On the Multigrid Method for 3D Mixed Macro-Elements

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Abstract

A natural mixed finite element method for the Stokes problem in the velocity-pressure formulation is used to approximate the velocity by continuous piecewise polynomials of degree $(k+1)$, and to approximate the pressure by discontinuous piecewise polynomials of degree k. This paper is devoted to proving constant rates of convergence for two nonnested multigrid methods when applied to solving some 3D stable $\mathcal{P}_{k+1}$-P_k $(k \geq 2)$ mixed-element equations where the underlying tetrahedral meshes have a macro-element structure. A numerical test is presented.

Key words: multigrid method, Stokes equations, mixed element, macro-element, tetrahedral mesh.

AMS subject classifications: 65M55, 65N30.

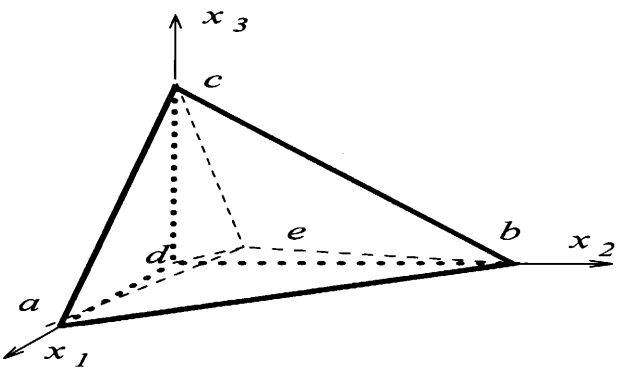

Figure 1: A macro-element consists of 4 tetrahedra.

1 Introduction

In the variation form of velocity-pressure formulation of the Stokes equations, the velocity and pressure are in the Sobolev spaces $H^1(\Omega)^d$ and $L_0^2(\Omega)$, respectively. The mixed element approximation spaces can be chosen to be the corresponding subspaces. A most natural approximation scheme would be, then, to choose continuous piecewise-polynomials of degree $(k+1)$ for the velocity and discontinuous piecewise-polynomials of degree k for the pressure. Such mixed element solutions satisfy the incompressibility condition. Scott and Vogelius [8] showed that the Babuška-Brezzi inequality holds for such $\mathcal{P}_{k+1}$-P_k triangular mixed-elements in 2D if the polynomial degree k is 3 or a higher, and if the meshes are singular-vertex free. This result is partially extended to 3D in [15]. It is shown that, when defined on tetrahedral meshes of a macro-element type, the above $\mathcal{P}_{k+1}$-P_k elements are stable if the polynomial degree for velocity is 3 or higher. In the method, starting with any quasi-uniform tetrahedral mesh, the mesh for computation is generated by subdividing each initial tetrahedron into 4 subtetrahedra by connecting the bary-center with 4 vertices (see Figure 1). The $\mathcal{P}_{k+1}$-P_k mixed elements are defined on this new mesh. The velocity in the mixed-element solution is divergence-free pointwise.

The multigrid method is an effective iteration method for solving linear systems of equations arising from discretizing partial differential equations. It is an optimal order algorithm in various cases ([1], [2]) and references therein. Verfürth has introduced two multigrid methods ([12] and [13]) for solving mixed element equations for Stokes problems and proved that the iterations converge with constant rates (independent of the number of unknowns in the linear system of equations). But the standard (nested) multigrid algorithm will not work for 3D $\mathcal{P}_{k+1}$-P_k tetrahedral mixed-elements. This is because the stability condition (see (2) below) will not hold any more at the multigrid refinement as the new interior edge of each tetrahedron is singular when the tetrahedron is subdivided into 8 half-sized tetrahedra (cf. [15]). Now, to get rid of

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singularity, if we cut each of the 8 subtetrahedra further (into 4) as depicted in Figure 1, then this process would lead to degenerate meshes which contain sharp and long tetrahedra. Given an initial tetrahedral grid, the correct way to build multilevel grids is to first refine the initial grid nestedly to the highest level. Then we can subdivide all tetrahedra on all levels to get the macro-element meshes. However, the multilevel grids created this way are not nested and the resulting mixed element spaces are not nested either. Some treatments are needed in defining the intergrid transfer operator, which is necessary in the multigrid method to transfer functions from a lower level to higher one (cf. [3] and [17]). A multigrid method for 2D, biharmonic, C^1 elements is studied in [16], where the meshes have the same macro-element structure.

