Estimates of $O(p^\alpha)$ Type for the Condition Number of Matrices in the p-Version of the Finite Element Method

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Abstract

We give general results on condition numbers of matrices in the p-version of the finite element method for elliptic problems. For the conform p-version approximation of $H_0^m(\Omega)$, $\Omega \subset R^d$, general m, we use a basis generalizing that used for m=1, and study the condition number $\kappa(.)$ of the elementary matrices $A_d^{m,k}$ corresponding to the Sobolev scalar product $(.,.)_{k,\Omega}, k \leq m$, and to the internal modes. We prove (theorem 2.1) that $\kappa(A_d^{m,k}) = O(p^{4(md-k)})$ for every $(d,m,k), 0 \leq k \leq m$. For $m = 1,2$, we prove moreover that these estimations are optimal and that the condition number after diagonal preconditioning is equivalent to $p^{2(md-k)}$). Finally, we compare these results with those obtained with the spectral element method associated with Gauss-Lobatto quadrature.

Key words: condition number, finite element, high degree.

AMS subject classifications: 65F35, 65M60, 65N22, 65N35.

1 Introduction

For problems of order 2m, we can use similar resolution methods as for problems of order 2 (see [1, 3]): Gaussian internal modes elimination or block diagonal preconditioning where each block corresponds to a geometrical part of the mesh (for example in 2 dimension: vertices, sides, interiors). In each case, it appears internal systems corresponding to an homogeneous Dirichlet problem on the reference element $\Omega =]-1,1[^d$, the condition number of which we study here.

We consider an elliptic problem of order 2m with homogeneous boundary conditions : $\frac{\partial^j u}{\partial n^j} = 0$ on $\partial\Omega$ for all j, $0 \leq j \leq m-1$. The functional space associated with this problem is

$$H_0^m(\Omega) = \left\{ v \in H^m(\Omega), \frac{\partial^j u}{\partial n^j}_{|\partial\Omega} = 0, \forall\, 0 \leq j \leq m-1 \right\}.$$

We define by $Q_p^{m,0}(\Omega) = Q_p(\Omega) \cap H_0^m(\Omega)$ the approximation space of polynomials of degree p in each variable which vanish on $\partial\Omega$.

Definition 1.1 For $Q_p^{m,0}(\Omega)$, we choose the basis $\{\mathcal{H}_{i_1}^m \otimes ... \otimes \mathcal{H}_{i_d}^m\}_{2m \leq i_1,...,i_d \leq p}$, which is made up of tensorial products of the 1-d basis $\{\mathcal{H}_i^m\}_{2m \leq i \leq p}$ where the $\mathcal{H}_i^m$, deduced from the Legendre polynomials by $\mathcal{H}_i^m(t) = \int_{-1}^t \int_{-1}^{t_1} ... \int_{-1}^{t_{m-1}} \frac{L_{i-m}(s)}{\|L_{i-m}\|_{0,(-1,1)}} ds dt_{m-1}...dt_1$, are orthonormal for the Sobolev scalar product $(.,.)_{m,\Omega}$. □

These m^{th} Legendre polynomial integrals induce very sparse matrices and generalize the most used basis for problem of order 2 (m=1)(see [2, 3, 8, 10] for example).

Definition 1.2 Denoting by D_α the derivation operator $D_\alpha = \frac{\partial^\alpha}{\partial x_1{}^{\alpha_1} ... \partial x_d{}^{\alpha_d}}$ for $\alpha = (\alpha_1, ..., \alpha_d)$ and $|\alpha| = \sum_{i=1}^d \alpha_i$, we consider for $0 \leq k \leq m$, the Sobolev scalar product: $(u,v)_{k,\Omega} = \sum_{\alpha, |\alpha|=k} (D_\alpha u, D_\alpha v)_{L^2(\Omega)}$, and the associated matrix $A_d^{m,k}$ for the basis $\{\mathcal{H}_{i_1}^m \otimes ... \otimes \mathcal{H}_{i_d}^m\}_{2m \leq i_1,...,i_d \leq p}$ of $Q_p^{m,0}(\Omega)$. □

