Chebyshev Pseudospectral Collocation for Parabolic Problems with Nonconstant Coefficients

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Abstract

This paper analyses a Chebyshev pseudospectral collocation semidiscrete (continuous in time) discretization of a variable coefficient parabolic problem. Optimal stability and convergence estimates are given. The analysis is based on an approximation property concerning the Gauss-Lobatto-Chebyshev interpolation operator.

Key words: spectral and pseudospectral methods, Chebyshev collocation, parabolic equations.

AMS subject classifications: 65N35, 65M15, 41A10, 65M10.

1 Introduction

Spectral approximations to the Dirichlet problem for linear parabolic equations with constant coefficients have been extensively investigated. Early references are [12], where Galerkin type discretizations are studied and [11], for pseudospectral approximations. A variational treatment for both Galerkin and collocations methods may be seen in [5]. Pseudospectral approximations for two dimensional problems are treated, among others, in [3].

However, less analysis has been done for the variable coefficient case. The paper [10] is one of the very few examples in the literature.

This paper analyses a Chebyshev pseudospectral approximation to a parabolic problem with nonconstant coefficients. More precisely, let us consider the parabolic problem

$$\text{(1)} \qquad \begin{aligned} u_t - (a(x)u_x)_x &= f(x,t), & x \in \Lambda, t \geq 0, \\ u(-1,t) = u(1,t) &= 0, & t \geq 0, \\ u(x,0) &= u_0(x), & x \in \Lambda, \end{aligned}$$

where $\Lambda = (-1,1)$, a is smooth function satisfying the classical assumption $0 < \underline{a} \leq a(x) \leq \overline{a}$ in $\overline{\Lambda}$, which ensure the parabolicity of the problem.

We follow the treatment given in [6] for a stationary advection-diffusion equation and discretize the equation by direct collocation at the Chebyshev-Gauss-Lobatto points. The leading term in our discretization is the derivative of the interpolating polynomial of $a(x)u_x$. Written in this form, the discretized equations have additional difficulties. In [10] the leading term was written as $a(x)u_{xx}$ and treated pseudospectrally by means of a special norm involving the function a. The approach we follow here has the advantage that the extension to advection-diffusion equations with nonconstant coefficient in both advective and diffusive terms is quite straightforward. Also, extensions to some problems in two dimensions, not reported in this paper, are possible along the lines presented here. Furthermore, our formulation is of interest from a practical point of view. The solution of the discrete equations obtained by spectral collocation needs of an appropriate time-stepping. Explicit time-integrators suffer from very severe stability time-step restrictions and implicit ones need the solution of linear systems of equations. The use of iterative solvers for such systems has become popular. As noted in [4], there is no effective preconditioning available for problems with large first derivative. So, one is impeled to reformulate non constant coefficient problems in the form of equation (1).

The next section is devoted to introduce the basic notations. The properties of problem (1) are stated in Section 3. Section 4 is devoted to the analysis of the collocation equations. The error estimates are presented in Section 5. In the last section we add some final remarks concern-

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ing the extension of the analysis to the advection-diffusion equation and to fully discrete schemes.

2 Preliminaries and notation

Let ω be the Chebyshev weight. All the functional spaces related to the spatial variable are defined over Λ. We use the notations, $L^2_\omega = \{v \mid \int_\Lambda v^2\omega\, dx < \infty\}$ and $H^1_{\omega,0} = \{v \in L^2_\omega \mid v_x \in L^2_\omega, v(1) = v(-1) = 0\}$ respectively endowed with the norms $\|.\|_{0,\omega}$ and $\|.\|_{1,\omega}$. The inner product in L^2_ω is represented by $(\cdot,\cdot)_{0,\omega}$. Sobolev spaces of higher order with respect the Chebyshev weight are denoted by H^ν_ω, $\nu > 0$ (see, for example,[1], [2, Chapter 1], for a definition). Let H^{-1}_ω be the dual space of $H^1_{\omega,0}$. The norm in H^{-1}_ω is denoted by $\|.\|_{-1,\omega}$. We shall identify the space L^2_ω with its dual, so we have $H^1_{\omega,0} \subset L^2_\omega = (L^2_\omega)^* \subset H^{-1}_\omega$, each space being dense in the next. We recall that, with this identification, for $g \in L^2_\omega$, we have $\|g\|_{-1,\omega} = \sup\{\int_\Lambda gv\omega\, dx/\|v\|_{1,\omega} \mid v \in H^1_{\omega,0}\}$.

