High-Order Entropy Conserving Difference Methods for Nonlinear Conservation Laws

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Abstract

It is demonstrated how entropy pairs can be obtained for symmetrizable systems of conservation laws by solving Euler's differential equation. This procedure suggests a way of discretizing the conservation laws such that the resulting difference scheme is entropy conservative. This technique will result in entropy conservative schemes of arbitrary order of accuracy as long as the difference operators satisfy a summation-by-parts rule. Having established entropy conservation, it is straightforward to prove generalized energy estimates for the semidiscrete equations by solving Euler's differential equation.

Key words: high-order, entropy, finite differences.

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1 The continuous problem

We shall consider one-dimensional hyperbolic systems of the form

$$(1) \qquad \begin{array}{ll} u_t + f_x = 0\,, & x \in (0,1)\,, \quad t > 0 \\ u(x,0) = \varphi(x)\,, & \end{array}$$

where $u, f(u) \in \mathrm{R}^d$. At the boundaries we prescribe data ψ for the ingoing characteristics, which will be defined later. Jacobians will be denoted by

$$f_u = \begin{pmatrix} \partial_{u_1} f_1 & \dots & \partial_{u_d} f_1 \\ \vdots & & \vdots \\ \partial_{u_1} f_d & \dots & \partial_{u_d} f_d \end{pmatrix}.$$

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A major concern when analyzing such systems is the existence of an entropy pair $(\eta(u), q(u))$, η, q scalar functions and η convex, satisfying the additional conservation law

$$(2) \qquad \eta_t + q_x = 0$$

for smooth solutions u of eq. (1). It is well known [2, 7] that such an entropy pair exists iff the hyperbolic system (1) is symmetrizable, i. e., there exists a change of variables $u = u(v)$ such that the Jacobians u_v, g_v are symmetric; $g(v) \equiv f(u(v))$. Furthermore, u_v is assumed to be positive definite, $u_v > 0$ for short.

It is easy to see that eq. (1) implies an additional conservation law [6] if the *entropy condition*

$$(3) \qquad \eta_u^T f_u = q_u^T$$

holds; η_u denotes the gradient

$$\eta_u = \begin{pmatrix} \partial_{u_1} \eta \\ \vdots \\ \partial_{u_d} \eta \end{pmatrix}$$

with a similar definition of q_u. Although theoretically very elegant, eq. (3) may be hard to solve explicitly for η and q. Therefore, we propose a simple solution procedure based on Euler's differential equation. Furthermore, the solutions will be obtained in a form that is suitable for proving the existence of a generalized energy estimate for eq. (1) and its semidiscrete counterpart.

Assuming that eq. (1) is symmetrizable by means of a change of variables $u = u(v)$ yields

$$f_x = g_x = (g - G)_x + G_x\,,$$

where $G = G(v)$ is an arbitrary function. Take G to be the solution of Euler's inhomogeneous differential equation

$$(4) \qquad G_v v = -G + g \Longleftrightarrow G(v) = \int_0^1 g(\theta v) d\theta\,.$$

Thus,

$$(5) \qquad f_x = (G_v v)_x + G_v v_x\,,$$

which is a variant of the skew-symmetric form described in [10]. It follows immediately that G_v is symmetric. Similarly,

$$u_t = (U_v v)_t + U_v v_t\,, \tag{6}$$

where $U(v)$ is the solution of

$$U_v v = -U + u \Longleftrightarrow U(v) = \int_0^1 u(\theta v)d\theta\,, \tag{7}$$

which clearly shows that U_v is symmetric positive definite (SPD). Hence,

$$\begin{aligned} 0 &= v^T(u_t + f_x) \\ &= v^T(U_v v)_t + v^T U_v v_t + v^T(G_v v)_x + v^T G_v v_x \\ &= (v^T U_v v)_t + (v^T G_v v)_x\,, \end{aligned}$$

where the last equality follows because of the symmetry of the Jacobians U_v and G_v. Thus, defining

$$\begin{aligned} \eta(u) &\equiv v^T U_v v \\ q(u) &\equiv v^T G_v v \end{aligned} \tag{8}$$

we obtain the additional conservation law

$$\eta_t + q_x = 0\,.$$

Now, by (4) and (8) it follows that

$$q_u = (v^T g - v^T G)_u = v_u^T (v^T g - v^T G)_v\,,$$

where the last equality is a consequence of the chain rule. Hence, the product rule yields

$$q_u = v_u^T(g_v^T v + g - G - G_v^T v) = v_u^T(g_v^T v + g - G - G_v v)$$

since G_v is symmetric. Invoking (4) once more yields

$$q_u = v_u^T g_v^T v = (g_v v_u)^T v = f_u^T v\,.$$

Similarly,

$$\eta_u = (u_v v_u)^T v = v\,,$$

whence

$$\eta_u^T f_u = q_u^T\,.$$

Furthermore, $\eta_{uu} = v_u > 0$. Consequently, (η, q) constitutes an entropy pair.

