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ON THE DYNAMICS OF SOME DIFFEOMORPHISMS OF $\mathbb{C}^2$ NEAR PARABOLIC FIXED POINTS

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ABSTRACT. In this paper we consider diffeomorphisms of $\mathbb{C}^2$ of the special form $F(z,w) = (w, -z + 2G(w))$. For such maps the origin is a parabolic fixed point. Under certain hypotheses on G we prove the existence of a domain $\Omega \subset \mathbb{C}$ with $0 \in \partial\Omega$ and of invariant complex curves $w = f(z)$ and $w = g(z)$, $z \in \Omega$, for F^{-1} and F, such that $F^{-n}(z, f(z)) \to 0$ and $F^n(z, g(z)) \to 0$ as $n \to \infty$.

1. INTRODUCTION AND STATEMENT OF RESULTS

The field of complex dynamics in several variables has dramatically developed in recent years. Global results in the theory, such as the study of the properties of Julia and Fatou sets are obtained in [BS], [FS1], [FS2], etc. Besides the global aspects, it is of interest to analyze the dynamics of holomorphic maps near fixed points. Results in this direction are obtained in [HP], [U1], [U2], etc. We are interested in the local behavior of the iterates of holomorphic maps near certain fixed points.

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a diffeomorphism which fixes 0 and is holomorphic near 0, and let λ_1, λ_2 be the eigenvalues of $F'(0)$. If $|\lambda_1| < 1 < |\lambda_2|$ then it is well known that for r sufficiently small the sets $W^s_{loc}(0) = \{(z,w) \in \mathbb{C}^2 : \|F^n(z,w)\| \leq r \text{ for all } n \geq 0, \lim_{n\to\infty} F^n(z,w) = 0\}$ and $W^u_{loc}(0) = \{(z,w) \in \mathbb{C}^2 : \|F^n(z,w)\| \leq r \text{ for all } n \leq 0, \lim_{n\to-\infty} F^n(z,w) = 0\}$ are invariant complex one dimensional manifolds called the local stable manifold and local unstable manifold of F at 0. In the case when $|\lambda_1| = 1$ or $|\lambda_2| = 1$ the above sets $W^s_{loc}(0)$ and $W^u_{loc}(0)$ are not necessarily manifolds anymore.

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In this paper we study diffeomorphisms F of $\mathbb{C}^2$ of the following special form

$$F(z,w) = (w, -z + 2G(w))\,, \tag{1.1}$$

where $G \in C^1(\mathbb{C})$ is holomorphic near 0, $G(0) = 0$, $G'(0) = 1$. For such maps the origin is a fixed point, the eigenvalues of $F'(0)$ are both equal to 1 and $F'(0)$ is nondiagonalizable (thus equivalent to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$). The real two dimensional case $(z,w) = (x,y) \in \mathbb{R}^2$, $F : \mathbb{R}^2 \to \mathbb{R}^2$, $G : \mathbb{R} \to \mathbb{R}$, is studied in [F]. Under the assumption that $G''(x) > 0$ for $x > 0$, it is shown in [F] that there are functions $f(x)$ and $g(x)$ defined for $x \geq 0$ such that the unique stable/unstable manifolds of F in $\{(x,y) : x \geq 0\}$ are precisely the graphs of g/f.

As described in [F], Section 2, maps F of form (1.1) have special symmetries. They come from the fact that $F = I \circ J$, where I and J are the involutions $I(z,w) = (w,z)$ and $J(z,w) = (z, -w + 2G(z))$, so $I \circ F \circ I = F^{-1}$ and $J \circ F \circ J = F^{-1}$. Assuming for the moment that there are unique stable and unstable manifolds for F at 0, given by the graphs of the univalent (i.e. one-to-one holomorphic) functions g and f respectively, it follows that I maps the stable manifold of F to the stable manifold of F^{-1}, which is the unstable manifold of F; thus for every z there is a ζ such that $(g(z), z) = I(z, g(z)) = (\zeta, f(\zeta))$, so $g = f^{-1}$. The same holds for J, so $(z, -g(z) + 2G(z)) = J(z, g(z)) = (\zeta, f(\zeta))$, and we conclude that $2G = f + f^{-1}$.