In this note, we apply the two multigrid methods of Verfürth to solve the new mixed-element equations and prove that the two methods retain their constant-rate of convergence and, consequently, their optimal order of computation. Section 2 provides some basics on the mixed element approximation of the stationary Stokes equations. We define a multigrid method for the mixed-element equations in Section 3, and prove its constant rate of convergence in Section 4. In Section 5, a combined conjugate gradient – multigrid method is defined, which also converges with a constant rate. A numerical test is presented in Section 6.

2 Preliminaries

We consider the stationary Stokes problem: Find functions $\mathbf{u}$ (the fluid velocity) and p (the pressure) on a 3D domain Ω such that

$$(1)\qquad \begin{aligned} -\nu\Delta\mathbf{u} + \mathrm{grad}p &= \mathbf{f} && \text{in } \Omega,\\ \mathrm{div}\mathbf{u} &= 0 && \text{in } \Omega,\\ \mathbf{u} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\mathbf{f}$ is the body force and ν is the kinematic viscosity.

Given an initial quasi-uniform (cf. [4]) tetrahedral mesh $\mathcal{M}_0 = \{M\}$ with mesh size h_0 on Ω, we can refine each tetrahedron M into 8 subtetrahedra (cf. [18]), nestedly, to generate multilevel meshes $\{\mathcal{M}_j\}_{j=0}^J$ where $h_{j-1} = 2h_j$. We let $\mathcal{T}_j = \{T\}$ denote the corresponding macro-element mesh where each tetrahedron M of $\mathcal{M}_j$ is cut into 4 by connecting the bary-center with 4 vertices, as depicted in Figure 1. Let $\mathcal{P}_{k,\mathcal{T}_j}$ and $P_{k,\mathcal{T}_j}$ denote the piecewise continuous and discontinuous polynomials of degree k on the mesh $\mathcal{T}_j$, respectively. Let $\mathcal{P}^0_{k,\mathcal{T}_j} = \mathcal{P}_{k,\mathcal{T}_j} \cap H^1_0(\Omega)$ and $P^0_{k,\mathcal{T}_j} = P_{k,\mathcal{T}_j} \cap L^2_0(\Omega)$, i.e., $P^0_{k,\mathcal{T}_j} = \{p \in P_{k,\mathcal{T}_j} \mid \int_\Omega p = 0\}$. To shorten the notation, we let $V_j = (\mathcal{P}^0_{k+1,\mathcal{T}_j})^3 \times P^0_{k,\mathcal{T}_j}$ and we will mention the dependence on the polynomial degree k when needed. It is shown in [15] that V_j satisfies the Babuška-Brezzi stability condition: there exists a constant $C > 0$ (independent of j, but depending on k) such that

$$(2)\qquad \sup_{\substack{\mathbf{v}\in(\mathcal{P}^0_{k+1,\mathcal{T}_j})^3,\\ \mathbf{v}\neq 0}} \frac{(\mathrm{div}\mathbf{v}, p)}{|\mathbf{v}|_1} \geq C\|p\|_0 \quad \forall p \in P^0_{k,\mathcal{T}_j},$$

which ensures the best order of convergence for the mixed elements solutions $\{[\mathbf{u}_j, p_j]\}$ defined below in (3). In this paper, we use the standard notation for Sobolev spaces and their norms, and we use C as a generic constant.

The mixed elements approximation to (1) in weak formulation is: Find $[\mathbf{u}_j, p_j] \in V_j$, such that

$$(3)\qquad L([\mathbf{u}_j, p_j], [\mathbf{v}, q]) = (\mathbf{f}, \mathbf{v}) \quad \forall [\mathbf{v}, q] \in V_j,$$

where $L([\mathbf{u}, p], [\mathbf{v}, q]) := a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q)$, $a(\mathbf{u}, \mathbf{v}) := \nu(\nabla\mathbf{u}, \nabla\mathbf{v})$ and $b(\mathbf{v}, p) := -(\mathrm{div}\mathbf{v}, p)$. We assume the boundary of Ω is regular enough such that if $\mathbf{f} \in L^2(\Omega)$ in (1), then the solution $[\mathbf{u}, p] \in H^2(\Omega)^3 \times H^1(\Omega)$ (cf. [5], [6] and [11]) and

$$(4)\qquad \|\mathbf{u}\|_2 + \|p\|_1 \leq C\|\mathbf{f}\|_0.$$

The analysis here can be extended to cover some domains with a curved boundary where (4) is known to hold. By examining (2) and (4), it follows that ([5, 4])

$$(5)\qquad \begin{aligned} \|\mathbf{u}_j - \mathbf{u}\|_1 + \|p_j - p\|_0 &\leq Ch_j\|\mathbf{f}\|_0,\\ \|\mathbf{u}_j - \mathbf{u}\|_0 &\leq Ch_j^2\|\mathbf{f}\|_0. \end{aligned}$$