Before proceeding to the semidiscrete case we make one more observation about the entropy condition (3). Following Mock [7] we define the new variable $v = \eta_u$. Then $v_u = \eta_{uu} > 0$ because of convexity, whence the transformation $v = v(u)$ is well defined. Clearly, v_u is symmetric. Furthermore,

$$g_v = f_u u_v = u_v(v_u f_u)u_v\,.$$

Being the inverse of the symmetric positive definite matrix v_u, it follows that u_v is SPD. Now, $v_u f_u = q_{uu} f_u$, which is symmetric [1]. The equation above thus shows that g_v is symmetric. The entropy condition (3) can now be expressed as (by applying the chain rule to $g(v) \equiv f(u(v))$ and $r(v) \equiv q(u(v))$)

$$v^T g_v v_u = (v_u^T r_v)^T\,,$$

i. e.,

$$v^T g_v = r_v^T\,,$$

which after transposition yields (g_v symmetric)

$$g_v v = r_v\,. \tag{9}$$

But this is Euler's differential equation for g with r_v as forcing [4]. Similarly, the chain rule implies

$$u_v v = u_v^T \eta_u = \zeta_v\,, \tag{10}$$

where $\zeta(v) \equiv \eta(u(v))$. Thus, neither $u(v)$ nor $g(v)$ will, in general, be homogeneous functions. The solution of eq. (10) can be written as

$$u(v) = \int_0^1 \frac{\zeta'(\theta v)}{\theta} d\theta = \phi_v\,, \tag{11}$$

where

$$\phi(v) \equiv \int_0^1 \frac{\zeta(\theta v)}{\theta^2} d\theta\,.$$

Suppose that $u(v)$ is homogeneous of degree p, i. e., $u(\lambda v) = \lambda^p u(v)$ for some $p \in \mathrm{R}$. Then by eq. (11) ($w = \lambda v$)

$$\phi_w(w) = \lambda^p \phi_v(v)\,,$$

that is,

$$\phi_v(\lambda v) = \lambda^{p+1}\phi_v(v)\,.$$

Upon normalizing $\phi(0) = 0$ we obtain

$$\phi(\lambda v) = \lambda^{p+1}\phi(v)\,. \tag{12}$$

Conversely, $\phi(\lambda v) = \lambda^{p+1}\phi(v) \Longrightarrow u(\lambda v) = \lambda^p u(v)$. Consequently, $u(v)$ is homogeneous of degree p iff $\phi(v)$ is homogeneous of degree $p+1$. Similarly, $g(v) = \psi_v(v)$ is homogeneous of degree p iff $\psi(v)$ is homogeneous of degree $p+1$, where

$$\psi(v) \equiv \int_0^1 \frac{r(\theta v)}{\theta^2} d\theta\,,$$

which follows from eq. (9). Thus, eq. (1) can be re-written as [7]

$$u_t + f_x = u_t + g_x = (\phi_v)_t + (\psi_v)_x = 0\,. \tag{13}$$

Harten [3] has constructed an explicit family of symmetrizing transformations satisfying $u_v v = pu$, $g_v v = pg$, for the Euler equations of gas dynamics. The homogeneity assumption will be of fundamental importance when analyzing the semidiscrete scheme.

2 The semidiscrete problem

We now turn to the semidiscrete problem. Rather than discretizing eq. (1) directly we first use eq. (5) to reformulate the space operator.

$$(14)\quad \begin{aligned} &u_t + (G_v v)_x + G_v v_x = 0\,, \quad x \in (0,1)\,, \quad t > 0 \\ &L_0^T(v)v(0,t) = 0 \\ &v(x,0) = \varphi(x)\,, \end{aligned}$$

where $L_0(v)$ represents the characteristic boundary conditions. For convenience we only consider solutions supported at $x = 0$. The generalization to include boundary conditions at $x = 1$ is straightforward. Furthermore, we confine ourselves to homogeneous boundary conditions. It will be shown in a future paper how to handle inhomogeneous boundary conditions.