We now state the results of the paper; the proofs are given in section 2. One of our goals is to find suitable conditions on G which ensure, in the complex case, the existence of invariant complex curves $w = f(z)$ and $w = g(z)$ for F such that $F^n(z,w) \to 0$ as $n \to -\infty$ for $(z,w) \in \{w = f(z)\}$ and $F^n(z,w) \to 0$ as $n \to \infty$ for $(z,w) \in \{w = g(z)\}$.

We let $\Omega \subset \mathbb{C}$ be a convex domain (not necessarily bounded) such that $0 \in \partial\Omega$ and we assume that the function $G \in C^1(\mathbb{C})$ is holomorphic in a neighborhood of $\overline{\Omega}$ and satisfies the following conditions

(C1) $G(0) = 0$, $G'(0) = 1$, $\Omega \subseteq G(\Omega)$.

(C2) *there exists* $\alpha \in (-\pi/2, \pi/2)$ *such that* $\Re[e^{i\alpha}(G'(z) - 1)] > 0$ *for* $z \in \Omega$.

(C3) *there is a ray* $L = \{re^{i\theta} : 0 < r \leq r_0\} \subset \Omega$ *such that* $G'(re^{i\theta}) \in \mathbb{R}$ *for all* $0 \leq r < r_0$.

For future references, let us denote by $X(\Omega, \alpha)$ the class of functions $G \in C^1(\mathbb{C}) \cap O(\overline{\Omega})$ satisfying condition (C1), (C2), and (C3) with $\theta = 0$ (i.e. the Taylor coefficients of G at 0 are real).

For the function F defined by (1.1) with G satisfying conditions (C1), (C2) and (C3) we introduce the following sets:

$$W^s_\Omega(0) = \{(z,w) \in \mathbb{C}^2 : \ F^n(z,w) \in \Omega \times \Omega \text{ for all } n \geq 0, \ \lim_{n\to\infty} F^n(z,w) = 0\} ,$$

$$W^u_\Omega(0) = \{(z,w) \in \mathbb{C}^2 : \ F^n(z,w) \in \Omega \times \Omega \text{ for all } n \leq 0, \ \lim_{n\to-\infty} F^n(z,w) = 0\} .$$

We have the following:

Theorem 1.1. *Let F, G and Ω be as above. There exists a function f univalent in Ω such that $\Omega \subseteq f(\Omega)$ and the functions f and $g = f^{-1}$ have the following properties:*

(i) the graphs $\{(z,g(z)) : \ z \in \Omega\}$ and $\{(z,f(z)) : \ z \in \Omega\}$ of g and f are invariant under of F and F^{-1} respectively, and $F^n(z,g(z)) \to 0$, $F^{-n}(z,f(z)) \to 0$, as $n \to \infty$, locally uniformly for $z \in \Omega$;

(ii) if condition (C2) on G holds with $\alpha = 0$ then $W^s_\Omega(0) =$ graph g and $W^u_\Omega(0) =$ graph f.

Following the proof of this theorem in section 2 we make some remarks regarding the analyticity of f and g at the origin.

We next apply Theorem 1.1 to the following general situation: $G \in C^1(\mathbb{C})$ is holomorphic near 0 and it has the expansion

$$G(z) = z + az^{j+1} + h(z) , \tag{1.2}$$

where $a > 0$, $j \geq 1$ and $h(z) = \sum_{k\geq j+2} \alpha_k z^k$, $\alpha_k \in \mathbb{R}$.