3 A multigrid algorithm

In this section we introduce a multigrid algorithm for solving the mixed-element equations (3). The algorithm is based on the general multigrid algorithm defined by Verfürth in [12]. We need to solve (3) on the highest level. Problems on all lower levels are auxiliary ones. We rewrite (3) in a more general form

$$(6)\qquad L([\mathbf{u}_j, p_j], [\mathbf{v}, q]) = G_j(\mathbf{v}, q) \quad \forall (\mathbf{v}, q) \in V_j.$$

G_j is a linear functional on V_j. In particular on the finest level, $G_J = (\mathbf{f}, \mathbf{v})$. Problem (6) can be written in matrix-vector form as $A_j x = b$ with a symmetric, indefinite matrix A_j. In the fine level smoothing of the multigrid method defined below, m steps of a Jacobi-like relaxation are applied to the squared system $A_j^2 x = A_j b$. The relaxation parameter ω_j, below, has to be less than or equal to the reciprocal of the spectral radius of A_j (cf. [1, 12]).

As the multilevel spaces $\{V_j\}$ are not nested, it is necessary to introduce an intergrid transfer operator $\mathrm{I}_j : V_{j-1} \to V_j$. For simplicity in analysis and in implementation (other intergrid transfer operators would lead to working algorithms too), we define

$$\mathrm{I}_j = \Pi_{\mathcal{T}_j} \times \mathrm{P}_{\mathcal{T}_j}, \quad \mathrm{I}_j[\mathbf{u}, p] = [\Pi_{\mathcal{T}_j}\mathbf{u}, \mathrm{P}_{\mathcal{T}_j}p] \in V_j \tag{7}$$

for any $\mathbf{u} \in C(\Omega)^3$ and $p \in L^2(\Omega)$. Here, in (7), we used the common notations $\Pi_{\mathcal{T}_j}$ and $\mathrm{P}_{\mathcal{T}_j}$ as the nodal value interpolation operator and the $L^2(\Omega)$ projection operator, respectively. We note that the computation of $\mathrm{P}_{\mathcal{T}_j} : P^0_{k,\mathcal{T}_{j-1}} \to P^0_{k,\mathcal{T}_j}$ is done element-wise since the pressure functions are discontinued. Because the grids $\{\mathcal{T}_j\}$ are constructed on $\{\mathcal{M}_j\}$, we can see that

$$V_j \cap V_{j+1} = (\mathcal{P}^0_{k+1,\mathcal{M}_j})^3 \times P^0_{k,\mathcal{M}_J} =: \tilde{V}_j, \tag{8}$$

the space of piecewise continuous and discontinuous polynomials on grid $\mathcal{M}_j$.

Definition 3.1 (Algorithm 3.1 in [12].)
1. *Smoothing.* Let $[\mathbf{u}_j^0, p_j^0] \in V_j$ be a given guess to the solution of Problem (6). For $l = 1, 2, \cdots, m$, compute the solutions of

$$(\mathbf{w}_j^l, \mathbf{v}) + h_j^2(r_j^l, q) = \omega_j^2\{G_j(\mathbf{v}, q) - L([\mathbf{u}_j^{l-1}, p_j^{l-1}], [\mathbf{v}, q])\} \quad \forall [\mathbf{v}, q] \in V_j \tag{9}$$

and

$$(\mathbf{u}_j^l - \mathbf{u}_j^{l-1}, \mathbf{v}) + h_j^2(p_j^l - p_j^{l-1}, q) = L([\mathbf{w}_j^l, r_j^l], [\mathbf{v}, q]) \quad \forall [\mathbf{v}, q] \in V_j. \tag{10}$$

2. *Correction.* Let $[\mathbf{u}_{j-1}^*, p_{j-1}^*] \in V_{j-1}$ be the exact solution of Problem (6) with

$$G_{j-1}(\mathbf{v}, q) := G_j(\mathrm{I}_j[\mathbf{v}, q]) - L([\mathbf{u}_j^m, p_j^m], \mathrm{I}_j[\mathbf{v}, q]) \quad \forall [\mathbf{v}, q] \in V_j. \tag{11}$$

