The Erfc-Log Filter and the Asymptotics of the Euler and Vandeven Sequence Accelerations

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Abstract

We introduce a new filter or sum acceleration method which is the complementary error function with a logarithmic argument. It was inspired by the large order asymptotics of the Euler and Vandeven accelerations, which we show are both proportional to the erfc function also. We also show the relationship between Vandeven's filter, the Erfc-Log filter and the "lagged-Euler" method. The theory for the last of these is used to predict a spatially-varying optimal order for filtering of a Fourier or Chebyshev series for a function with a discontinuity, front or shock.

Key words: sequence acceleration, filtering, Fourier spectral, Chebyshev spectral.

AMS subject classifications: 41A60, 42A24, 65B10, 76L05.

1 Introduction

When the solution u(x,t) develops a shock or other region of very rapid variation, the convergence of all types of spectral series is slowed to a crawl. Chebyshev, Fourier and Legendre exhibit Gibbs' Phenomenon: the N-term truncation of the series has O(1) errors with rapid, unphysical oscillations in a boundary layer of width O(1/N) centered on the shock or frontal zone [5, 9, 3]. "Filtering" or "sum acceleration" is an important tool for reducing Gibbs' oscillations. If the original unfiltered (and slowly converging) series is

$$u_N(x) = \sum_{j=0}^{N} a_j \ \phi_j(x) \tag{1}$$

where the $\phi_j(x)$ are the basis functions, then a smoother and more physical approximation is the filtered partial sum

$$u(x) = \sum_{j=0}^{N} \sigma(j/N) \ a_j \ \phi_j(x) \tag{2}$$

where σ is symmetric with respect to $\theta = 0$, that is, $\sigma(-\theta) = \sigma(\theta)$ for all θ. Unfortunately, there is as yet no theory that identifies a unique, optimum filtering function $\sigma(\theta)$ for each situation. However, some general considerations are known.

One is the concept of the "order" p of a filter, which will be made more precise in the next section. A high order filter is one which modifies u(x) only slightly in the smooth regions away from the shock. Almost by definition, large p is desirable far from the frontal zone. Majda, McDonough and Osher [11] show that it is possible to recover spectral accuracy away from the shock, even when u(x) is discontinuous, by filtering of sufficiently large order.

In the vicinity of the front, however, *low* filtering order is desirable because large p gives a filtered function $u_\sigma(x)$ which is very oscillatory and in the limit $p \to \infty$ displays Gibbs' Phenomenon even worse than the unfiltered series. It follows the Holy Grail of filtering is one which is *spatially-adaptive* with an order $p(x)$ that varies from small values around the shock to large values in the smooth regions far away from the discontinuity in u(x).

To carry out such adapative filtering, we need a tool for identifying shocks or regions of very high gradient. Local error estimates have been well-developed for spatially-adaptive finite difference, finite element and finite volume codes, so we shall not discuss them further here. We shall instead simply assume that we have identified the points where low order filtering is needed.

It is beyond the scope of this article to apply a filter with a spatially-adaptive p to a real fluid flow. Our goal is more modest: to define a new filter, the "Erfc-Log" acceleration,

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and to derive a theory for how the order p should vary with nearness to the front. As a kind of extended prologue, we shall derive asymptotic approximations to two widely used filters, due to Vandeven and Euler, to show that these are asymptotically equivalent to each other and to the Erfc-Log filter in the limit $p \to \infty$. This asymptotic equality allows us to connect the theory of the Erfc-Log filter with earlier work of Boyd on the "lag-averaged Euler" acceleration, which supplies a theory for optimizing $p(x)$. Simple numerical experiments show that the theory is quite effective.

For simplicity, our illustrations use only Fourier series. In the next to last section, we show how the Fourier results generalize almost trivially to Chebyshev and Legendre expansions, too.

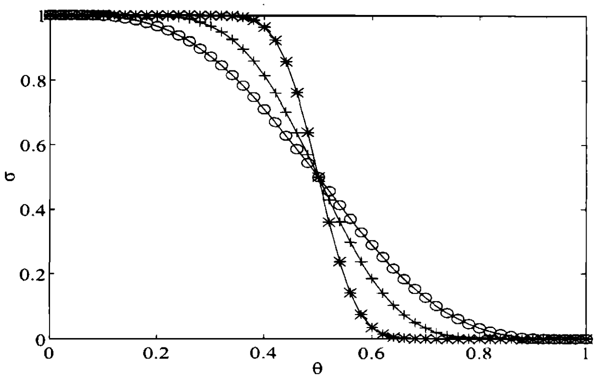

Figure 1: The Vandeven filter for different orders p. Circles: $p = 4$. Plus signs: $p = 10$. Asterisks: $p = 40$ (steepest slope).