Theorem 1.2. *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be of form (1.1) with G as above. There exist a domain $D \subset \mathbb{C}$ and functions f, g holomorphic on D with the following properties:*

(i) $0 \in \partial D$, D is starlike with respect to 0, $D \subseteq \{z : \ |\arg z| < \pi/j\}$ and the rays $\{z : \ |\arg z| = \pi/j\}$ are tangent to ∂D at 0;

(ii) $g(D) \subseteq D \subseteq f(D)$ and $f \circ g = id$ on D (hence g is univalent on D);

(iii) the graphs $\{(z,g(z)) : \ z \in D\}$ and $\{(z,f(z)) : \ z \in D\}$ are invariant under F and F^{-1} respectively, and $F^n(z,g(z)) \to 0$, $F^{-n}(z,f(z)) \to 0$, as $n \to \infty$, locally uniformly for $z \in D$.

Finally, we consider some Henon maps F_j of form (1.1), obtained for $G(z) = G_j(z) = z + az^{j+1}$, where $j \geq 1$ and $a > 0$. For such maps we can use Theorem 1.1 to obtain results of a global nature. We introduce the following domains $D_k \subset \mathbb{C}^k$:

$$D_k = \left\{ re^{i\phi} : \ 0 < r < \infty, \ \frac{(2k-1)\pi}{j} < \phi < \frac{(2k+1)\pi}{j} \right\} , \tag{1.3}$$

where $k \in \{0, 1, ..., j-1\}$. Note that the union $\bigcup_{k=0}^{j-1} D_k$ equals the complex plane minus the union of j rays joining 0 to ∞. We have the following

Theorem 1.3. *In the above setting, there exist functions f_k and g_k holomorphic on D_k, $k \in \{0, 1, ..., j-1\}$, with the following properties*

(i) $g_k(D_k) \subset D_k \subset f_k(D_k)$ (properly) and $f_k \circ g_k = id$ on D_k;

(ii) the graphs $\{(z, g_k(z)) : z \in D_k\}$ and $\{(z, f_k(z)) : z \in D_k\}$ are invariant under F_j and F_j^{-1} respectively, and $F_j^n(z, g_k(z)) \to 0$, $F_j^{-n}(z, f_k(z)) \to 0$, as $n \to \infty$, locally uniformly for $z \in D_k$;

(iii) $f_k\left(\exp\left(\frac{2k\pi i}{j}\right) z\right) = \exp\left(\frac{2k\pi i}{j}\right) f_0(z)$ for $z \in D_0$, and the same holds for g_k and g_0.

This theorem improves a result of [F] (see the remark in section 2 which follows the proof of the theorem).

2. Proofs

We will use the following classic results in geometric function theory (see [D]):

Lemma 2.1. *Let h be a holomorphic function in a neighborhood of the closed line segment $[z_0, z_1] \subset \mathbb{C}$. Then there exists a point Z in the closed convex hull of the set $h'([z_0, z_1])$ such that $h(z_1) - h(z_0) = Z(z_1 - z_0)$.*

Lemma 2.2. (Noshiro-Warschawski Theorem) *Let $D \subseteq \mathbb{C}$ be a convex domain and let h be a holomorphic function in D satisfying $\Re[e^{i\alpha} h'(z)] > 0$ in D, for some $\alpha \in \mathbb{R}$. Then h is univalent in D.*

We also need the following lemma, which is essentially proved in [F] (note that we do not make any assumptions on the second derivative of H):

Lemma 2.3. *Let $H : [0, x_0] \to \mathbb{R}$ be a continuously differentiable function satisfying $H(0) = 0$ and $H'(x) > 1$ for $0 < x < x_0$. Then the sequences of functions $\{h_n\}_{n>0}$ and $\{k_n\}_{n>0}$ given by $h_1 = H$, $k_1 = H^{-1}$, $h_{n+1} = 2H - k_n$, $k_{n+1} = (h_{n+1})^{-1}$ are well defined and satisfy the following conditions for all $0 < x < x_0$:*

(i) $0 \leq h_n(x) \leq h_{n+1}(x) \leq 2H(x)$;

(ii) $0 \leq k_{n+1}(x) \leq k_n(x)$;

(iii) if $k(x) = \lim_{n\to\infty} k_n(x)$ then the sequence of iterates $\{k^j\}_{j>0}$ converges pointwise to 0 on $[0, x_0]$ as $j \to \infty$.