If $j = 0$, put $[\tilde{\mathbf{u}}_{j-1}, \tilde{p}_{j-1}] := [\mathbf{u}_{j-1}^*, p_{j-1}^*]$.
If $j > 0$, compute an approximation $[\tilde{\mathbf{u}}_{j-1}, \tilde{p}_{j-1}]$ by applying $\alpha(\geq 2)$ iterations of the $(j-1)$-level scheme to (6) with starting value zero. Put

$$[\mathbf{u}_j^{m+1}, p_j^{m+1}] = [\mathbf{u}_j^m, p_j^m] + \mathrm{I}_j[\tilde{\mathbf{u}}_{j-1}, \tilde{p}_{j-1}]. \tag{12}$$

□

In real computation the L^2 inner products in (9)–(10) are replaced by equivalent discrete L^2 inner products. This does not affect the analysis (cf. [1]).

4 Convergence analysis

In this section, we prove the constant rate of convergence for the multigrid algorithm defined in Definition 3.1. Let $\{[\phi_l^j, \psi_l^j]\}_{l=1}^{\dim(V_j)}$ be the complete set of eigenfunctions for the symmetric bilinear functional $L(\cdot, \cdot)$:

$$L([\phi_l^j, \psi_l^j], [\mathbf{v}, q]) = \lambda_l^j\{(\phi_l^j, \mathbf{v}) + h_j^2(\psi_l^j, q)\}. \tag{13}$$

We can assume the eigenfunctions are normalized and that $0 < |\lambda_1^j| \leq \cdots \leq |\lambda_{\dim(V_j)}^j|$. Given $[\mathbf{u}_j, p_j] = \sum_l c_l[\phi_l^j, \psi_l^j] \in V_j$, we define the $|||\cdot|||_s$ norm by

$$|||[\mathbf{u}_j, p_j]|||_s := \left\{ \sum_{l=1}^{\dim(V_j)} c_l^2 |\lambda_l^j|^s \right\}^{1/2}. \tag{14}$$

We note that $|||\cdot|||_0$ is defined for all functions in $H^1(\Omega)^3 \times L^2(\Omega)$, while the other norms $|||\cdot|||_s$ are defined only in V_j.

Let $[\mathbf{u}_j^*, p_j^*] \in V_j$ denote the solution of Problem (6) and $[\mathbf{e}_j^l, \epsilon_j^l] := [\mathbf{u}_j^* - \mathbf{u}_j^l, p_j^* - p_j^l]$ be the error of the lth iterate, $0 \leq l \leq m+1$. The following fine-level smoothing property is shown in [12] under the conditions (4) and (2).

$$|||[\mathbf{e}_j^m, \epsilon_j^m]|||_2 \leq Ch_j^{-2}m^{-1/2}|||[\mathbf{u}_j^0, p_j^0]|||_0. \tag{15}$$

Lemma 4.1 *Let $[\mathbf{u}_j, p_j] \in V_j$ be L^2 orthogonal to $\tilde{V}_{j-1}$ in the sense that*

$$(\mathbf{u}_j, \mathbf{v}) + h_j^2(p_j, q) = 0 \quad \forall [\mathbf{v}, q] \in \tilde{V}_{j-1}. \tag{16}$$

Then

$$|||[\mathbf{u}_j, p_j]|||_{-2} \leq Ch_j^2|||[\mathbf{u}_j, p_j]|||_0. \tag{17}$$

Proof This lemma is almost identical to Lemma 4.2 in [12]. We note that the space $\tilde{V}_{j-1}$ has the same order of approximation as that for V_{j-1} and that $\tilde{V}_{j-1} \subset V_j$. Therefore, the proof there remains the same. □

Theorem 4.1 *Let $\delta_{j,m}$ be the convergence rate measured in the $|||\cdot|||_0$-norm of one iteration of multigrid algorithm defined in Definition 3.1 at level j with m relaxations. For every $\kappa \in (0, 4^{-1/(\alpha-1)})$ there is a number m_κ, which depends on k and κ, but not on j, such that*

$$\delta_{j,m} \leq \kappa \quad \forall j, m \geq m_\kappa. \tag{18}$$

Proof Following the frameworks of Verfürth [12] and Bank-Dupont [1], we need to prove the case of two-level multigrid algorithm, i.e., $[\tilde{\mathbf{u}}_{j-1}, \tilde{p}_{j-1}] := [\mathbf{u}_{j-1}^*, p_{j-1}^*]$ in (12). Let $[\mathbf{w}_{j-1}, r_{j-1}]$ be the orthogonal projection of the