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Definition 1.3 With a tensorial numbering, we define the canonical scalar product of $R^{(p+1-2m)^d}$ by:

$$(u,v)_d = \sum_{i_1,\dots,i_d=2m}^{p} u_{i_1,\dots,i_d} v_{i_1,\dots,i_d}. \qquad \square$$

Definition 1.4 For any matrix M, similar to a symmetric positive definite one, we define $\kappa(M)$ by $\kappa(M) = \frac{\lambda_{max}(M)}{\lambda_{min}(M)}$. $\square$

2 Majoration of condition numbers

We begin by a short lemma showing that the matrices $A_d^{m,k}$ of the d-dimensional case can be expressed in function of the only $A_1^{m,\cdot}$ of the 1-dimensional case.

Lemma 2.1 *The matrix $A_d^{m,k}$ associated with the $(.,.)_{k,\Omega}$ scalar product can be deduced from the $A_1^{m,\alpha}$ matrices, for $0 \le \alpha \le k$, associated with the $(.,.)_{\alpha,]-1,1[}$ scalar product, by:*

$$(1) \qquad (A_d^{m,k})_{i_1,\dots,i_d,j_1,\dots,j_d} = \sum_{\alpha,|\alpha|=k} \prod_{n=1}^{d} (A_1^{m,\alpha_n})_{i_n,j_n}$$

Proof We have
$(A_d^{m,k})_{i_1,\dots,i_d,j_1,\dots,j_d}$
$= \sum_{\alpha,|\alpha|=k} (D_\alpha(\mathcal{H}_{i_1}^m \otimes \dots \otimes \mathcal{H}_{i_d}^m), D_\alpha(\mathcal{H}_{j_1}^m \otimes \dots \otimes \mathcal{H}_{j_d}^m))$
$= \sum_{\alpha,|\alpha|=k} \prod_{n=1}^{d} \int_{-1}^{1} \frac{d^{\alpha_n}\mathcal{H}_{i_n}^m}{dt^{\alpha_n}} \frac{d^{\alpha_n}\mathcal{H}_{j_n}^m}{dt^{\alpha_n}} dt$, which proves (1). $\square$

Now, we can give the main result on the condition numbers $\kappa(.)$:

Theorem 2.1
For $A_d^{m,k}$, the matrix associated with the $(.,.)_{k,\Omega}$ $(0 \le k \le m)$ scalar product and corresponding to the basis (definition 1.1) built for the homogeneous elliptic problems of order 2m on $]-1,1[^d$, we have:
$\kappa(A_d^{m,k}) = O(p^{4(md-k)}), \forall (d,m,k), 0 \le k \le m.$

Proof The following inverse inequality ([5]) and Poincare type inequality

$$(2) \qquad \begin{aligned} \forall v \in Q_p^{1,0}([-1,1]) \quad &, \quad \forall (k,l), 0 \le k \le l \\ \| v \|_{k,(-1,1)}^2 \quad &\le \quad C_2 \| v \|_{l,(-1,1)}^2 \\ &and \\ \| v \|_{l,(-1,1)}^2 \quad &\le \quad C_1 p^{4(l-k)} \| v \|_{k,(-1,1)}^2 \end{aligned}$$

give for the matrices (l=m):

$$\begin{aligned} \forall x \in R^{p+1-2m}, (A_1^{m,k}x, x) \quad &\le \quad C_2 (A_1^{m,m}x, x) \\ and \ (A_1^{m,m}x, x) \quad &\le \quad C_1 p^{4(m-k)} (A_1^{m,k}x, x). \end{aligned}$$

Since $A_1^{m,m} = \mathrm{I}$, we deduce $\lambda_{min}(A_1^{m,k}) \ge C_1^{-1} p^{4(k-m)}$ and $\lambda_{max}(A_1^{m,k}) \le C_2$ so that $\kappa(A_1^{m,k}) \le C p^{4(m-k)}$. This proves the result for d=1 which can be extended to general d thanks to lemma 2.1 and to the following result, the proof of which will be given in the appendix.