The space of polynomials of degree N satisfying the boundary conditions will be denoted by $\mathbb{P}^N_0$. Let ω_j and x_j, $0 \le j \le N$, be the weights and nodes of the Gauss-Lobatto-Chebyshev quadrature in Λ. We denote by I_N the corresponding interpolation operator. We use the discrete norm $\|v^N\|_N = \left\{\sum_{j=0}^N v^N(x_j)^2\omega_j\right\}^{1/2}$ defined over $\mathbb{P}^N_0$ and denote by $(\cdot,\cdot)_N$ the corresponding inner product.

Let T_k be the k-th Chebyshev polynomial in Λ. Each polynomial in $\mathbb{P}^N_0$ has a (unique) representation in terms of the derivatives of the $T_k, k = 1,\ldots,N-1$:

$$v^N(x) = \sum_{k=1}^{N-1} \hat{v}_k(1-x^2)T'_k(x). \tag{2}$$

Throughout, we use the norm $|||v||| = \left\{\int_\Lambda [(v\omega)_x]^2\,\omega^{-1}dx\right\}^{1/2}$. It is well known that $|||\cdot|||$ is a norm over $H^1_{\omega,0}$ equivalent to $\|\cdot\|_{1,\omega}$. Furthermore, a calculation shows that, for polynomials $v^N \in \mathbb{P}^N_0$,

$$|||v^N|||^2 = \frac{\pi}{2}\sum_{k=1}^{N-1} k^4|\hat{v}_k|^2, \tag{3}$$

where $\hat{v}_k$ denotes the k-th coefficient in the representation (2) of v^N.

3 The continuous problem

The problem (1) can be written in variational form as

$$(u_t,\varphi)_{0,\omega} + a(u,\varphi) = (f(t),\varphi)_{0,\omega}, \quad \forall\varphi \in H^1_{\omega,0}, \tag{4}$$

where $a(u,v)$ is the bilinear form over $H^1_{\omega,0}$ defined by

$$a(u,v) = \int_\Lambda a(x)u_x(v\omega)_x dx, \quad \forall u,v \in H^1_{\omega,0}, \tag{5}$$

supplemented with the initial condition $u(\cdot,0) = u_0(\cdot)$.

In this section we analyze the continuous problem (4). We start the analysis by stating a coercivity inequality for the bilinear form $a(u,v)$ in (5). More precisely we have the following

Theorem 3.1 *Let $a(x)$ be a function with continuous first order derivative in $\overline{\Lambda}$. There exists a positive constant μ_0 such that if $\mu > \mu_0$*

$$a(u,u) + \mu(u,u)_{0,\omega} \ge \gamma_\mu\|u\|^2_{1,\omega} \tag{6}$$

for all $u \in H^1_{\omega,0}$. Here $\gamma_\mu > 0$ is a suitable positive constant depending on μ.

Proof Following the techniques in [4, Theorem 11.1], we first obtain the auxiliary result

$$a(u,u) \ge \frac{1}{4}\int_\Lambda a(x)u_x^2\omega\, dx + \frac{1}{4}\int_\Lambda a_x u^2\omega_x dx, \tag{7}$$

and, as a consequence, the inequality

$$a(u,u) \ge \frac{1}{4}\int_\Lambda a(x)u_x^2\omega\, dx - \frac{1}{4}\int_\Lambda [xa_x]^- u^2\omega^3 dx, \tag{8}$$

where $[xa_x]^-$ denotes the negative part of the function xa_x. Let M be an upper bound of $[xa_x]^-$; the second term in (8) is bounded by

$$\int_{-1}^1 [xa_x]^- u^2\omega^3 dx \le M\left\{\int_\Lambda u^2\omega^5 dx\right\}^{1/2}\left\{\int_\Lambda u^2\omega\, dx\right\}^{1/2}, \tag{9}$$

and, using a Hardy inequality [2, Lemma 2.3] and the well-known fact that $ab \le \epsilon a^2 + (1/4\epsilon)b^2$ we have

$$a(u,u) \ge \frac{1}{4}\left[\underline{a} - M\epsilon\right]\|u_x\|^2_{0,\omega} - \frac{M}{16\epsilon}\|u\|^2_{0,\omega}. \tag{10}$$