iterative error $[\mathbf{e}_j^m, \epsilon_j^m]$ in $\tilde{V}_{j-1}$ Since $\tilde{V}_{j-1} \subset V_{j-1}$, by (4), it follows by a duality argument (cf. [12] and [1]) that

$$\begin{aligned}
&|||[\mathbf{e}_j^m, \epsilon_j^m] - [\mathbf{u}_{j-1}^*, p_{j-1}^*]|||_0 \\
&\quad \le Ch_j \inf_{[v,q]\in V_{j-1}} \{\|\mathbf{e}_j^m - v\|_1 + \|\epsilon_j^m - q\|_0\} \\
&\quad \le Ch_j\{\|\mathbf{e}_j^m - \mathbf{w}_{j-1}\|_1 + \|\epsilon_j^m - r_{j-1}\|_0\} \\
&\quad \le C|||[\mathbf{e}_j^m, \epsilon_j^m] - [\mathbf{w}_{j-1}, r_{j-1}]|||_0,
\end{aligned}$$

where in the last step we used an inverse inequality (cf. [4]). Because of the nonnestedness of spaces, the $m+1$-st iterative error is actually

$$\begin{aligned}
&|||[\mathbf{e}_j^{m+1}, \epsilon_j^{m+1}]|||_0 \\
&\quad = |||[\mathbf{e}_j^m, \epsilon_j^m] - \mathrm{I}_j[\mathbf{u}_{j-1}^*, p_{j-1}^*]|||_0 \\
&\quad \le |||[\mathbf{e}_j^m, \epsilon_j^m] - \mathrm{I}_j[\mathbf{w}_{j-1}, r_{j-1}]|||_0 \\
&\qquad + |||\mathrm{I}_j[\mathbf{w}_{j-1}, r_{j-1}] - \mathrm{I}_j[\mathbf{u}_{j-1}^*, p_{j-1}^*]|||_0 \\
&\quad = |||[\mathbf{e}_j^m, \epsilon_j^m] - [\mathbf{w}_{j-1}, r_{j-1}]|||_0 \\
&\qquad + |||\mathrm{I}_j([\mathbf{w}_{j-1}, r_{j-1}] - [\mathbf{u}_{j-1}^*, p_{j-1}^*])|||_0 \\
&\quad \le |||[\mathbf{e}_j^m, \epsilon_j^m] - [\mathbf{w}_{j-1}, r_{j-1}]|||_0 \\
&\qquad + C|||[\mathbf{w}_{j-1}, r_{j-1}] - [\mathbf{u}_{j-1}^*, p_{j-1}^*]|||_0,
\end{aligned}$$

where in the last step we used the L^2 stability of nodal-value interpolation operator when restricted on V_{j-1} (see [10] for a proof, where the averaging interpolation operator can be designed to be identical to $\Pi_{\mathcal{T}_j}$ when restricted to V_{j-1}). Combining above two estimates we obtain

$$(19)\quad |||[\mathbf{e}_j^{m+1}, \epsilon_j^{m+1}]|||_0 \le C|||[\mathbf{e}_j^m, \epsilon_j^m] - [\mathbf{w}_{j-1}, r_{j-1}]|||_0.$$

Now, applying Lemma 4.1 and (15) we can get (cf. [12]) that

$$\begin{aligned}
&|||[\mathbf{e}_j^m, \epsilon_j^m] - [\mathbf{w}_{j-1}, r_{j-1}]|||_0^2 \\
&\quad \le C|||[\mathbf{e}_j^m, \epsilon_j^m]|||_2 |||[\mathbf{e}_j^m, \epsilon_j^m] - [\mathbf{w}_{j-1}, r_{j-1}]|||_{-2} \\
&\quad \le \frac{C}{\sqrt{m}} |||[\mathbf{e}_j^0, \epsilon_j^0]|||_2 |||[\mathbf{e}_j^m, \epsilon_j^m] - [\mathbf{w}_{j-1}, r_{j-1}]|||_0.
\end{aligned}$$

Therefore, the proof is completed by choose m_κ large enough as

$$|||[\mathbf{e}_j^{m+1}, \epsilon_j^{m+1}]|||_0 \le \frac{C}{\sqrt{m}} |||[\mathbf{e}_j^0, \epsilon_j^0]|||_0.$$