Lemma 2.2 *Let $(B^k)_{1 \le k \le d}$ be d symmetric definite matrices of dimension n, then we have for all $\theta \in R^n$:*

$$\begin{cases} \theta_{i_1,\dots,i_d} \prod_{k=1}^{d} (B^k)_{i_k,j_k} \theta_{j_1,\dots,j_d} \le \prod_{k=1}^{d} \lambda_{max}(B^k)(\theta,\theta)_d \\ \theta_{i_1,\dots,i_d} \prod_{k=1}^{d} (B^k)_{i_k,j_k} \theta_{j_1,\dots,j_d} \ge \prod_{k=1}^{d} \lambda_{min}(B^k)(\theta,\theta)_d \end{cases}$$

We apply this lemma to each term of the sum $\sum_{\alpha,|\alpha|=k} \prod_{n=1}^{d} (A_1^{m,\alpha_n})_{i_n,j_n}$. Then, we obtain for λ_{min} for example:

$$\theta_{i_1,\dots,i_d} \prod_{n=1}^{d} (A_1^{m,\alpha_n})_{i_n,j_n} \theta_{j_1,\dots,j_d} \ge \prod_{n=1}^{d} \lambda_{min}(A_1^{m,\alpha_n})(\theta,\theta)_d.$$

Since $\lambda_{min}(A_1^{m,\alpha_n}) \ge C p^{4(\alpha_n - m)}$,

and $\lambda_{min}(A_d^{m,k}) \ge \sum_{\alpha,|\alpha|=k} \prod_{n=1}^{d} \lambda_{min}(A_1^{m,\alpha_n})$, we have

$$\lambda_{min}(A_d^{m,k}) \ \ge \ C \sum_{\alpha,|\alpha|=k} \prod_{n=1}^{d} p^{4(\alpha_n - m)} \ = \ C p^{4(k-md)},$$

so that $\lambda_{min}(A_d^{m,k}) \ge C(k) p^{4(k-md)}$.
We get also $\lambda_{max}(A_d^{m,k}) \le C$, and the theorem is proved $\square$

3 Cases m=1 and m=2

3.1 Equivalence of condition numbers

We have proved for m=1 (see [6]) and for m=2 (see [7]), that the condition numbers $\kappa(A_d^{m,k})$ are equivalent to $p^{4(md-k)}$. We precise these results in the following theorem.

Theorem 3.1
For $A_d^{m,k}$, the matrix associated with the $(.,.)_{k,\Omega}$ $(0 \le k \le m)$ scalar product and corresponding to the basis (definition 1.1) built for the homogeneous elliptic problems of order 2m, m=1 or m=2, on $]-1,1[^d$, we have:
$\kappa(A_d^{m,k}) \sim p^{4(md-k)}), \forall (d,k), 0 \le k \le m.$

Proof For m=1, we have simple relations between the extreme eigenvalues of $A_d^{1,k}$ and $A_1^{1,0}$, given below without proof

Lemma 3.1 *For m=1, the extreme eigenvalues of the d-dimensional mass and stiffness matrices can be expressed in function of those of the only 1-dimensional mass matrix by:*

$$\begin{aligned}\lambda_{max}(A_d^{1,0}) &= \lambda_{max}(A_1^{1,0})^d\\ \lambda_{min}(A_d^{1,0}) &= \lambda_{min}(A_1^{1,0})^d\\ \lambda_{max}(A_d^{1,1}) &= \lambda_{max}(A_1^{1,0})^{d-1}\\ \lambda_{min}(A_d^{1,1}) &= \lambda_{min}(A_1^{1,0})^{d-1}.\end{aligned}$$

Thanks to these results, the proof of the optimality of the bounds in theorem 2.1 has only to be done for the 1-d mass matrix $A_1^{1,0}$. For that we have to find a particular vector for which the Raleigh quotient attains the bound.