Define the grid functions $v^T = (v_0^T \ \ldots v_N^T)$ and $u^T = (u_0^T \ \ldots u_N^T)$, $u_j = u(v_j)$. The semidiscrete scheme is formulated as

$$(15)\quad \begin{aligned} &u_t + P(DG'v + G'Dv) = -L(L^TV'L)^{-1}(L^TV')_t u \\ &v(0) = \varphi\,, \end{aligned}$$

where $G' = \operatorname{diag}(G_v(v_j))$, $G_v(v_j)$ defined by eq. (4), $L^T = (L_0^T(v)\ 0\ \ldots 0)$, $V' = \operatorname{diag}(v_u(u_j))$, $v_u(u_j)$ being the Jacobian of the transformation $v = v(u)$; P is the projection operator defined by

$$P = I - L(L^TV'L)^{-1}L^TV'$$

and D is a difference operator satisfying a summation-by-parts rule [5, 8, 9]

$$(16)\qquad (u, Dv)_h = u_N^T v_N^T - u_0^T v_0^T - (Du, v)_h\,,$$

where (h denotes the mesh size)

$$(17)\qquad (u,v)_h = h\sum_{j=0}^{N} \sigma_j u_j^T v_j\,.$$

Multiply eq. (15) by P. Hence,

$$(18)\qquad Pu_t + P(DG'v + G'Dv) = 0$$

Subtraction of eq. (18) from (15) yields

$$(I - P)u_t = -L(L^TV'L)^{-1}(L^TV')_t u\,.$$

which upon multiplication by L^TV' from the left implies

$$(L^TV'u)_t = 0\,.$$

Only the first element of the left hand side is nonzero. Thus,

$$(19)\qquad (L_0^T(v_0)v_u(u_0)u_0)_t = 0\,.$$

We now assume that the transformation $u = u(v)$ satisfies (cf. [3])

$$u_v v = su\,, \quad s \neq 0\,.$$

Hence,

$$v_u u = \frac{1}{s}v\,.$$

Substituting this into eq. (19) leads to

$$(L_0^T(v_0)v_0/s)_t = 0\,.$$

Consequently, the analytic boundary conditions $L_0^T(v_0)\cdot v_0 = 0$ are fulfilled for $t > 0$ if the initial data satisfy the boundary conditions. We have thus shown that the semidiscrete solution satisfies

$$(20)\qquad L^Tv = 0, \quad t > 0\,,$$

which is equivalent to

$$(21)\qquad v = P^Tv\,.$$

We also note that P has the following structure:

$$P = \begin{pmatrix} P_0 & & \\ & I & \\ & & \ddots \end{pmatrix}, \qquad P_0, I \in \mathrm{R}^{d\times d}\,,$$

which implies $(u, Pv)_h = (P^Tu, v)_h$.

Scalar multiplication of eq. (15) yields

$$(v, u_t)_h = -(v, P(DG'v + G'Dv))_h - (v, Lw)_h\,,$$

where $w \equiv (L^TV'L)^{-1}(L^TV')_t u$. Hence,

$$(v, u_t)_h = -(P^Tv, (DG'v + G'Dv))_h - (v, Lw)_h\,.$$

But $v = P^Tv$, whence

$$(v, u_t)_h = -(v, (DG'v + G'Dv))_h - (P^Tv, Lw)_h\,.$$

Now,

$$(P^Tv, Lw)_h = (v, PLw)_h = 0$$

since $PL \equiv 0$. Consequently (v is assumed to have compact support for convenience),

$$(22)\qquad \begin{aligned} (v, u_t)_h &= -(v, DG'v)_h - (v, G'Dv)_h \\ &= v_0^T G_v(v_0)v_0\,, \end{aligned}$$

where we used summation by parts and $(u, G'v)_h = (G'u, v)_h$. Next we employ eq. (6)

$$(23)\quad (v, u_t)_h = (v, (U'v)_t)_h + (v, U'v_t)_h = \frac{d}{dt}(v, U'v)_h\,,$$

where

$$U' \equiv \mathrm{diag}(U_v(v_j)).$$

Introduce the notation (cf. eq. (8))

$$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} v_0^T U_v(v_0) v_0 \\ \vdots \\ v_N^T U_v(v_N) v_N \end{pmatrix},$$

and

$$q = \begin{pmatrix} v_0^T G_v(v_0) v_0 \\ \vdots \\ v_N^T G_v(v_N) v_N \end{pmatrix}.$$