□

5 A combined conjugate gradient – multigrid algorithm

In this section we define the second algorithm which is also based on an algorithm of Verfürth [13]. We show that the algorithm retains the constant rate of convergence. Even the multilevel spaces are nonnested in the present case. First, we define an operator $\mathrm{L} : P_{k,\mathcal{T}_J}^0 \to P_{k,\mathcal{T}_J}^0$ as follows. Given $p \in P_{k,\mathcal{T}_J}^0$ let $\mathbf{u}_p \in (\mathcal{P}_{k+1,\mathcal{T}_J}^0)^3$ and $\mathrm{L}p \in P_{k,\mathcal{T}_J}^0$ be the unique solutions of the equations

$$\begin{aligned}
(20)\quad & a(\mathbf{u}_p, \mathbf{v}) &&= b(\mathbf{v}, p) \quad \forall \mathbf{v} \in (\mathcal{P}_{k+1,\mathcal{T}_J}^0)^3, \\
(21)\quad & (\mathrm{L}p, q) &&= b(\mathbf{u}_p, q) \quad \forall q \in P_{k,\mathcal{T}_J}^0.
\end{aligned}$$

We remark that solving two linear systems is required for each evaluation of L. The first system consists of three discrete Laplace equations for continuous piecewise polynomials, where we apply the multigrid method. The second linear system is a discrete mass equation which is uncoupled on each tetrahedron. This system can be solved locally with a cost proportional to the number of unknowns (k fixed). Next, let $\mathbf{u}_f \in (\mathcal{P}_{k+1,\mathcal{T}_J}^0)^3$ and $g \in P_{k,\mathcal{T}_J}^0$ be the unique solutions of the equations

$$\begin{aligned}
(22)\quad & a(\mathbf{u}_f, \mathbf{v}) &&= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in (\mathcal{P}_{k+1,\mathcal{T}_J}^0)^3, \\
(23)\quad & (g, q) &&= b(\mathbf{u}_f, q) \quad \forall q \in P_{k,\mathcal{T}_J}^0.
\end{aligned}$$

It is shown in [13] that the pair $[\mathbf{u}, p] \in V_J$ is the solution of (3) on the top level if, and only if,

$$(24)\quad \mathrm{L}p = g,$$

where $a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - b(\mathbf{v}, p)$, for all $\mathbf{v} \in (\mathcal{P}_{k+1,\mathcal{T}_J}^0)^3$. We can write the method (20–24) in operator forms:

$$(25)\quad \mathrm{L} = \mathrm{B}\mathrm{A}^{-1}\mathrm{B}^*, \quad g = \mathrm{B}\mathrm{A}^{-1}\mathrm{P}_{\mathcal{T}_J}\mathbf{f},$$

where the operators are defined by $a(\mathrm{A}_j^{-1}\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \mathbf{v})$ (recall the notation $\mathrm{A}_J = \mathrm{A}$), $(\mathrm{B}\mathbf{u}, q) := b(\mathbf{u}, q)$ and $(\mathrm{B}^* p, \mathbf{v}) := b(\mathbf{v}, p)$, for all $[\mathbf{v}, q] \in V_j$. For simplicity, we drop the index j, if $j = J$; and, if there is no confusion. Here, $\mathrm{P}_{\mathcal{T}_J}$ is also used to denote the L^2 orthogonal projection operator on the space $(\mathcal{P}_{k+1,\mathcal{T}_j}^0)^3$. We let ω_j denote the reciprocal of the spectral radius of A_j. We now define a sequence of symmetric operators to approximate A^{-1} by the multigrid method (cf. [3]).

Definition 5.1
Set $\mathrm{K}_{0,n} = \mathrm{A}_0^{-1}$, $n = 1, 2, \cdots$.
Assume that $\mathrm{K}_{j-1,n}$ has been defined; define $\mathrm{K}_{j,1}\mathbf{z}$ for $\mathbf{z} \in (\mathcal{P}_{k+1,\mathcal{T}_j}^0)^3$ as follows:
1. Set $\mathbf{w}^0 = 0$ and $\mathbf{e}^0 = 0$.
2. Define $\mathbf{w}^l$ for $l = 1, \cdots, m$ by

$$(26)\quad \mathbf{w}^l = \mathbf{w}^{l-1} + \omega_j(\mathbf{z} - \mathrm{A}_j\mathbf{w}^{l-1}).$$