For the maximal eigenvalue, we have $\lambda_{max}(A_1^{1,0}) \leq C$ (C independent of p) from the proof of theorem 2.1, and we have for the particular vector γ corresponding to the first basis function (the same for every p due to the hierarchical character of the basis):

$$(A_1^{1,0}\gamma,\gamma)_1 \quad = \quad C(\gamma,\gamma)_1\ ,$$

where C is independent of p.

For the minimal eigenvalue, we give without proof the following result:

Lemma 3.2 *The function* $\psi_p = L_{p+2}'' - \frac{(p+3)(p+4)}{4} L_{p+1}'$ *of* $Q_p^{1,0}([-1,1])$ *satisfies:*

$$40 \mid \psi_p \mid^2_{1,(-1,1)} \sim p^4 \parallel \psi_p \parallel^2_{0,(-1,1)}$$

This lemma shows that there exists a particular vector $\tilde{\theta}$ in R^{p-1} such that:

$$(A_1^{1,0}\tilde{\theta},\tilde{\theta})_1 \quad \leq \quad Cp^{-4}(\tilde{\theta},\tilde{\theta})_1\ ,$$

that is $\lambda_{min}(A_1^{1,0}) \sim p^{-4}$ thanks to theorem 2.1, which proves theorem 3.1 for m=1 thanks to lemma 3.1.

For the maximal eigenvalue (m=2), we have $\lambda_{max}(A_d^{2,k}) \leq C$ (independent of p) from the proof of theorem 2.1, and we have for the particular vector γ corresponding to the basis function $\mathcal{H}_4^2$:

$$(A_1^{2,k}\gamma,\gamma)_1 \quad = \quad C(k)(\gamma,\gamma)_1\ , k=1,2\ ,$$

where C(k) is independent of p. So, from Rayleigh quotient of Γ associated to $\mathcal{H}_4^2 \otimes ... \otimes \mathcal{H}_4^2$, we obtain $\lambda_{max}(A_d^{2,k}) \geq C(k)$.

For the minimal eigenvalue of $(A_d^{2,k})$, it is more technical. We give the following lemma without proof:

Lemma 3.3 *The function* $\phi_p = L_{p+3}''' - \frac{(p+5)(p+6)}{4} L_{p+2}'' + \frac{(p+3)(p+4)(p+5)(p+6)}{48} L_{p+1}'$ *of* $Q_p^{2,0}([-1,1])$ *satisfies:*

$$\begin{aligned}\mid \phi_p \mid^2_{2,(-1,1)} &\sim p^8 \parallel \phi_p \parallel^2_{0,(-1,1)}\ ,\\ \mid \phi_p \mid^2_{1,(-1,1)} &\sim p^4 \parallel \phi_p \parallel^2_{0,(-1,1)}\ ,\\ \parallel \phi_p \parallel^2_{0,(-1,1)} &\sim p^{10}\ .\end{aligned}$$

This lemma proves that there exists a particular vector $\tilde{\tilde{\theta}}$ in R^{p-3} such that:

$$(A_1^{2,0}\tilde{\tilde{\theta}},\tilde{\tilde{\theta}})_1 \leq Cp^{-8} \quad , \quad (A_1^{2,1}\tilde{\tilde{\theta}},\tilde{\tilde{\theta}})_1 \leq Cp^{-4}\ .$$

Considering the Rayleigh quotient of the vector

$$\tilde{\tilde{\Theta}}_{i_1,\ldots,i_d} = \prod_{n=1}^{d} \tilde{\tilde{\theta}}_{i_n},$$

and using theorem 2.1, we can prove the equivalences of theorem 3.1.

□

3.2 Equivalence of condition numbers with diagonal preconditioning

Definition 3.1 We note $\tilde{A}_d^{m,k}$, the scaled matrix:

$$\tilde{A}_d^{m,k} = diag(A_d^{m,k})^{-1/2} A_d^{m,k} diag(A_d^{m,k})^{-1/2}\ ,$$

and $\tilde{\kappa}(A_d^{m,k}) = \tilde{\lambda}_{max}(A_d^{m,k})/\tilde{\lambda}_{min}(A_d^{m,k})$ the condition number of $\tilde{A}_d^{m,k}$. □

We prove for m=1 (see [6]) and for m=2 (see [7]), that the condition number after diagonal preconditioning $\tilde{\kappa}(A_d^{m,k})$ of $A_d^{m,k}$ is equivalent to $p^{2(md-k)}$. We precise this result in the following theorem.