Taking $\epsilon_0 = \underline{a}/M$ and $\mu_0 = M/16\epsilon_0$ we have that, for $\mu > \mu_0$,

$$a(u,u) + \mu(u,u)_{0,\omega} \ge \gamma_\mu\|u_x\|^2_{0,\omega}$$

with $\gamma_\mu = \frac{1}{4}(\underline{a} - \frac{M^2}{16\mu}) = \frac{M^2}{64}\left[\frac{1}{\mu_0} - \frac{1}{\mu}\right]$. □

The coercivity of $A(u,v) = a(u,v) + \mu(u,v)_{0,\omega}$ ensures that, for data $u_0 \in L^2_\omega$ and $f \in L^2([0,T];H^{-1}_\omega)$, $0 < T < \infty$, the problem (4) has a unique solution $u \in L^2([0,T];H^1_{\omega,0}) \bigcap C([0,T];L^2_\omega)$ such that $u_t \in L^2([0,T];H^{-1}_\omega)$ (see [8, Cap. XVIII, §3, Ths. 1-2]). Furthermore, it is now straightforward to show that the problem is well-posed.

Theorem 3.2 *In the above conditions, the solution of (4) satisfies the stability estimate*

$$(11)\ \|u(t)\|_{0,\omega} + \left(\int_0^t \|u(s)\|^2_{1,\omega} ds\right)^{\frac{1}{2}} \leq C(\gamma)e^{\mu t}\left[\|u_0\|_{0,\omega} + \left(\int_0^t \|f(s)\|^2_{-1,\omega} ds\right)^{\frac{1}{2}}\right],$$

where $C(\gamma)$ is a constant depending on $\gamma = \gamma_\mu$.

4 The collocation approximation

Problem (1) is discretized, in a standard way, by collocation at the Gauss-Lobatto points. Following [6], the collocation equations are written in variational form as

$$(12)\ (u^N_t(t), v^N)_N + a_N(u^N(t), v^N) = (I_N(f), v^N)_N,$$

for all $v^N \in \mathbb{P}^N_0$, where the discrete bilinear form a_N is defined, for polynomials φ^N, ψ^N in $\mathbb{P}^N_0$, by

$$(13)\qquad a_N(\varphi^N, \psi^N) = \int_\Lambda I_N(a(x)\varphi^N_x)(\psi^N\omega)_x dx.$$

Formula (12) is supplemented with the initial condition

$$(14)\qquad u^N(0) = I_N(u_0).$$

In order to prove the stability of the semidiscrete approximation (12)–(14), we first establish a coercivity property similar to (6). The proof is based on a approximation property for the interpolation operator I_N that we next state as a lemma.

Lemma 4.1 *Let $b(x)$ be a function in $C^\sigma = C^{m,\nu}$, $m + \nu = \sigma$, $0 \leq 0 < 1$, the class of functions whose m-th derivative is Hölder continuous in Λ with exponent ν. The interpolation operator I_N based on the Gauss-Lobatto points satisfies, for each polynomial $v^N \in \mathbb{P}^N_0$,*

$$(15)\quad \left\|I_N\left[\frac{b(x)v^N(x)}{1-x^2}\right] - \frac{b(x)v^N(x)}{1-x^2}\right\|_{0,\omega} \leq CN^{\frac{-\sigma}{1+2\sigma}}[\log N]^{1/2}\|v^N\|_{1,\omega}.$$

Proof Using (2), (3) and the equivalence between the norms $\|\cdot\|_{1,\omega}$ and $|||\cdot|||$, we arrive at

$$(16)\quad \left\|I_N\left[\frac{b(x)v^N(x)}{1-x^2}\right] - \frac{b(x)v^N(x)}{1-x^2}\right\|_{0,\omega} \leq C\|v^N\|_{1,\omega}\left\{\sum_{k=1}^{N-1} k^{-4}\|I_N[bT'_k] - bT'_k\|^2_{0,\omega}\right\}^{1/2}.$$