2 Vandeven's theorem

Theorem 2.1 (Vandeven Filter Order) *(Simplified from the original). If $\sigma(\theta)$ is a sufficient smooth function such that*

$$
\begin{aligned}
\sigma(0) &= 1 \\
\sigma^{(m)}(0) &= 0, \qquad m = 1, 2, \ldots, (p-1) \\
\sigma^{(m)}(1) &= 0, \qquad m = 0, 1, \ldots, (p-1)
\end{aligned} \tag{3}
$$

where $\sigma^{(m)}$ denotes the m-th derivative of σ, then the filter function $\sigma(\theta)$ is of "order p in Vandeven's sense" and he proved the following:

1. *If $u(x)$ is smooth in the sense of possessing at least p continuous derivatives, then*

$$|u(x) - u_\sigma(x)| \leq constant \ \frac{1}{N^{p-1/2}} \tag{4}$$

2. *If $u(x)$ has a jump discontinuity at one or more points $x = c_m$, i. e.,*

$$\lim_{\epsilon \to 0} [u(c_m + \epsilon) - u(c_m - \epsilon)] = j_m \neq 0, \tag{5}$$

then

$$|u(x) - u_\sigma(x)| \leq constant \ \frac{1}{[d(x)]^{p-1} \ N^{p-1}} \tag{6}$$

where $d(x)$ is the distance from x to the nearest singularity, that is,

$$d(x) = \inf \{|x - (c_m + 2k\pi|\}, \tag{7}$$

for all m and any integer k.

The first part of the theorem implies that prior to the development of front, we can drive the error to zero faster than any finite power of N (i. e., achieve "spectral accuracy") by using a filter of sufficiently high order p. The second part of the theorem shows, in a more precise reaffirmation of Madja *et al.* [11], that spectral accuracy is still possible even with a discontinuity in $u(x)$ provided x is not too close to the shock. The factor of $d(x)$ shows that the error estimate falls apart – to O(1) errors – when $d(x)$ is $O(1/N)$. Sadly, this is not a flaw in the proof, but rather an intrinsic limitation of the class of filters described by Eq.(2).

The conditions for small σ were known long before Vandeven, but the need to impose conditions on the filter function for $\theta \approx 1$, that is, near the truncation or aliasing limit, was new and surprising.

Vandeven's Theorem provides some constraints on filters, but not specify a unique form. In the next section, we shall describe a filter first proposed by Vandeven himself.

3 Vandeven's filter

This acceleration is defined [12] by

$$\sigma_V(\theta) \equiv 1 - \frac{\Gamma(2p)}{[\Gamma(p)]^2} \int_0^\theta [t(1-t)]^{p-1} \, dt \tag{8}$$

and illustrated in Figure 1. For integer order p, this can be alternatively defined as the unique Hermite interpolating polynomial of degree $(2p-1)$ which satisfies the $2p$ conditions to be a filter of order p in Vandeven's sense.

The filter can be evaluated for general p by exploiting the fact that it is a special case of the incomplete beta-function, which in turn is a special case of the hypergeometric function:

$$\sigma_V = 1 - I_\theta(p,p) = 1 - \frac{\Gamma(2p)}{[\Gamma(p)]^2}\frac{\theta^p}{p}F(p, 1-p; p+1; \theta) \tag{9}$$

in the notation of the *The NBS Handbook of Functions* [1]. Because of the identity

$$I_\theta(p,p) = 1 - I_{1-\theta}(p,p) \tag{10}$$

it is only necessary to apply the hypergeometric power series, which has a unit radius of convergence, for $\theta \leq 1/2$:

$$I_\theta(p,p) = \frac{\theta^p(1-\theta)^p\Gamma(2p)}{p[\Gamma(p)]^2}\left\{1 + \sum_{n=0}^{\infty}\frac{\Gamma(p+1)\Gamma(2p+n+1)}{\Gamma(2p)\Gamma(p+n+2)}\theta^{n+1}\right\} \tag{11}$$

4 Steepest descent approximations for large order p

4.1 Nonuniform approximation

Figure 2 shows the integrand of the integral in σ_V for different orders. The most striking conclusion is that the integrand becomes *narrower* and *narrower* as the order p increases. This suggests that the integrand can be more and more accurately approximated for large p by writing the integrand as an exponential and then making a Taylor approximation. To simplify, let $P \equiv (p-1)$ and change the integration variable to $y \equiv (t - 1/2)$ so that the integrand is centered on y=0. Then, without approximation,

$$\begin{aligned} t^P\,(1-t)^P &= \exp(P\log(t[1-t])) \\ &= \exp\left\{P\log\left(\frac{1-4y^2}{4}\right)\right\} \\ &= 2^{-2P}\exp\left\{P\log(1-4y^2)\right\} \end{aligned} \tag{12}$$