Theorem 3.2

For $A_d^{m,k}$, *the matrix associated with the* $(.,.)_{k,\Omega}$ $(0 \leq k \leq m)$ *scalar product and corresponding to the basis (definition 1.1) built for the homogeneous elliptic problems of order 2m, m=1 or m=2, on* $]-1,1[^d$, *we have:* $\tilde{\kappa}(A_d^{m,k}) \sim p^{2(md-k)}), \forall(d,k), 0 \leq k \leq m.$

Proof For m=1, we have again simple relations between the extreme eigenvalues of $\tilde{A}_d^{1,k}$ and $\tilde{A}_1^{1,0}$, given in the following lemma:

Lemma 3.4 *For m=1, the extreme eigenvalues of the d-dimensional diagonal preconditioned mass and stiffness matrix can be expressed in function of those of the only 1-dimensional diagonal preconditioned mass matrix by:*

$$\begin{aligned}\tilde\lambda_{max}(A_d^{1,0}) &= \tilde\lambda_{max}(A_1^{1,0})^d\\ \tilde\lambda_{min}(A_d^{1,0}) &= \tilde\lambda_{min}(A_1^{1,0})^d\\ \tilde\lambda_{max}(A_d^{1,1}) &= \tilde\lambda_{max}(A_1^{1,0})^{d-1}\\ \tilde\lambda_{min}(A_d^{1,1}) &= \tilde\lambda_{min}(A_1^{1,0})^{d-1}.\end{aligned}$$

Proof The scaled matrix $\tilde A_1^{1,0}$ is the mass matrix corresponding to the L^2 normalized basis $\left\{\tilde\phi\right\}_{2\le i\le p}$ with $\tilde\phi_i = \frac{\phi_i}{\|\phi_i\|_{L^2(-1,1)}}$. Thus we can apply lemma 3.1 here too and obtain the result for the mass matrices $\tilde A_d^{1,0}$.

From

$$(A_d^{1,1})_{i_1,\dots,i_d,j_1,\dots,j_d} = \sum_{k=1}^{d} \delta_{i_k,j_k} \prod_{l=1,l\neq k}^{d} (A_1^{1,0})_{i_l,j_l} \ ,$$

where δ is the Kronecker symbol, we see that $(A_d^{1,1}\theta,\theta)_d$ is made up of d terms. Each term is of the type $(A_{d-1}^{1,0}\beta,\beta)_{d-1}$ where β is for example $\beta_{i_2,\dots,i_d} = \theta_{i_1,i_2,\dots,i_d}$, and can be bounded as follows:

$$(A_d^{1,1}\theta,\theta)_d \le \tilde\lambda_{max}(A_{d-1}^{1,0})(diag(A_{d-1}^{1,0})\beta,\beta)_{d-1} \ ,$$

since $diag(A_{d-1}^{1,0})^{-1}A_{d-1}^{1,0}$ and $\tilde A_{d-1}^{1,0}$ have the same eigenvalues.

Summing the d terms and identifying the diagonal of $A_d^{1,1}$, we obtain:

$$(A_d^{1,1}\theta,\theta)_d \le \tilde\lambda_{max}(A_{d-1}^{1,0})(diag(A_d^{1,1})\theta,\theta)_d \ .$$

In the same way, we can prove:

$$(A_d^{1,1}\theta,\theta)_d \ge \tilde\lambda_{min}(A_{d-1}^{1,0})(diag(A_d^{1,1})\theta,\theta)_d \ .$$

The preceding inequalities imply an upper (resp. lower) bound for $\tilde\lambda_{max}(A_1^{1,0})$ (resp. $\tilde\lambda_{min}(A_1^{1,0})$). To prove that these bounds are attained, we exhibit the special $\tilde\theta$ defined by:

$$\tilde\theta_{i_1,\dots,i_d} = \prod_{k=1}^{d} \frac{v_{i_k}}{m_{i_k}^{1/2}}$$

where v is an eigenvector of the matrix $\tilde A_1^{1,0}$ associated with $\tilde\lambda_{max}(A_1^{1,0})$ (resp. $\tilde\lambda_{min}(A_1^{1,0})$) and $m_{i_k} = (A_1^{1,0})_{i_k,i_k}$.