In order to estimate the terms $\|I_N[bT'_k] - bT'_k\|_{0,\omega}$, let us introduce β_r, the best maximum-norm approximation polynomial of degree r of $b(x)$ over Λ. A classical result [13, Theorem 1.5], states

$$(17)\qquad \|b - \beta_r\|_\infty = \inf_{q\in\mathbb{P}_r} \|b - q\|_\infty \leq C_\sigma r^{-\sigma}\|b\|_{C^\sigma},$$

where $\|b\|_{C^\sigma}$ denotes the norm in C^σ

$$\|b\|_{C^\sigma} = \sup_{x\in\Lambda, 0\leq k\leq m} |b^{(k)}(x)| + \sup_{x,y\in\Lambda} \frac{|b^{(m)}(x) - b^{(m)}(y)|}{|x-y|^\nu}.$$

Then, we have that

$$\|I_N[bT'_k] - bT'_k\|_{0,\omega} \leq \|I_N[bT'_k] - \beta_{N+1-k}T'_k\|_{0,\omega} + \|(\beta_{N+1-k} - b)T'_k\|_{0,\omega}.$$

Using that $\|T'_k\|_{0,\omega} = \sqrt{\pi}k^{3/2}$ in the second term in the right-hand side, the equivalence between the norms $\|\cdot\|_{0,\omega}$ and $\|\cdot\|_N$ in the first and (17), we obtain, for $1 \leq k \leq N-1$,

$$(18)\qquad \|I_N[bT'_k] - bT'_k\|_{0,\omega} \leq 2\sqrt{\pi}C_\sigma\|b\|_{C^\sigma}(N+1-k)^{-\sigma}k^{3/2}.$$

Even though the above inequality can be used for all possible values of k, it only provides a useful estimate for k bounded away from N. On the other hand, we also have, for $1 \leq k \leq N-1$,

$$(19)\qquad \|I_N[bT'_k] - bT'_k\|_{0,\omega} \leq 2\sqrt{\pi}\|b\|_{C^\sigma}k^{3/2},$$

So, our purpose is to use (19) for the first values of k, and (19) for the rest. With this in mind, we take ν, $0 < \nu < 1$, and write ($\lfloor\cdot\rfloor$ denotes integer part),

$$S = \sum_{k=1}^{N-1} k^{-4}\|I_N[bT'_k] - bT'_k\|^2_{0,\omega} = \sum_{k=1}^{\lfloor\nu N\rfloor} k^{-4}\|I_N[bT'_k] - bT'_k\|^2_{0,\omega} + \sum_{k=\lfloor\nu N\rfloor}^{N-1} k^{-4}\|I_N[bT'_k] - bT'_k\|^2_{0,\omega}.$$

Using (19) and (19) in order to handle the first and second terms respectively, we get

$$S \leq C\sum_{k=1}^{\lfloor\nu N\rfloor} k^{-1}(N+1-k)^{-2\sigma} + C\sum_{\lfloor\nu N\rfloor+1}^{N-1} k^{-1},$$

where C is a constant depending only on b. Using now that $N+1-k \geq (1-\nu)N$ for $k \leq \lfloor \nu N \rfloor$, we get

$$S \leq C\left((1-\nu)^{-2\sigma}N^{-2\sigma}(1+\log N)+\log \nu^{-1}+\log \frac{\nu N}{\lfloor \nu N \rfloor}\right).$$

Now we restrict ν to be in $[\frac{1}{2},1]$. Using the boundedness of the function $\log \nu/(1-\nu)$ on that interval, we obtain

$$S \leq C\left((1-\nu)^{-2\sigma}N^{-2\sigma}(1+\log N)+(1-\nu)+N^{-1}\right)$$

for all $\nu \in [\frac{1}{2},1]$ and for all $N \geq 4$ with a constant C independent of both N and ν. In order to make the bound optimal we choose $\nu = \nu(N) = 1-N^{-2\sigma/(1+2\sigma)}$ (which indeed belongs to $[\frac{1}{2},1]$ for N large enough) and get the final bound

$$S \leq CN^{\frac{-2\sigma}{1+2\sigma}}\log N.$$

□

Theorem 4.1 *Let us suppose that the function a has continuous first order derivatives in $\overline{\Lambda}$. There exist positive constants μ_1 and N_0 such that, if $\mu > \mu_1$, and $N > N_0$,*

$$(20) \qquad a_N(v^N,v^N)+\mu(v^N,v^N)_N \geq \gamma_\mu \|v^N\|^2_{1,\omega},$$

for all $v^N \in \mathbb{P}_0^N$. Here γ_μ is a positive constant depending on μ but not on N.