If we expand the logarithm as a Taylor series, then the integrand is approximated by the Gaussian so that

$$\begin{aligned} \sigma_V(\theta;p) &\sim 1 - \frac{\Gamma(2p)}{2^{2P}[\Gamma(p)]^2}\int_{-1/2}^{\theta-1/2}\exp(-4Py^2)dy \\ &\sim 1 - (1/2)\mathrm{erf}\left\{2p^{1/2}(|\theta| - 1/2)\right\} \\ &\sim (1/2)\mathrm{erfc}\left\{2p^{1/2}(|\theta| - 1/2)\right\}, \qquad p >> 1 \end{aligned} \tag{13}$$

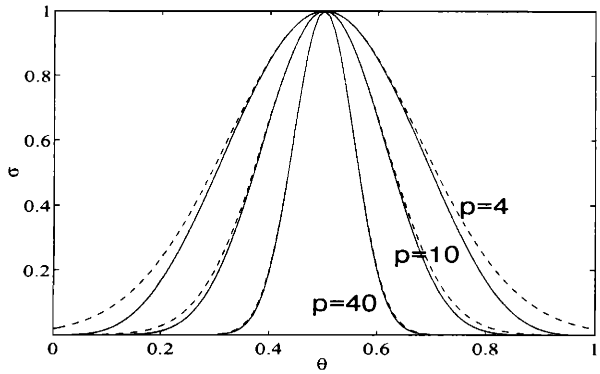

Figure 2: The integrand of the integral in Vandeven's filter, scaled to have unit maximum, for three different p (solid) compared with corresponding approximation by the Gaussian function $\exp(-4p[t-1/2]^2)$ (dashed)

No approximations have been made in Eq. 13 except for the Taylor expansion of the argument of the logarithm, and also the replacement of $P(= p-1)$ by p, consistent with $p >> 1$. Unfortunately, the erfc approximation is not uniformly valid as evident from Figure 2 because the expansion is about $t = 1/2$, but the integration is only over a small range of t far from this point when θ is small.

4.2 Uniform, improved approximation

A uniform approximation can be derived by consistently applying the method of steepest descent. The first step is to make an exact change of variable so that the argument of the exponential is quadratic in the new variable $z(y)$:

$$-z^2 \equiv P\log(1-4y^2) \tag{14}$$

Expanding the metric factor dy/dz in the transformed integrand and retaining only the lowest order in $1/p$ gives the approximation, uniformly valid in $\theta \in [0,1]$ for $p >> 1$,

$$\sigma_V(\theta;p) \sim \frac{1}{2}\mathrm{erfc}\left\{2p^{1/2}\left(|\theta| - \frac{1}{2}\right)\sqrt{\frac{-\log(1-4[\theta-1/2]^2)}{4[\theta-1/2]^2}}\right\} \tag{15}$$

The error in approximating the Vandeven filter by these two approximations is shown (on a logarithmic scale) in Figure 3. The maximum error for various orders is illustrated in Table 1 and is roughly $0.045/p$ – quite small even for low order p.

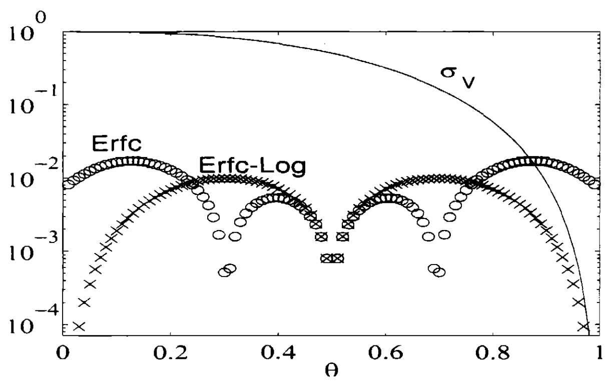

Figure 3: Solid $\sigma_V(\theta, p=3)$. Circles: Absolute value of the error in the Erfc approximation. Pluses: Error in Erfc-Log approximation.