Thus we have for example:

$$(A_d^{1,1}\tilde\theta,\tilde\theta)_d = \left\{\tilde\lambda_{max}(A_1^{1,0})\right\}^{d-1}(diag(A_d^{1,1})\tilde\theta,\tilde\theta)_d \ .$$

□

For the one dimensional scaled mass matrix, we have (m=1):

Lemma 3.5 *For m=1, the extreme eigenvalues of the 1-d mass matrix preconditioned by its diagonal satisfy*

$$\begin{aligned}\tilde\lambda_{max}(A_1^{1,0}) &\sim 1\\ \tilde\lambda_{min}(D^{-1}A_1^{1,0}) &\sim p^{-2}.\end{aligned}$$

Proof The explicit knowledge of the mass matrix $A_1^{1,0}$ permits to explicit $\tilde A_1^{1,0}$ as follows:

$$\begin{aligned}(\tilde A_1^{1,0})_{i,i} &= 1 \ , \ \forall\, 2\le i\le p\\ (\tilde A_1^{1,0})_{i,i+2} &= \frac{-1}{2}\sqrt{\frac{(2i-3)}{(2i-1)}\frac{(2i+5)}{(2i+3)}} \ , \ \forall\, 2\le i\le p-2\\ (\tilde A_1^{1,0})_{i,i-2} &= (\tilde A_1^{1,0})_{i-2,i} \ , \ \forall\, 4\le i\le p-2\end{aligned}$$

By Gerschgorin theorem one has:

$$\max_{4\le i\le p-2}\left\{\sum_{j=2,j\neq i}^{p} |\,(\tilde A_1^{1,0})_{i,j}\,|\right\} \le 1 - \frac{6}{(2p-1)(2p+9)}$$

and then $\lambda_{max}(D^{-1}A_1^{1,0}) \le 2$.

The scaled matrix $\tilde A_1^{1,0}$ is diagonal dominant, although $A_1^{1,0}$ is not, since:

$$|\,(\tilde A_1^{1,0})_{i,i+2}\,| \le \frac{1}{2} - \frac{3}{(2i-1)(2i+3)} < \frac{1}{2}.$$

Thus we can use again Gerschgorin theorem to obtain:
$\tilde\lambda_{min}(A_1^{1,0}) \ge \frac{6}{(2p-1)(2p-9)} \ge \frac{3}{2p^2}$.

Since $\lambda_{max}(\tilde A_1^{1,0}) \ge \max_{2\le i\le p}\left\{(\tilde A_1^{1,0})_{i,i}\right\} = 1$, we have the lower bound of $\lambda_{max}(\tilde A_1^{1,0})$.

For $\lambda_{min}(\tilde A_1^{1,0})$, we have the following upper bound:
$\tilde\lambda_{min}(A_1^{1,0}) \le \frac{\lambda_{min}(A_1^{1,0})}{\lambda_{min}(Diag(A_1^{1,0}))}$
where
$\lambda_{min}(Diag(A_1^{1,0})) = (A_1^{1,0})_{p,p} = \frac{2}{(2p+1)(2p-3)}$
and
$\lambda_{min}(A_1^{1,0}) \sim p^{-4}$. □

We have given the proof of theorem 3.2 only for the case m=1.

For m=2, the proof can be done with similar techniques (see [7]).