3. Define $\mathbf{w}^{m+1} = \mathbf{w}^m + \Pi_{\mathcal{T}_j}\mathbf{e}^\alpha$, $\alpha \ge 2$, where $\mathbf{e}^l$ for $l = 1, \cdots, \alpha$ is defined by

$$\begin{aligned}
(27)\quad & \mathbf{e}^l = \mathbf{e}^{l-1} \\
& + \mathrm{K}_{j-1,1}\{\mathrm{P}_{\mathcal{T}_{j-1}}(\mathbf{z} - \mathrm{A}_j\mathbf{w}^m) - \mathrm{A}_{j-1}\mathbf{e}^{l-1}\}.
\end{aligned}$$

4. Define $\mathbf{w}^l$ for $l = m+2, \cdots, 2m+1$ by (26).
5. Set $\mathrm{K}_{j,1}\mathbf{z} = \mathbf{w}^{2m+1}$.

Finally, $\mathrm{K}_{j,n}\mathbf{z}$ is defined by doing n cycles of the above iterations:

$$\mathrm{K}_{j,n}\mathbf{z} = \mathrm{K}_{j,n-1}\mathbf{z} + \mathrm{K}_{j,1}(\mathbf{z} - \mathrm{A}_j\mathrm{K}_{j,n-1}\mathbf{z}) \tag{28}$$

□

Theorem 5.1 *Let (4) hold. For any $0 < \delta < 1$, there is an integer m independent of j, but depending upon α and k, such that*

$$\|(\mathrm{A}_j^{-1} - \mathrm{K}_{j,n})\mathbf{z}\|_1 \le \delta^n \|\mathbf{z}\|_1 \quad \forall \mathbf{z} \in (\mathcal{P}^0_{k+1,\mathcal{T}_j})^3. \tag{29}$$

Proof This theorem is a corollary of Theorem 4.1 in [9], which proves the convergence of W-cycle nonnested multigrid methods. □

We remark that the multilevel spaces

$$(\mathcal{P}^0_{k+1,\mathcal{T}_0})^3 \not\subset (\mathcal{P}^0_{k+1,\mathcal{T}_1})^3 \not\subset \cdots \not\subset (\mathcal{P}^0_{k+1,\mathcal{T}_J})^3 \tag{30}$$

are used in our algorithm. This results in a nonnested multigrid method. One could use the following multilevel spaces

$$(\mathcal{P}^0_{k+1,\mathcal{M}_0})^3 \subset \cdots \subset (\mathcal{P}^0_{k+1,\mathcal{M}_J})^3 \subset (\mathcal{P}^0_{k+1,\mathcal{T}_J})^3 \tag{31}$$

to compute A_J^{-1} approximately. The advantage of the latter is that one gets better convergence rate and less computational work inside each iteration (the convergence theory in this case is standard and covered by [1]). The disadvantage is that one has to set up two data structures to handle two different families of multilevel finite element spaces.

We now define a conjugate gradient algorithm for solving Problem (24), which is an equivalent problem of the Stokes equations (3). When we solve (24), we approximate $\mathrm{L}p$ for $p \in P^0_{k,\mathcal{T}_J}$ by

$$\mathrm{L}_n p := \mathrm{B}\mathrm{K}_{J,n}\mathrm{B}^* p.$$

Definition 5.2 (Algorithm 5.1 in [13])

1. Pre-processing: Compute

$$g^* := \mathrm{B}\mathrm{K}_{J,n}\mathrm{P}_{\mathcal{T}_J}\mathbf{f}.$$

2. Start: Given an initial guess $p^0 \in P^0_{k,\mathcal{T}_J}$ for the pressure p_J solving (3). Compute

$$q^0 = \mathrm{L}_n p^0$$

and put

$$r^0 = q^0 - g^*, \qquad d^0 = -r^0.$$

Set $i = 0$, and ϵ to a small positive tolerance.

3. Iteration step: If $\|r^i\|_0 \le \epsilon$ go to step 4. Otherwise compute

$$q^{i+1} = \mathrm{L}_n d^i$$

and set

$$\begin{aligned}
\alpha^{i+1} &= -\frac{(r^i, d^i)}{(q^{i+1}, d^i)}, \\
p^{i+1} &= p^i + \alpha^{i+1} d^i, \\
r^{i+1} &= r^i + \alpha^{i+1} q^{i+1}, \\
\beta^{i+1} &= \frac{(r^{i+1}, r^{i+1})}{(r^i, r^i)}, \\
d^{i+1} &= -r^{i+1} + \beta^{i+1} d^i.
\end{aligned}$$

Replace i by $i+1$ and return to the beginning of the this step.