Proof. An easy induction on n shows that $\{h_n\}$ and $\{k_n\}$ are well defined and satisfy (i) and (ii). Clearly $k(x) < x$ and $k_n'(x) < 1$ for all $x \in (0, x_0)$, so

$|k_n(x) - k_n(x')| \leq |x - x'|$ and hence k is Lipschitz. Then for any $x \in (0, x_0)$ $\{k^j(x)\}$ decreases to 0, which is the only fixed point of k. □

Proof of Theorem 1.1. Let P_α be the half plane $P_\alpha = \{z \in \mathbb{C} : \Re[e^{i\alpha}(z-1)] > 0\}$ and let D_α be the disc $D_\alpha = j(P_\alpha)$, where $j(z) = 1/z$ (D_α has 0 and 1 on its boundary and is tangent to ∂P_α at 1). Set $\lambda_\alpha = \sup\{|z| : z \in D_\alpha\} = 1/\cos\alpha$.

We first construct by induction sequences of holomorphic functions $\{f_n\}_{n>0}$ and $\{g_n\}_{n>0}$ with the following properties

$$(2.1) \qquad \begin{cases} f_n \in O(\overline{\Omega}), & f_n(0) = 0,\ f_n'(0) = 1 \\ \Re[e^{i\alpha}(f_n'(z) - 1)] > 0 & \text{for } z \in \Omega \\ f_n : \Omega \to f_n(\Omega) & \text{is univalent, } \Omega \subseteq f_n(\Omega), f_n(L) \supseteq L \end{cases}$$

$$(2.2) \qquad \begin{cases} g_n \in O(\overline{\Omega}), & g_n(0) = 0,\ g_n'(0) = 1 \\ g_n'(z) \in D_\alpha & \text{for } z \in f_n(\Omega) \\ g_n : f_n(\Omega) \to \Omega & \text{is the inverse of } f_n,\ g_n(L) \subseteq L \end{cases}$$

$$(2.3) \qquad 2G(z) = f_{n+1}(z) + g_n(z) \text{ for } n \geq 1 \text{ and } z \in \Omega\ .$$

These sequences are constructed in analogy to the real two dimensional case [F].

Let $f_1 = G$. By Lemma 2.2 f_1 is univalent on Ω. Clearly conditions (C2) and (C3) imply that $f_1(L) \supseteq L$, so f_1 satisfies (2.1). We assume now by induction that f_n is defined so that it satisfies (2.1) and construct g_n and f_{n+1} such that (2.1), (2.2) and (2.3) hold.

Let $g_n = (f_n)^{-1} : f_n(\Omega) \supseteq \Omega \to \Omega$. Then $g_n'(z) = 1/f_n'(g_n(z))$ for $z \in f_n(\Omega)$, so $g_n'(z) \in D_\alpha$. In order to show that g_n extends holomorphically to a neighborhood of $\overline{\Omega}$ it is enough to notice that for any $\zeta \in \partial\Omega \cap \partial f_n(\Omega)$ there is a disc Δ_ζ centered at ζ such that g_n extends holomorphically to Δ_ζ. This follows since $\zeta = f_n(\xi)$ for some $\xi \in \partial\Omega$ and f_n is univalent in a neighborhood of ξ, as $f_n'(\xi) \neq 0$. Clearly $g_n(0) = 0$, $g_n'(0) = 1$ and $g_n(L) \subseteq L$. We also have

$$(2.4) \qquad |g_n(z)| = \left| \int_0^z g_n'(\zeta) d\zeta \right| \leq \lambda_\alpha |z|\ ,$$

for all $z \in \Omega$. Set $\widetilde{f}_n(r) = e^{-i\theta} f_n(re^{i\theta})$ and $\widetilde{g}_n(r) = e^{-i\theta} g_n(re^{i\theta})$, $0 \leq r \leq r_0$. Then $\widetilde{f}_n$ and $\widetilde{g}_n$ are real valued, $\widetilde{f}_n(0) = \widetilde{g}_n(0) = 0$, $\widetilde{f}_n'(r) \geq 1$ and $\widetilde{g}_n = (\widetilde{f}_n)^{-1}$.