□

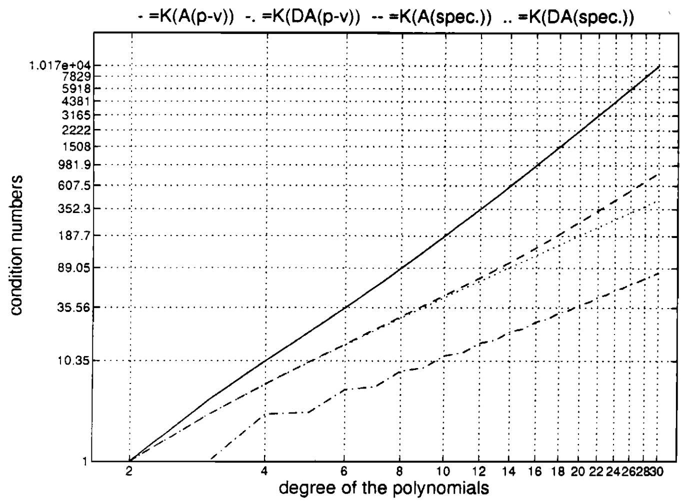

Figure 1: Comparison in 2-D

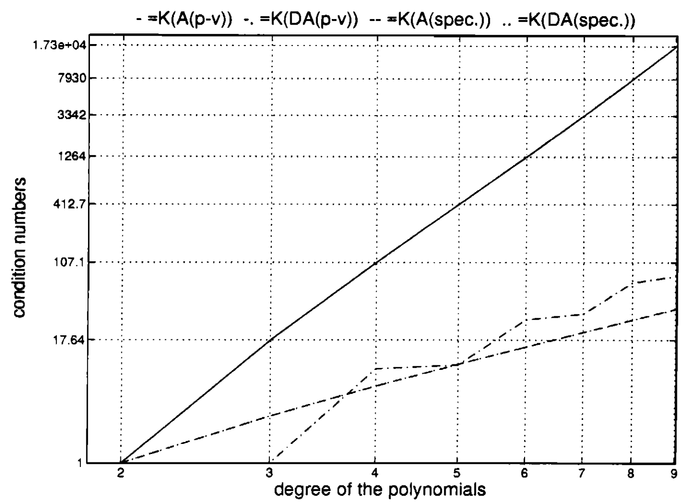

Figure 2: Comparison in 3-D

4 Comparison with the spectral element matrices

For the spectral element method, the condition number for the matrix built with bubble functions on the reference element $[-1,1]^d$ have been studied by C. Bernardi and Y. Maday, which have given the following result:

Theorem 4.1

For d = 1 to 3, the condition number of the stiffness matrix $A_d^{spec.}$ obtained with Gauss-Lobatto numerical quadrature, and built with the Lagrangian interpolants on p-1 Gauss-Lobatto interior points satisfies

$$\kappa(A_d^{spec.}) \sim p^3$$

For $A_d^{spec.}$, we have no theoretical results for the diagonal preconditioning.

Following figures give, in 2-d and 3-d, a numerical comparison between p-version (p-v) and spectral (spec.) element for the condition number of the stiffness matrix with (DA) or without (A) diagonal preconditioning.

Remark 4.1 *In 2-d, we see that, the condition number of the stiffness matrix for the p-version is better (resp. worse) with (resp. without) diagonal preconditioning than for the spectral element. Moreover we note that diagonal preconditioning is efficient for all degrees in p-version, but only for degrees greater than ten for the spectral element method.*

Remark 4.2 *In 3-d, we remark that the condition number of the spectral element method is lower than that of the p-version with or without diagonal preconditioning.*

5 Conclusion

We have proved that in the p-version of the finite element method, the condition numbers grow as a power op the degree p, the power of p depending of the dimension d of the problem. Moreover, we have proved that for all matrices obtained for problem of order 2 or 4, the diagonal preconditioning divides by 2 the power of p in the condition numbers.

Appendix

We present a proof of lemma 2.2 which can be considered as a generalization of one given in [9].

We shall denote by I the identity matrix on R^n and use for the sums the convention of the repeated indices.

We make d successive linear variable transformations by $(B^j)^{1/2}$ in order to simplify the central term of lemma 2.2, and note $\theta(k)$ the vector obtained after k such transformations.