Order p	$max\|\sigma_V(\theta;p) - \sigma_{Erfc-Log}(\theta;p)\|$
1	0.0787
2	0.0287
3	0.0170
4	0.0120
5	0.0093
6	0.0076
7	0.0064
8	0.0055
9	0.0049
10	0.0043

Table 1: Maximum error for $\theta \in [0,1]$ of the Erfc-Log approximation to Vandeven's filter

5 Euler acceleration

The Euler filter of order M is defined by [4, 6, 7]

$$(16) \qquad \begin{aligned} \sigma_E(0) &= 1 \\ \sigma_E\left(\frac{j}{M+1}\right) &= \sum_{k=j}^{M} \mu_{Mk}, \qquad j = 1, 2, \ldots M \\ \sigma_E(1) &= 0 \end{aligned}$$

where the "partial sum weights" are

$$(17) \qquad \mu_{Mk} = \frac{M!}{2^M} \frac{1}{k!(M-k)!}$$

The summation from $k = j$ to M is analogous to the indefinite intnegral in Vandeven's method while the partial sum weight μ_{Mk} plays the role of the integrand $t^{p-1}(1-t)^{p-1}$. In sharp contrast, however, the Euler acceleration is defined only for *discrete* values of θ.

Like Vandeven's integrand, the partial sum weights become increasingly concentrated with respect to the middle of the range as the order increases. By applying the method of steepest descent for *sums* [2], we find

$$(18) \; \sigma_E(\theta; M) \approx \frac{1}{2}\text{erfc}\left(\sqrt{2M+4}\left[|\theta| - \frac{1}{2}\right]\right), \quad M >> 1$$

This has exactly the same form as the large-order approximation to the Vandeven filter. Indeed, the two filters are asymptotically *identical* if the orders of the two methods are related by

$$(19) \qquad M = 2p - 2$$

Table 2 shows that the erfc-approximation is very accurate. The maximum error in any of the weights for a given order is roughly $0.03/M$.

The Euler acceleration does have one major weakness compared to Vandeven's: because the filter of order M is defined only at $(M+2)$ discrete points, the Euler filter can only be applied to $(M+2)$ terms of a series. In contrast, one has the option (a useful one, it turns out) of applying σ_V for fixed order p to an arbitrarily large number of terms.

The lag-averaged Euler acceleration, described in the section after next, generalizes the classical Euler filter to obtain most of the advantages of Vandeven's acceleration.

6 The Erfc and Erfc-log filters

Asymptotic approximations are usually only imperfect reflections of an underlying reality. Filters, however, are only

Order M	$max\|\sigma_E(\theta;M) - \sigma_{Erfc-Log}(\theta;M)\|$
1	0.0416
2	0.0223
3	0.0127
4	0.0083
5	0.0076
6	0.0061
7	0.0044
8	0.0043
9	0.0038
10	0.0030
15	0.00205
20	0.00151
25	0.00119
30	0.00097

Table 2: Maximum error for $\theta \in [0,1]$ of the Erfc-Log approximation to $\sigma(j/(M+1))$, Euler's acceleration

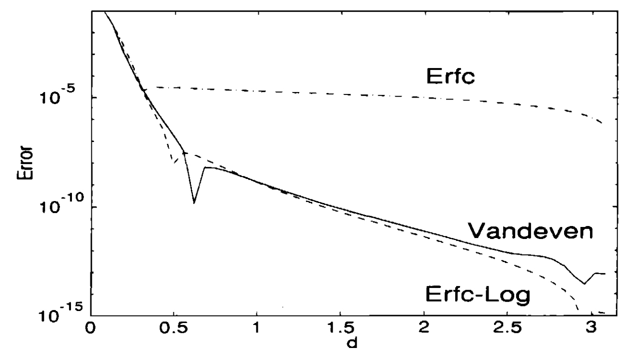

Figure 4: Absolute value of the error in the sine series for the sawtooth function, truncated at $N = 100$, after application of the Vandeven filter (solid). the Erfc-Log filter (dashed) and the Erfc filter (dotted) for order $p = 8$. The abscissa is $d(x)$, the distance to the nearest jump discontinuity.

a means to an end, a tool for improving other approximations. In the absence of a theory to identify "the" optimum filter, a slavish affection for a particular filter, such as Vandeven's, seems silly. The Vandeven Filter Theorem does not identify a unique filter, but only suggests a whole class of filters. In practice, some filters which nearly but not exactly satisfy the conditions of the theorem work well in applications [11].