We next define $f_{n+1}(z) = 2G(z) - g_n(z)$. Then $f_{n+1}(0) = 0$, $f'_{n+1}(0) = 1$ and f_{n+1} is holomorphic on $\overline{\Omega}$, since G and g_n are holomorphic on $\overline{\Omega}$. Now $g'_n(z) \in D_\alpha$ implies $\Re\,[e^{i\alpha}(g'_n(z) - 1)] \le 0$, so

$$\Re\,[e^{i\alpha}(f'_{n+1}(z) - 1)] = 2\Re\,[e^{i\alpha}(G'(z) - 1)] - \Re\,[e^{i\alpha}(g'_n(z) - 1)] > 0\ ,$$

for $z \in \Omega$. Thus f_{n+1} is univalent in Ω and $\widetilde{f}'_{n+1}(r) \ge 1$, so $\widetilde{f}_{n+1}([0, r_0]) \supseteq [0, r_0]$ and $f_{n+1}(L) \supseteq L$. Finally we let $\zeta \in \partial\Omega$ and assume $f_{n+1}(\zeta) \in \Omega$. As $g_n(\zeta) \in \overline{\Omega}$ and Ω is convex it follows that $G(\zeta) = f_{n+1}(\zeta)/2 + g_n(\zeta)/2 \in \Omega$, which contradicts $\Omega \subseteq G(\Omega)$. We conclude that $\Omega \subseteq f_{n+1}(\Omega)$.

Relation (2.4) now shows that $\{g_n\}$ is a normal family in Ω and so is $\{f_n\}$, by (2.3). If we let $\widetilde{G}(r) = e^{-i\theta}G(re^{i\theta})$ then $\widetilde{f}_{n+1}(r) = 2\widetilde{G}(r) - \widetilde{g}_n(r)$, $\widetilde{g}_n = (\widetilde{f}_n)^{-1}$ and, by (C2) and (C3), $\widetilde{G}'(r) > 1$ for $r \in (0, r_0)$. Thus Lemma 2.3 implies that $\{\widetilde{f}_n\}$ increases to a function $\widetilde{f}$ and $\{\widetilde{g}_n\}$ decreases to a function $\widetilde{g}$ which satisfies $\widetilde{g}^n(r) \to 0$ as $n \to \infty$, for all $0 < r < r_0$. It follows that any two subsequential limits of $\{f_n\}$ and $\{g_n\}$ respectively agree on L, so there are functions f and g holomorphic on Ω such that $f_n \to f$ and $g_n \to g$ locally uniformly in Ω. Since $\Omega \subseteq f_n(\Omega)$, $g_n = (f_n)^{-1}$ and $f_{n+1} + g_n = 2G$ we have $\Omega \subseteq f(\Omega)$, $g = f^{-1}$ and $f + g = 2G$ on Ω. By (2.4) $|g(z)| \le \lambda_\alpha |z|$, so g, and hence $f = 2G - g$, extend continuously at 0 by $f(0) = g(0) = 0$. As $g(\Omega) \subseteq \Omega$ the iterates $\{g^n\}$ form a normal family (in the larger sense that subsequences may diverge locally uniformly to infinity). But $\widetilde{g}^n \to 0$ implies that $g^n \to 0$ locally uniformly on Ω.