Proof We denote by C a uniform positive constant not necessarily the same at each occurrence. Let $v^N \in \mathbb{P}_0^N$. Our aim is to prove the inequality

$$(21) \quad a_N(v^N,v^N) \geq \frac{1}{4}\int_\Lambda I_N[av_x^N]v_x^N\omega\,dx+\frac{1}{4}\int_\Lambda a_x(v^N)^2\omega_x dx - E_N[v^N],$$

with

$$E_N[v^N] = \frac{1}{4}\int_\Lambda\left(2v_x^N+\frac{3xv^N}{1-x^2}\right)\left(I_N\left[\frac{xav^N}{1-x^2}\right]-\frac{xav^N}{1-x^2}\right)\omega\,dx.$$

Using a Hardy inequality to handle the term E_N, we get

$$|E_N[v^N]| \leq C\|v^N\|_{1,\omega}\left\|I_N\left[\frac{xav^N}{1-x^2}\right]-\frac{xav^N}{1-x^2}\right\|_{0,\omega}.$$

The conclusion is then reached by applying (16) and arguing as in Theorem 3.1. In order to get (21), we start by writing a_N in the form

$$(22)\, a_N(v^N,v^N) = \int_\Lambda I_N[av_x^N]v_x^N\omega\,dx - \frac{1}{2}\int_\Lambda (v^N)^2a_x\omega_x dx - \frac{1}{2}\int_\Lambda (v^N)^2a\omega_{xx}dx + E_N^{(1)}[v^N],$$

with

$$E_N^{(1)}[v^N] = \int_\Lambda v_x^N\left[I_N[\frac{xav^N}{1-x^2}]-\frac{xav^N}{1-x^2}\right]\omega\,dx.$$

We next observe that

$$(23) \quad a_N(v^N,v^N)+\int_\Lambda v_x^N\frac{xav^N}{1-x^2}\omega\,dx + \int_\Lambda \frac{x^2a(v^N)^2}{(1-x^2)^2}\omega\,dx + E^{(1)}[v^N]+E^{(2)}[v^N] \geq 0,$$

where

$$E^{(2)}[v^N] = \int_\Lambda \frac{xv^N}{1-x^2}\left[I_N[\frac{xav^N}{1-x^2}]-\frac{xav^N}{1-x^2}\right]\omega\,dx.$$

Integrating by parts the second term in (23) and using $\omega_{xx}-2\omega_x^2/\omega = \omega^5$, it is easy to obtain

$$(24) \quad \frac{1}{2}\int_\Lambda \omega^5a(v^N)^2dx \leq a_N(v^N,v^N) - \frac{1}{2}\int_\Lambda \omega_x a_x(v^N)^2dx + E^{(1)}[v^N]+E^{(2)}[v^N].$$

Finally, using (22) and (24) and following once more the techniques in [4, Theorem 11.1], we obtain the desired inequality (21) □

5 Error estimates

In order to get the optimal error estimates, we need to state the stability of the discrete problem for a slightly more general class of discrete right-hand sides. Namely, we consider the following generalization of (12)

$$(25) \quad (u_t^N(t),v^N)_N + a_N(u^N(t),v^N) = < l_N(t),v^N >$$

for all $v^N \in \mathbb{P}_0^N$, where $l_N \in L^2(0,T;(\mathbb{P}_0^N)^*)$. Here $<\cdot,\cdot>$ stands for the duality between $(\mathbb{P}_0^N)^*$ and $\mathbb{P}_0^N$. For $g \in (\mathbb{P}_0^N)^*$ we write $\|g\|_{-1,N} = \sup\{<g,v>/\|v\|_{1,\omega} \mid v \in \mathbb{P}_0^N\}$. As (25) is a finite dimension problem, it has a unique solution $u^N(t) \in \mathbb{P}_0^N$ for data $u_0^N \in \mathbb{P}_0^N$ and $l_N \in L^2(0,T;(\mathbb{P}_0^N)^*)$, $T>0$. Further, the uniform coercivity of the bilinear forms $a_N(\varphi^N,\psi^N)+\mu(\varphi^N,\psi^N)_N$ ensures that we have the discrete analogue of Theorem 3.2:

Theorem 5.1 *In the above conditions, the solution of (25) satisfies, for $N > N_0$, the following uniform stability estimate:*

$$(26)\, \|u^N(t)\|_{0,\omega}+\left(\int_0^t \|u^N(s)\|^2_{1,\omega}ds\right)^{\frac{1}{2}} \leq C(\gamma)e^{\mu t}\left[\|u_0^N\|_{0,\omega}+\left(\int_0^t \|l_N(s)\|^2_{-1,N}ds\right)^{\frac{1}{2}}\right]$$

where $C(\gamma)$ is a constant depending on the constant $\gamma = \gamma_\mu$ in Theorem 4.1 but not on N.

For the error between the exact solution u and its collocation approximation u^N, we have the following result.

Theorem 5.2 *Assume the function $a(x)$ is in C^σ, $\sigma \geq 1$, and $f \in L^2([0,T], H^\tau_\omega)$, $\tau > \frac{1}{2}$. Let us suppose further that $u \in C([0,T], H^m_\omega)$ and $u_t \in L^2([0,T], H^{m-2}_\omega)$, $m \geq 3$ (not necessarily an integer). Then we have, for all $0 < t \leq T$ and N large enough, the error estimate:*

$$(27)\ \|u(t) - u^N(t)\|_{0,\omega} + \left(\int_0^t \|u(s) - u^N(s)\|^2_{1,\omega} ds\right)^{\frac{1}{2}} \leq$$

$$Ce^{\mu t}\left[N^{-m}\|u_0\|_{m,\omega} + N^{1-m}\left(\int_0^t \|u(s)\|^2_{m,\omega} ds\right)^{\frac{1}{2}}\right.$$

$$+N^{-m}\|u(t)\|_{m,\omega} + N^{1-m}\left(\int_0^t \|u_t(s)\|^2_{m-2,\omega} ds\right)^{\frac{1}{2}}$$

$$\left. +N^{-\tau}\left(\int_0^t \|f(s)\|^2_{\tau,\omega} ds\right)^{\frac{1}{2}} + N^{-\sigma}\left(\int_0^t \|u(s)\|^2_{1,\omega} ds\right)^{\frac{1}{2}}\right],$$

where C is a constant depending on the function $a(x)$ but not on N.

Proof Let $\pi^*_N : H^1_{\omega,0} \mapsto \mathbb{P}^N_0$ be the projection operator defined by:

$$\int_{-1}^1 \psi^N_x \left((\varphi - \pi^*_N\varphi)\omega\right)_x dx = 0, \quad \forall \psi^N \in \mathbb{P}^N_0.$$

From [1, Theorem 4.3] we have the estimate

$$\|\varphi - \pi^*_N\varphi\|_{r,\omega} \leq CN^{r-s}\|\varphi\|_{s,\omega} \tag{28}$$

for $0 \leq r \leq 1 \leq s$. Indeed, the duality argument used therein to prove the estimate for $r = 0$ also works when $r = -1$. Let $e^N(t) = \pi^*_{\lfloor \frac{N}{2} \rfloor} u(t) - u^N(t)$. From (4) and (12), we conclude that $e^N(t)$ satisfies

$$(e^N_t(t), v^N)_N + a_N(e^N(t), v^N) = < l^1_N(t) + l^2_N(t) + l^3_N(t), v^N >$$

for all $v^N \in \mathbb{P}^N_0$, where

$$\begin{aligned} < l^1_N(t), v^N > &= (\pi^*_{\lfloor \frac{N}{2} \rfloor} u_t(t), v^N)_N - (u_t(t), v^N)_{0,\omega} \\ < l^2_N(t), v^N > &= a_N(\pi^*_{\lfloor \frac{N}{2} \rfloor} u(t), v^N) - a(u(t), v^N) \\ < l^3_N(t), v^N > &= (f(t), v^N)_{0,\omega} - (f(t), v^N)_N \end{aligned}$$