Consequently, it is sensible to regard the Erfc and Erfc-Log formulas as something more than mere approximations. These expressions themselves define new filters, coequal in status with the Euler and Vandeven filters:

$$\bar{\theta} \equiv |\theta| - \frac{1}{2} \tag{20}$$

$$\sigma_{Erfc}(\theta;p) \equiv (1/2)\mathrm{erfc}\left\{2p^{1/2}\,\bar{\theta}\right\} \tag{21}$$

$$\sigma_{Erfc-Log}(\theta;p) \equiv \frac{1}{2}\mathrm{erfc}\left\{2p^{1/2}\,\bar{\theta}\sqrt{\frac{-\log(1-4\bar{\theta}^2)}{4\bar{\theta}^2}}\right\} \tag{22}$$

The Erfc filter is simple, but it does not satisfy the conditions of Vandeven's theorem. Does it matter?

To test this, we applied the Erfc, Erfc-Log and Vandeven filters to accelerate the Fourier series for the piecewise linear or "sawtooth" function, which is defined by

$$\mathrm{Sw}(x) \equiv \begin{cases} x/\pi & x \in [-\pi,\pi] \\ \mathrm{Sw}(x+2\pi m) & |x| \geq \pi, m = integer \end{cases} \tag{23}$$

or equivalently by

$$\mathrm{Sw}(x) \equiv \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\sin(jx) \qquad \forall\ x \tag{24}$$

This function has a jump discontinuity at $\pm\pi$ and is thus a good model of a function with a shock wave, or of a frontal zone too narrow to be resolved by N Fourier terms.

Fig. 4 shows that in the vicinity of the front, all three methods are about equally bad. Away from the front, however, the Erfc filter is awful compared to both the Vandeven and Erfc-Log filters, whose results are indistinguishable. With regret, we must abandon the Erfc filter, in spite of its highly desirable simplicity, because it is too inaccurate in consequence of its violation of Vandeven's Theorem.

The Erfc-Log filter, however, is just as good as Vandeven's, but simpler. The numerical results of later sections will all be generated using the Erfc-Log filter.

The Euler, Vandeven, and Erfc-Log filters are identical triplets with in the sense of asymptotic equivalence. (Recall that the asymptotic approximations of the Euler and Vandeven filters by the Erfc-Log filter are accurate even for order p or M as small as 2.) The filters are not exactly the same, but then, identical triplets have individual personalities. We can pick whichever personality is convenient.

For computation, the Erfc-Log filter is the most convenient. For theoretical purposes, the Euler filter has some advantages because we can tap into a couple of centuries

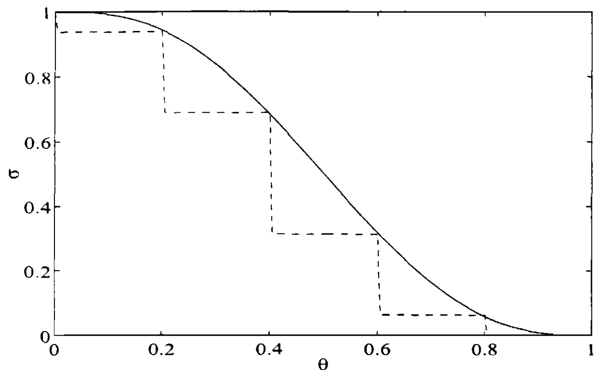

Figure 5: Solid: $\sigma_V(\theta;p)$ for $p=3$. Dashed: $\sigma_{LE}([(M+1)/M)]\theta;4)$, the lag-averaged Euler scheme derived from the standard Euler method of order 4. The argument of θ in σ_{LE} has been scaled so its graph touches the Vandeven filter at the edge of each "step".

of analysis that has grown around it, as we shall do in the next section.

7 Lag-averaged Euler acceleration

The lag-averaged Euler method is a very simple (one might unkindly say "simple-minded") generalization of Euler's method. Since the latter (at order M) is defined only at $(M+2)$ discrete points, extend it to a larger number of terms by applying each of the M "non-trivial" weights to λ terms in succession where λ is the "lag parameter". (The "non-trivial" weights are those which are not equal to 1 or 0.) The ordinary Euler acceleration is the special case $\lambda = 1$. The weight function is [8]:

$$\sigma_{LE}(\theta;\lambda,N=1+M\lambda) = \begin{cases} 1 & \theta = 0 \\ \sigma_E\left(\frac{k}{M};M\right), & \frac{(k-1)}{M} < \theta \leq \frac{k}{M} \end{cases} \tag{25}$$

as illustrated schematically in Fig. 5.

For $M = 4$, for example, a_0 is weighted by 1, $\{a_1,\ldots,a_\lambda\}$ are multiplied by 15/16, $\{a_{\lambda+1},\ldots,a_{2\lambda}\}$ by 11/16, the next quarter of the series is multiplied by 5/16, and final fourth of the terms is weighted by 1/16.