To prove conclusion (i) of the theorem we use the facts that $f \circ g = id$ and $f + g = 2G$ on Ω to see that

$$F(z, g(z)) = (g(z), -z + 2G(g(z))) = (g(z), g^2(z))$$

and

$$F^{-1}(z, f(z)) = (2G(z) - f(z), z) = (g(z), z) = (g(z), f(g(z)))$$

for all $z \in \Omega$. We then get by induction that $F^n(z, g(z)) = (g^n(z), g^{n+1}(z))$ and $F^{-n}(z, f(z)) = (g^n(z), f(g^n(z)))$, for $z \in \Omega$ and for $n \ge 0$. Since $g^n \to 0$ as $n \to \infty$, locally uniformly in Ω, and since f extends continuously at 0 by $f(0) = 0$, this shows that $F^n(z, g(z)) \to 0$ and $F^{-n}(z, f(z)) \to 0$ as $n \to \infty$, locally uniformly for $z \in \Omega$.

Finally, to prove (ii) we assume that $\alpha = 0$ and we notice by conclusion (i) that *graph* $g \subseteq W^s_\Omega(0)$ and *graph* $f \subseteq W^u_\Omega(0)$. Let $F^n(z, w) = (z_n, w_n)$ for $n \in \mathbb{Z}$. Assuming that $(z, w) \in W^s_\Omega(0)$ and $n > 0$ we have

$$F^n(z, w) = F(z_{n-1}, w_{n-1}) = (w_{n-1}, g(w_{n-1}) + f(w_{n-1}) - z_{n-1})$$

so, since by (2.1) $\Re f'(z) > 1$ in Ω, we see using Lemma 2.1 that there is $\beta \in \mathbb{C}$ with $\Re\beta \geq 1$ such that

$$\begin{aligned} f(w_n) - z_n &= f(g(w_{n-1}) + f(w_{n-1}) - z_{n-1}) - w_{n-1} \\ &= f(g(w_{n-1})) + \beta(f(w_{n-1}) - z_{n-1}) - w_{n-1} \\ &= \beta(f(w_{n-1}) - z_{n-1})\,. \end{aligned}$$

Thus by induction we get $|f(w_n) - z_n| \geq |f(w) - z|$ and since $(z_n, w_n) \to 0$ and f is continuous at 0 we conclude that $f(w) = z$, so $(z, w) \in graph\ g$.

Similarly, if $(z, w) \in W^u_\Omega(0)$ and $n < 0$ we have $|f(z_n) - w_n| \geq |f(z) - w|$, hence $(z, w) \in graph\ f$ and the theorem is completely proved.

Remark. We now make some remarks on the analyticity of f and g at 0. Without loss of generality we may assume that $\theta = 0$ in condition (C3), so G has the following expansion at 0

$$G(z) = z + \alpha_{j+1}z^{j+1} + \alpha_{j+2}z^{j+2} + \dots\,, \quad \alpha_{j+1} \neq 0\,,$$

where $j \geq 1$ and $\alpha_n \in \mathbb{R}$.

If f and g are analytic at 0 we write

$$\begin{aligned} f(z) &= a_1 z + a_2 z^2 + \dots\,, \\ g(z) &= 2G(z) - f(z) \\ &= (2 - a_1)z - a_2 z^2 - \dots + (2\alpha_{j+1} - a_{j+1})z^{j+1} + \dots\,, \end{aligned}$$

expand $g \circ f$ around 0 and use $g \circ f(z) = z$ to find a_n inductively. Clearly $a_1 = 1$.

We first notice by induction on n that if $n \geq 2$ and $2n - 1 < j + 1$ then $a_n = 0$. Indeed, the coefficient of z^3 in the expansion of $g \circ f$ is $a_3 - 2a_2^2 - a_3 = 0$, so $a_2 = 0$; moreover, if n is such that $2n - 1 < j + 1$ and $a_2 = \dots = a_{n-1} = 0$ then the coefficient of z^{2n-1} in $g \circ f$ is $a_{2n-1} - na_n^2 - a_{2n-1} = 0$, so $a_n = 0$.