From (28) with $r = -1$ and $s = m - 2$, we get

$$\|l^1_N\|_{-1,N} \leq CN^{1-m}\|u_t\|_{m-2,\omega}. \tag{29}$$

To handle l^2_N, we write the decomposition

$$\begin{aligned} < l^2_N(t), v^N > = &\left((a - \alpha_{\lfloor \frac{N}{2} \rfloor})_x \pi^*_{\lfloor \frac{N}{2} \rfloor} u(t)_x, v^N_x + \frac{xv^N}{1-x^2}\right)_N \\ &+ \left((\alpha_{\lfloor \frac{N}{2} \rfloor}) - a)_x \pi^*_{\lfloor \frac{N}{2} \rfloor} u(t)_x, v^N_x + \frac{xv^N}{1-x^2}\right)_{0,\omega} \\ &+ \left(a(u - \pi^*_{\lfloor \frac{N}{2} \rfloor} u(t)_x)_x, v^N_x + \frac{xv^N}{1-x^2}\right)_{0,\omega}, \end{aligned}$$

where $\alpha_{\lfloor \frac{N}{2} \rfloor}$ stands for the best maximum norm approximation polynomial of degree $\lfloor \frac{N}{2} \rfloor$ of the function $a(x)$ over Λ. So, recalling (17) and using (28) with $r = s = 1$, we obtain

$$\begin{aligned} \|l^2_N\|_{-1,N} \leq\ &CN^{-\sigma}\|a\|_{C^\sigma}\|u(t)\|_{1,\omega} \\ &+CN^{-m}\|u(t)\|_{m,\omega}. \end{aligned} \tag{30}$$

Using the estimate for the interpolation operator

$$\|f - I_N f\|_{0,\omega} \leq CN^{-\tau}\|f\|_{\tau,\omega}$$

(see [7]) and standard arguments, we obtain the bound:

$$\|l^3_N\|_{-1,N} \leq CN^{-\tau}\|f\|_{\tau,\omega}. \tag{31}$$

The desired error estimate follows using (29)-(31) in the bound obtained by replacing in (26) e^N by u^N and $l^1_N + l^2_N + l^3_N$ by l_N together with (28) applied to u, first with $r = 0$ and $s = m$ and then with $r = 1$ and $s = m - 2$. □

6 Final remarks and conclusions

Once the coercivity of the discrete bilinear form a_N has been proved, the analysis can be easily extended to cover a number of other situations. For reason of brevity we only point out one of such extensions: the case of the advection-diffusion equation in one dimension, with nonconstant coefficients in both the advective and diffusive terms. Our analysis may also be extended to some other cases in dimension two that we do not report here. The last subsection is devoted to remarking how the fully discrete case can be analyzed along the preceding lines.

6.1 Advection-diffusion equation

Let us consider the following advection-diffusion equation

$$\begin{aligned} u_t - (a(x)u_x)_x + b(x)u_x + c(x)u &= f(x,t), & x \in \Lambda, t \geq 0, \\ u(-1,t) = u(1,t) &= 0, & t \geq 0, \\ u(x,0) &= u_0(x), & x \in \Lambda. \end{aligned} \tag{32}$$

Let we assume that the functions b and c are bounded in Λ. Associated with (32) we have the bilinear form defined by

$$(33)\quad A(u,v) = a(u,v) + b(u,v) + c(u,v), \quad u,v \in H^1_{\omega,0}$$

where $a(u,v)$ is defined in (5) and $b(u,v)$ and $c(u,v)$ are respectively defined by $b(u,v) = \int_\Lambda b(x)u_x v\omega\, dx$ and $c(u,v) = \int_\Lambda c(x)uv\omega\, dx$. It is readily shown that

$$(34)\quad \begin{aligned} |b(u,u)| &\le b_\infty \epsilon \|u_x\|^2_{o,\omega} + \frac{b_\infty}{4\epsilon}\|v\|^2_{0,\omega}, \\ |c(u,u)| &\le c_\infty \|u\|^2_{0,\omega}, \end{aligned}$$

where b_∞ and c_∞ are the maximum value in Λ of the functions $b(x)$ and $c(x)$ respectively, and ϵ is a parameter to be determined later. So, one can consider the terms $b(u,v)$ and $c(u,v)$ in (33) as perturbations of the coercive form $a(u.v)$. Using then (6) we get the inequality

$$(35)\quad A(u,u) + (\mu + \frac{b_\infty}{4\epsilon} + c_\infty)\|u\|^2_{0,\omega} \ge (\gamma_\mu - b_\infty\epsilon)\|u\|^2_{1,\omega}.$$

Taking $\epsilon < \gamma_\mu / b_\infty$, we get a coercivity property for the perturbed form $A(u,v)$.