The reason this seemingly obvious generalization is interesting is that both it and the Euler acceleration can be derived from averaging successive partial sums. The partial sums are

$$S_k \equiv \sum_{j=0}^{k} a_j \tag{26}$$

Suppose the coefficients a_j *oscillate* in *degree* j with period P. The shortest possible period is $P=2$ which corresponds to a strictly alternating series: if a_j is positive, then a_{j+1} is negative, a_{j+2} is positive, a_{j+3} is negative and so on. An elementary theorem of first-year calculus shows that the partial sums will successively overshoot and undershoot the true sum S.

In this case, the sequence of "once-averaged" partial sums

$$T_j \equiv (S_j + S_{j-1})/2 \tag{27}$$

should be a better approximation, for each j, than either of the two partial sums from which it was formed. The overshoot of S_{j+1} is largely cancelled by the undershoot S_{j-1} when the two are averaged.

The once-averaged partial sums often oscillate, too. In this case, the rate of convergence to S can be further accelerated by averaging the averages T_j to form a sequence of *twice*-averaged sums. Continuing this "averaging-of-averages" until all $M+1$ terms in a given series have been exhausted gives the standard Euler acceleration.

If the coefficients oscillate with a different period P, then Boyd [8] suggested lag-averaging of partial sums, that is, generalizing the fundamental averaging by averaging partial sums which differ in degree by an integer λ, i. e.,

$$T_j \equiv (S_j + S_{j-\lambda})/2 \tag{28}$$

where the optimal lag is

$$\lambda = [P/2] \tag{29}$$

where $[P/2]$ denotes the integer *nearest* half the period in degree. With this choice of λ, the "crest" of a "wave" in a_j is averaged with the "trough" for maximum cancellation. It is shown in [8] that repeating the lag-averaging until all N terms in the truncated series have been exhausted gives the weight in Eq. 25.

8 Optimizing filter order

The reason the lag-averaged Euler theory is intriguing is that at least for some classes of functions, it is possible to determine how the period-in-degree varies with d(x), the distance to the nearest singularity of u(x), and thereby optimize the lag λ as a function of x. Because of the close connection between the Euler and lag-Euler methods and the Vandeven and Erfc-Log filters, i. e., asymptotically these methods are all described by a single formula, the lag-averaged Euler theory should work equally well for $\sigma_V(\theta;p)$ and $\sigma_{Erfc-Log}(\theta;p)$ as well.

The sawtooth function, defined earlier, is the simplest function with a jump discontinuity. However, it is more than a mere exemplar. If we adjust the strength and location of the singularity, the difference between the shifted-and-scaled sawtooth function and an arbitrary u(x) with a single discontinuity per period is continuous. It follows that the Fourier series of the difference converges more rapidly than that of the sawtooth function, implying in turn that the Fourier coefficients of the sawtooth function *asymptotically* approximate those of $u(x)$, a_j, as degree $j \to \infty$. It follows that what is optimum for the sawtooth function should also be optimum for other functions with one discontinuity, at least for sufficiently large N where N is the truncation of the Fourier series.

Boyd [8] shows, in an analysis not repeated here, that

$$P(x) = \frac{2\pi}{d(x)} \tag{30}$$

where $d(x)$, as in Vandeven's Theorem, is the difference between x and the singularity. Thus, the period of the Fourier coefficients a_j in j varies from P=2 at the point farthest from the singularity (and its periodic images) where $d(x) = \pi$, its maximum value, to ∞ where $x = x_s$, the location of the discontinuity in u(x).

The optimum lag λ in the lag-averaged Euler method is simply the integer nearest $P(x)/2$. Translating this to the Euler and Erfc-Log filters so these filters, applied to $N = 1 + M\lambda$ terms, are the envelope of the lag-averaged Euler method gives

$$\boxed{p_{optimum}(x) = 1 + N\ \frac{d(x)}{2\pi}} \tag{31}$$

9 Numerical tests

Figure 6 illustrates how the error in approximating a function with a discontinuity varies with both nearness to the singularity $d(x)$ and filter order p. Along the left edge of the figure where $d(x)$ is small, i. e., close to the discontinuity, the error is mediocre $(O(10^{-2}) \approx O(1/N))$ for all orders p. Very close to the discontinuity in $\mathrm{Sw}(x)$, filtering helps little.

