\tilde{W}_j(x,y) = Im\{((a_j - x) + i(b_j - y))^2 \, log((a_j - x) + i(b_j - y)\}.$$

Therefore the solution u belongs to $H^s_\alpha(\Omega)$ for all $0 < s < min\{\rho + 2, 3 + \alpha\}$. We denote u_N the discrete solution provided by the Gauss-Lobatto-Jacobi collocation spectral method, namely:

$$(30)\qquad \begin{aligned} U_N &\in P_N(\Omega) \\ -\Delta U_N(x) &= f(x) \qquad x \in \Xi \bigcap \Omega \\ U_N(x) &= 0 \qquad x \in \Xi \bigcap \partial\Omega \end{aligned}$$

where Ξ stands for the set of Gauss-Lobatto-Jacobi nodes. So, if only standard approximation results (see [4] and [6]) are used, the following convergence estimation is obtained:

$$(31)\quad \|u - u_N\|_{1,\alpha,\Omega} \leq C_\delta\{N^{1-\sigma+\delta}\|u\|_{\sigma-\delta,\alpha,\Omega} + N^{-\rho}\|f\|_{\rho,\alpha,\Omega}\}$$

where $\sigma = min\{3+\alpha, \rho+2\}$.
However, following the same lines as in section 4 of [5] we can use the previous results in this paper to approximate the functions $\tilde{W}_j$ in the framework of the Z^s_α spaces, and use standard results to approximate the regular part u_r. Finally, to enforce the homogeneous boundary conditions we use [4, proposition V.1] and we obtain the estimation

$$(32)\quad \|u - u_N\|_{1,\alpha,\Omega} \leq C_\delta\{N^{1-\sigma+\delta}\|u\|_{\sigma-\delta,\alpha,\Omega} + N^{-\rho}\|f\|_{\rho,\alpha,\Omega}\}$$

where now $\sigma = min\{5+\alpha, 5+2\alpha, \rho+2\}$.
In (29) and (30) δ can be taken arbitrarily small.
For instance, if $\alpha = -1/2$ and $f \in C^\infty(\bar{\Omega})$, the standard results state

$$\|u - u_N\|_{1,\alpha,\Omega} = O(N^{-3/2+\delta})$$

while the ideas here proposed conclude an improvement in the estimate to obtain

$$\|u - u_N\|_{1,\alpha,\Omega} = O(N^{-3+\delta})$$

7 Conclusions

We have improved the classical results for the polynomial approximation in weighted Sobolev spaces of Jacobi type by introducing a new family of weighted spaces. We have shown in a model problem that our results enable us to improve the error estimates for Jacobi spectral methods by using the knowledge of the singularities of the solution near the corners. Our analysis includes, in particular, the important case of the Chebyshev weight. Although we have considered the analysis of a simple model problem, the results and techniques can be used in the general case of Jacobi spectral discretization of elliptic problems. In the future we will also apply the patching method with these techniques in order to state results about spectral Chebyshev approximations in more general domains.

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