The usual collocation discretization of (32) leads to the problem

$$(36)\quad \begin{aligned} (u^N_t(t), v^N)_N + A_N(u^N(t), v^N) = (I_N(f), v^N)_N, \\ \forall v^N \in \mathbb{P}^N_0 \end{aligned}$$

with

$$A_N(u^N, v^N) = a_N(u^N, v^N) + b_N(u^N, v^N) + c_N(u^N, v^N),$$

for polynomials $u^N, v^N \in \mathbb{P}^N_0$. The discrete bilinear form a_N has been defined in (13) and the perturbations b_N and c_N are given by,

$$\begin{aligned} b_N(u^N, v^N) &= \sum_{n=1}^{N-1} b(x_j)u^N_x(x_j)v^N(x_j)\omega_j, \\ c_N(u^N, v^N) &= \sum_{n=1}^{N-1} c(x_j)u^N(x_j)v^N(x_j)\omega_j, \end{aligned}$$

where $x_j, \omega_j, j = 0, 1, \ldots, N$, are the nodes and weights of the Gauss-Lobatto-Chebyshev quadrature formula. It is readily shown that b_N and c_N satisfy similar estimates as their continuous counterparts b and c. Then, taking into account the coercivity of the form a_N, one can obtain the same property for the perturbed form A_N. As a consequence the stability and convergence results of the previous section also apply to the convection-diffusion case. In particular, the bilinear form A_N satisfies a uniformly on N coercivity inequality

$$(37)\quad A_N(v^N, v^N) + \mu(v^N, v^N) \ge \gamma_\mu \|v^N\|^2_{1,\omega},$$

for suitable constants μ and γ_μ.

6.2 Fully-discrete schemes

A semidiscrete approximation to a partial differential equation yields a system of ordinary differential equations that can be numerically integrated by means of a standard ODE solver. As it is well known, explicit finite-difference methods for the time integration of spectral discretization of second order parabolic equations may have a restriction on the time step Δt of the form $\Delta t \le C/N^4$, see for example [4, Chapter 4]. For the case of a Legendre collocation spatial discretization of a constant coefficient advection-diffusion equation, it has been recently proved that, if a general rational approximation to the exponential is used for the time integration a $\mathcal{O}(N^{-4})$ time-step restriction guarantees stability. In order to avoid such a severe restriction, A-stable time discretization are often used.

The usual way to analyze a finite-difference time-discretization is by resorting to the energy method and therefore following the same lines we have previously presented. We shall give no details but as an example, let us consider the backward Euler method for the problem (32),

$$(38)\quad \begin{aligned} ((u^N_{k+1} - u^N_k)/\Delta t, v^N)_N + A_N(u^N_{k+1}, v^N) = \\ (I_N[f_{k+1}], v^N)_N, \end{aligned}$$

where $f_k = f(k\Delta t)$ and u^N_k is meant to be an approximation to the solution of (36) at $t = k\Delta t$. Taking $v^N = u^N_{k+1}$, using (37) and summing, we get the stability estimate

$$\|u^N_{k+1}\|_N + \left(\Delta t \sum_{j=1}^{k+1} \|u_j\|^2_{1,\omega}\right)^{1/2} \le$$
$$C(\gamma)e^{\mu t}\left\{ \|I_N[u(0)]\|_N + \Delta t \left(\sum_{j=1}^{k+1} \|I_N[f_j]\|^2_{0,\omega}\right)^{1/2}\right\},$$

for $0 \le (k+1)\Delta t \le t$. The error estimate for the semidiscrete approximation together with consistency properties of the scheme and standard procedures (see, for example, [14]) allow to derive an $\mathcal{O}(\Delta t + N^{-m})$ bound for enough smooth data and coefficients.

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