With

$$\theta(1)_{.,j_2,\ldots,j_d} = (B^1)^{1/2}\theta_{.,j_2,\ldots,j_d}\ ,\ \forall 1 \le j_2,\ldots,j_d \le n\ ,$$

we obtain :

$$\theta_{i_1,\dots,i_d}\prod_{k=1}^{d}(B^k)_{i_k,j_k}\theta_{j_1,\dots,j_d} =$$
$$((B^1)^{-1/2})_{i_1,r}\theta(1)_{r,i_2,\dots,i_d}\prod_{k=1}^{d}(B^k)_{i_k,j_k}((B^1)^{-1/2})_{j_1,s}$$
$$\theta(1)_{s,j_2,\dots,j_d} .$$

Since

$$\begin{aligned}((B^1)^{-1/2})_{i_1,r} \quad (B^1) \quad {}_{i_1,j_1}((B^1)^{-1/2})_{j_1,s} &= ((B^1)^{-1/2})_{i_1,r}((B^1)^{1/2})_{i_1,s} \\ &= ((B^1)^{-1/2})_{i_1,r}((B^1)^{1/2})_{s,i_1} = I_{s,r} ,\end{aligned}$$

we have

$$\begin{aligned}\theta_{i_1,\dots,i_d}\prod_{k=1}^{d} \quad (B^k) \quad {}_{i_k,j_k}\theta_{j_1,\dots,j_d} \\ &= \theta(1)_{i_1,i_2\dots,i_d}\prod_{k=2}^{d}(B^k)_{i_k,j_k}\theta(1)_{i_1,j_2,\dots,j_d} ,\end{aligned}$$

because of the identity

$$\theta(1)_{r.i_2\dots,i_d}I_{s,r}\theta(1)_{s,j_2\dots,j_d} = \theta(1)_{i_1,i_2\dots,i_d}\theta(1)_{i_1,j_2\dots,j_d} .$$

For $\theta(2)$, we have for all $1 \leq j_3, \dots, j_d \leq n$,

$$\theta(2)_{i_1,.,j_3,\dots,j_d} = (B^2)^{1/2}\theta(1)_{i_1,.,j_3,\dots,j_d} ,$$

which gives :

$$\begin{aligned}\theta_{i_1,\dots,i_d} \quad \prod \quad {}_{k=1}^{d}(B^k)_{i_k,j_k}\theta_{j_1,\dots,j_d} \\ &= \theta(2)_{i_1,i_2,i_3\dots,i_d}\prod_{k=3}^{d}(B^k)_{i_k,j_k}\theta(2)_{i_1,i_2,j_3\dots,j_d} .\end{aligned}$$

We obtain finally:

$$\theta_{i_1,\dots,i_d}\prod_{k=1}^{d}(B^k)_{i_k,j_k}\theta_{j_1,\dots,j_d} = (\theta(d),\theta(d))_d .$$

Now, we return successively to $(\theta,\theta)_d$:

$$\begin{aligned}(\theta(d),\theta(d))_d &= \theta(d)_{i_1,\dots,i_d}\theta(d)_{i_1,\dots,i_d} \\ &= ((B^d)^{1/2})_{i_d,r}\theta(d-1)_{i_1,\dots,i_{d-1},r} \\ &\quad ((B^d)^{1/2})_{i_d,s}\theta(d-1)_{i_1,\dots,i_{d-1},s},\end{aligned}$$

$$\begin{aligned}(\theta(d) \quad , \quad \theta(d))_d &= \\ &= (B^d)_{r,s}\theta(d-1)_{i_1,\dots,i_{d-1},r}\theta(d-1)_{i_1,\dots,i_{d-1},s} \\ &= ((B^d)\theta(d-1)_{i_1,\dots,i_{d-1},.},\theta(d-1)_{i_1,\dots,i_{d-1},.})_1,\end{aligned}$$

so that

$$\begin{aligned}\lambda_{min}(B^d)(\theta(d-1),\theta(d-1))_d &\leq (\theta,\theta)_d, \\ \lambda_{max}(B^d)(\theta(d-1),\theta(d-1))_d &\geq (\theta,\theta)_d.\end{aligned}$$

The final result follows recursively.

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