Eqs. (22) and (23) can then be combined into

$$(e, \eta_t + Dq)_h = 0, \tag{24}$$

which is the discrete counterpart of eq. (2). The following proposition has thus been established:

Proposition 2.1 *Suppose that the conservation law (1) can be symmetrized by means of a variable transformation* $u(v)$, $g(v) \equiv f(u(v))$, *satisfying* $u_v v = su$, $g_v v = sg$, $s \neq 0$. *Then the semidiscrete initial-boundary value problem (15) is entropy conservative.*

It is obvious from eq. (7) that $U_v > 0$ (and symmetric). We thus have a generalized scalar product

$$(u, U'v)_h. \tag{25}$$

We use "generalized" to emphasize that the scalar product itself may depend on the solution u. For symmetric hyperbolic systems, however, one has $U' = (1/2)I$ (cf. the linear case). Combining eqs. (22), (23) yields

$$\frac{d}{dt}(v, U'v)_h = v_0^T G_v(v_0) v_0. \tag{26}$$

The homogeneity assumptions $u_v v = su$ and $g_v v = sg$, $s \neq 0$ imply that

$$U(v) = \frac{1}{s+1} u(v), \quad G(v) = \frac{1}{s+1} g(v), \tag{27}$$

where $s > -1, s \neq 0$. Define the generalized scalar product

$$\langle u, v \rangle_h \equiv (u, u'v)_h, \quad u' = \mathrm{diag}(u_v(v_j)).$$

Eq. (26) can thus be written as

$$\frac{d}{dt}\langle v, v \rangle_h = v_0^T g_v(v_0) v_0. \tag{28}$$

It should be remarked that eq. (28) holds for any $s \neq -1, 0$, although $U(v)$ and $G(v)$ are well defined only for $s > -1$. The reason for this is that the homogeneity assumptions actually allow us to derive eq. (28) without introducing $U(v)$ and $G(v)$. Since g_v is symmetric we have $\Lambda = Q^T g_v Q$, where the columns of Q are the (orthogonal) eigenvectors of g_v. Partition

$$\Lambda = \begin{pmatrix} \Lambda_I & \\ & \Lambda_O \end{pmatrix},$$

where the elements of Λ_I are given by $\lambda_j > 0$; the elements of Λ_O satisfy $\lambda_j \leq 0$. The characteristic variables χ are defined as ($v \in \mathrm{R}^d$)

$$\chi = Q^T v.$$

Let $\chi_I = Q_I^T v$ and $\chi_O = Q_O^T v$ denote the in- and outgoing characteristic variables. The characteristic boundary conditions can thus be expressed as

$$\chi_I = 0 \iff L_0^T(v) v = 0, \quad L_0(v) \equiv Q_I(v).$$

Eq. (28) can thus be expressed as ($v \in \mathrm{R}^{d(N+1)}$ now denotes a grid vector)

$$\frac{d}{dt}\langle v, v \rangle_h + \chi_O^T |\Lambda_O| \chi_O = \chi_I^T \Lambda_I \chi_I$$

But (cf. (20), (21))

$$P^T v = v \iff L_0^T(v_0) v_0 = 0 \iff \chi_I = 0.$$

Integration with respect to t thus proves

Proposition 2.2 *Suppose that the conservation law (1) can be symmetrized by means of a variable transformation* $u(v)$, $g(v) \equiv f(u(v))$, *satisfying* $u_v v = su$, $g_v v = sg$, $s \neq 0, -1$. *Then the solution of the semidiscrete problem (15) satisfies the generalized energy estimate*

$$\langle v, v \rangle_h + \int_0^t \chi_O^T |\Lambda_O| \chi_O(\tau) d\tau = \langle \varphi, \varphi \rangle_h$$

for homogeneous characteristic boundary conditions.

Remark 2.1 *It should be noted that the energy estimate follows for any discrete difference operator D satisfying a summation-by-parts rule (16) with respect to a scalar product of the form (17). Explicit 3rd, 4th, and 5th order accurate examples can be found in [9].*

The Euler equations of gas dynamics satisfy all of the hypotheses of proposition 2.2 [3]. Thus, it is possible to generate entropy-conservative finite difference approximations of arbitrary order for the Euler equations. Furthermore, if shocks are present one can add artificial viscosity to (15) such that the resulting scheme will satisfy an *entropy inequality.* This is the topic of a forthcoming paper.

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