On the right of Figure 6, far from the singularity, we see confirmation of Madja, McDonough and Osher's contention [11] that it is possible to retrieve spectral accuracy: for sufficiently large p, we obtain errors smaller than $O(10^{-12})$ [to the upper right of the contour labelled "-12"] in spite of the nasty singularity in Sw(x). Further, for fixed $d(x)$, the error decreases roughly geometrically with p.

Close to the singularity, Figure 6 shows little except that no filter works particularly well. Figure 7 is a blowup of

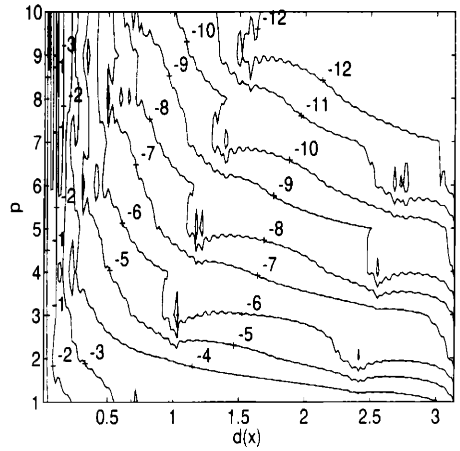

Figure 6: Contours of the logarithm (base 10) of the error as a function of order p (vertical) and distance $d(x)$ to the discontinuity in the function (horizontal). Erfc-Log filter applied to the first 100 terms of the sine series for the sawtooth function.

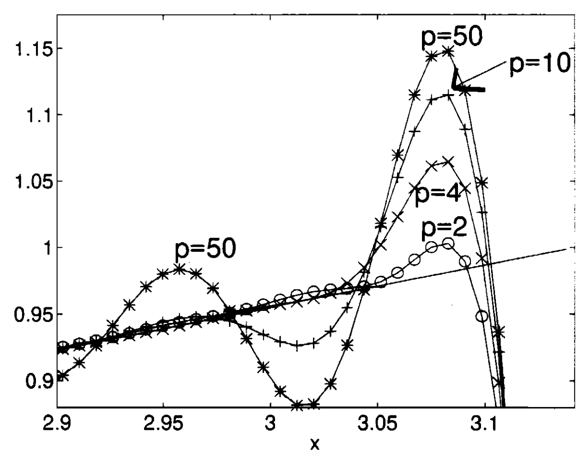

Figure 7: Comparison of the sawtooth function (straight line without symbols) with Erfc-Log filtered 100-term sine series for various orders p. Circles: p=2. x's: $p = 4$. Pluses: $p = 10$. Asterisks: $p = 50$.

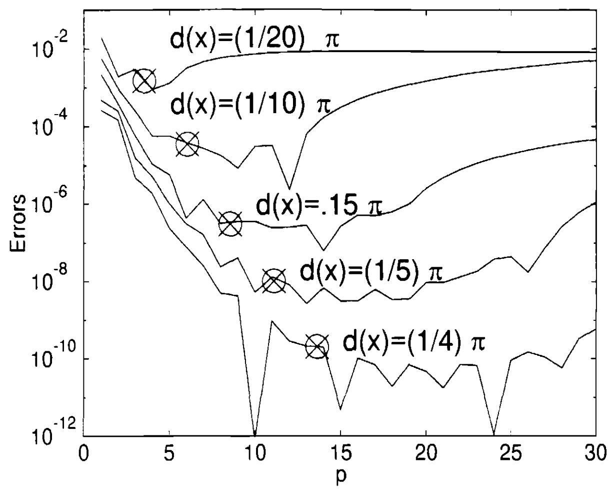

Figure 8: The error in approximating Sw(x) by 100 terms of its Fourier series after application of the Erfc-Log filter of various orders p at five different distances $d(x)$ from the singularity of the sawtooth function. The large x-in-circle symbols mark the predicted optimal order.

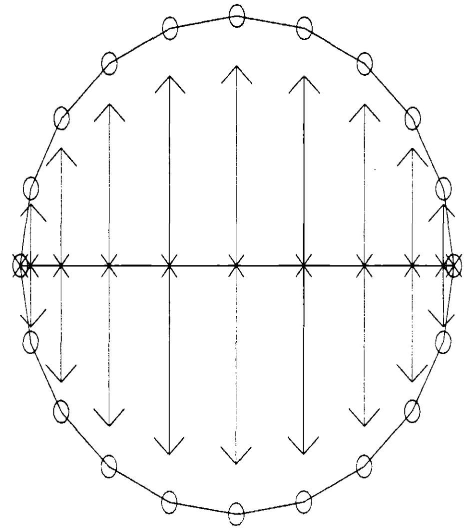

Figure 9: A graphical interpretation of the Chebyshev → Fourier mapping. Each point on the Chebyshev grid (crosses on the horizontal line bisecting the circle is mapped by t=arccos(x) into two points on the corresponding evenly spaced Fourier grid (circles on the unit circle) as indicated by the arrows. The Fourier theory for optimizing filter order can be applied to the Chebyshev case if distance d to the singularity is measured on the circle, not on the Chebyshev grid itself.