4. Post-processing: Compute

$$\mathbf{u}^i := \mathrm{K}_{J,n}(\mathrm{P}_{\mathcal{T}_J}\mathbf{f} - \mathrm{B}^* p^i)$$

and take $[\mathbf{u}^i, p^i] \in V_J$ as the final approximation to the solution of (3).

□

Theorem 5.2 *Let $[\mathbf{u}^i, p^i]$ be defined in Definition 5.2 and $[\mathbf{u}_J, p_J]$ the solution of (3). Then*

$$\begin{aligned}
&\|\mathbf{u}_J - \mathbf{u}^i\|_1 + \|p_J - p^i\|_0 \\
&\le \frac{1}{C - \delta^n}\{\epsilon C + \delta^n C\|\mathbf{f}\|_0 + \delta^n C\|p_J\|_0\},
\end{aligned}$$

where ϵ is defined in Definition 5.2 and δ is defined in Theorem 5.1.

Proof This is Proposition 5.1 in [13]. □

6 A numerical test

We test the combined conjugate gradient-multigrid method defined in Definition 5.1 in solving a model 2D Stokes problem on a unit right triangle (mesh for $\mathcal{M}_0$). We use 2D mixed elements on macro-element meshes where each triangle of $\mathcal{M}_j$ is subdivided into three triangles when generating $\mathcal{T}_i$, similarly to the 3D case depicted by Figure 1. For $k = 1$ and $k = 2$, such mixed elements have been shown stable in [7]. The analysis for the multigrid methods provided in this manuscript remains the same for the 2D case. But, we should point out that for high degree ($k > 2$)

degree	on $\mathcal{T}_0$		on $\mathcal{T}_1$	
$k+1$	CG	$C(BA^{-1}B^*)$	CG	$C(BA^{-1}B^*)$
2	8	14.4559	27	17.7363
3	14	14.1227	25	15.6759
4	21	15.3867	26	16.2885
5	23	15.8065	28	16.7038
6	25	16.4055	29	17.0055
7	15	16.5612	29	17.2318
8	28	16.9165	32	17.4086

Table 1: Iteration numbers and condition numbers.

degree	on $\mathcal{T}_1$	
$k+1$	CG	$C(BK_{J,n}B^*)$
2	22	17.9957
3	25	22.3300
4	32	48.3480
5	32	119.6085
6	45	323.0922
7	96	889.0082
8	175	2541.4468

Table 2: Iteration numbers and condition numbers.

polynomials, such macro-element meshes have no advantage as the regular meshes would provide stable mixed elements ([8]). The table 1 lists the numbers of the outer conjugate iterations when applying the algorithm of Definition 5.2 where $K_{J,n} = A^{-1}$. Also, the condition numbers of operator $BA^{-1}B^*$ in the L^2 inner product are listed in Table 1. We remark that in the conjugate iteration, we have to use the L^2 mass matrix for pressure functions as a preconditioner. One can find such a preconditioned conjugate iteration in [14]. One can see from the numerical data that the operator $BA^{-1}B^*$ is well conditioned, independently of the polynomial degree k. Numerical data also indicate that $BA^{-1}B^*$ remains well conditioned when we refine meshes. For example, for $k = 1$, $C(BA^{-1}B^*)$= 18.8885 and 19.3166 on $\mathcal{T}_2$ and $\mathcal{T}_3$, respectively.

However, when A^{-1} is replaced by the multigrid approximation $K_{J,n}$, the number of conjugate iterations and the condition number of $BK_{J,n}B^*$ both increase with the polynomial degree k and the mesh level J. This can be observed by the data listed in the table 2. Here, we apply the two-level nonnested multigrid method defined in Definition 5.1 where $m = 4$ (doing 4 pre-smoothings and 4 post-smoothings), and $n = 4$ (doing 4 cycles of multigrid iterations). When we increase m, or n, or both to get a better approximation $K_{J,n}$ for A^{-1}, the data in Table 2 will approach those listed in Table 1. Unlike the case of $BA^{-1}B^*$, if we fix the polynomial degree and refine the mesh, the condition number of $BK_{J,n}B^*$ would become worse (for example, using quadratic polynomials for the velocity, $C(BK_{J,n}B^*)$= 25.7506 and 70.7792 on $\mathcal{T}_2$ and $\mathcal{T}_3$, respectively).

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