a comparison between the filtered sine series and the sawtooth function, showing only a small portion of the periodicity interval near the discontinuity at $x = \pi$. All of the filtered series are way off for $x > 3.1$. For smaller x, however, low order ($p = 2$) is best because the approximation is nearly monotonic with only a slight overshoot. In contrast, $p = 50$ gives back the wild oscillations of Gibbs' oscillation. The approximation is poor for $x < 3.1$, too.

Neither the high accuracy possible far from the singularity for high order p, nor the desirability of low order p in the neighborhood of the jump, are novelties; Figs. 6 and 7 have been included merely for completeness. The intriguing question is: how well does our theory *predict* the optimal p as a function of nearest to the singularity?

Figure 8 shows the answer is: pretty well. For each value of d, the distance to the singularity, there is a minimum in the error as a function of filter order p. The minimum is very flat so that the error is insensitive to the choice of p within a factor of (3/2) either too large or too small. The predictions of Eq. 31 are at the low-p edge of the flat part of each curve, but this is quite acceptable. It seems likely that for actual fluid dynamics calculations, which will be much more contaminated by aliasing and other noise than the sawtooth function, that erring on the side of *low* order – stronger filtering – is desirable anyway.

10 Chebyshev and Legendre series

A Chebyshev polynomial expansion on $x \in [-1, 1]$ is mapped into a Fourier cosine series in $\tau \in [-\pi, \pi]$ by the change-of-variable

$$(32) \qquad x = cos(\tau) \qquad \Longleftrightarrow \qquad \tau = arccos(x)$$

as shown schematically in Fig. 9. Because a Chebyshev series is just a Fourier series in disguise, all earlier results carry over to Chebyshev polynomials with only minor modifications.

The important change is that in order to borrow the Fourier theory that relates optimal order $p(x)$ to distance from the front $d(x)$, this distance to the singularity should now be measure in terms of the *trigonometric* argument τ instead of x, the argument of the Chebyshev polynomials:

$$(33) \quad d(x) \equiv \arccos(x) - \arccos(x_s) \qquad \text{[Chebyshev]}$$

where x_s is the location of the singularity. (For multiple singularities, take the minimum of the difference of arccosines over all the singularities of u(x).)

Legendre polynomials are very popular in spectral elements [3, 10]. Unfortunately, no simple transformation is known to turn a Legendre expansion into a Fourier series. However, the Legendre grid is very similar to the Chebyshev grid and the two types of series converge within the same region in the complex-plane. It seems likely, though we offer no proof, that $p(x)$ can be optimized for Legendre series by the same formulas as for Chebyshev.

11 Conclusions

Our long-term goal, which is to compute fluid flows with shocks by using spectral methods with a filter of spatially-varying order, must be left for the future. In this work, we have accomplished a couple of useful preliminaries.

First, we have introduced a new filter defined in terms of the complementary error function with a logarithmic argument. The Erfc-Log filter is closely related to both the Vandeven and Euler filters, for which we have shown it furnishes an asymptotic approximation for large order. However, the Erfc-Log filter is preferable because it avoids the messiness of the incomplete beta-function, which is the Vandeven filter, and is defined for continuous order in contrast to the Euler acceleration, which is defined only when its order parameter is an integer.

Second, we have exploited the connection between the erfc-log and Euler filters to apply existing theory for the latter to make a prediction of the optimal order for the erfc-log [and Vandeven] filters, Eq. poptimal. Fig. 8 shows that theory is acceptably accurate at least for simple models. Untested here is whether our prediction of optimal filter order is as good for complicated functions, such as the solutions to real engineering problems, as we have shown it to be for the sawtooth function.

The Erfc-log filter (or any filter) requires an algorithm for to identify shock or frontal regions. However, this first need can be met by borrowing from a large body of work on data-adaptive numerical algorithms. Adaptive regridding, for example, has the same need as filtering to identify the location of the shock.

The strategy we propose has similarities to Flux-Corrected Transport algorithms, which we use a high order approximation to the flux away from the front and a low-order but smooth approximation in the shock neighborhood. The key difference is that our filter order is allowed to vary *continuously* with x. This avoids filter-induced, spurious discontinuities in the numerical solution.

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