There are two cases:

Case 1. $j + 1 = 2l$, $l \geq 1$. By above $f(z) = z + a_{l+1}z^{l+1} + \dots$. The coefficient of z^{2l} in $g \circ f$ is $a_{2l} + 2\alpha_{2l} - a_{2l} = 0$, so $\alpha_{j+1} = 0$, a contradiction. Thus in the case when $j + 1$ is even there are no functions f, g holomorphic around 0 such that $g = f^{-1}$ and $f + g = 2G$. Consequently the functions f, g of Theorem 1.1 do not extend analytically at 0.

Case 2. $j + 1 = 2l - 1$, $l \geq 2$. By above $f(z) = z + a_l z^l + \dots$, and computing the coefficients as indicated before it is not hard to see that $a_l^2 = 2\alpha_{j+1}/l \neq 0$ and the coefficient of z^{2l+n-1} in the expansion of $g \circ f$ yields a formula for a_{l+n}, $n > 0$, of the type $a_l a_{l+n} = E(a_l, \dots, a_{l+n-1}; \alpha_k, k \leq 2l + n - 1)$. So a formal power series at 0 exists for f (and hence for g) such that $g = f^{-1}$ and $f + g = 2G$. Moreover the coefficients a_n are real, since α_n are real.

Remark. The proof of Theorem 1.1 shows that the operator $T : X(\Omega, \alpha) \to X(\Omega, \alpha)$, $Tu = 2G - u^{-1}$ is well defined and that f is a fixed point of T, obtained as the limit of $\{T^n G\}_{n \geq 0}$.

Proof of Theorem 1.2. For $r > 0$ small enough we let

$$\begin{aligned} S(r) &= \{z \in \mathbb{C} : \ |z| < r, \ |\arg z| < \pi/j\} , \\ Q(r) &= S(r)^{-j} = \{w \in \mathbb{C} : \ |w| > r^{-j}, \ \arg w \neq \pi\} , \end{aligned}$$

and we consider the holomorphic function

$$H(w) = \left[G\left(w^{-1/j}\right)\right]^{-j} = w - aj + O(|w|^{-1/j}) , \tag{2.5}$$

defined for $w \in Q(r)$. Here G has the form (1.2). We need the following

Lemma 2.4. *There exist positive constants r_0 and C_0 such that*
(i) $r_0 C_0 > a(j+1)$;
(ii) $|h(z)| < C_0|z|^{j+2}$, $|h'(z)| < C_0|z|^{j+1}$, for z with $|z| \leq r_0$;
(iii) $|H(w) - w + aj| < C_0|w|^{-1/j}$, for $w \in Q(r_0)$;
(iv) $|H(w)|^{\frac{j-1}{j}} > |w|^{\frac{j-1}{j}} - C_0|w|^{-1/j}$, for $w \in Q(r_0)$.

PROOF. By (1.2) and (2.5) one can clearly choose some constants r_0, C_0 so that (i), (ii) and (iii) hold. If $z = w^{-1/j}$ for $w \in Q(r_0)$ then

$$\begin{aligned} |H(w)|^{\frac{j-1}{j}} &= |G(z)|^{-(j-1)} = \left|\frac{1}{z^{j-1}}(1 - (j-1)az^j + O(|z|^{j+1}))\right| \\ &> \frac{1}{|z|^{j-1}} - O(|z|) = |w|^{\frac{j-1}{j}} - O(|w|^{-1/j}) , \end{aligned}$$

so (iv) also holds for r_0, C_0 suitably chosen. □

For $\theta \in \left(-\frac{\pi}{2j}, \frac{\pi}{2j}\right)$ we define

$$C(\theta) = \left[\frac{C_0\left(1 + \frac{C_0}{a(j+1)}\right)}{aj\cos j\theta}\right]^j , \tag{2.6}$$

$$\Omega(\theta) = \left\{z = re^{i\phi} \neq 0 : \ \cos j(\phi - \theta) > \max\left(\frac{C_0}{a(j+1)}r, \ C(\theta)r^j\right)\right\} , \tag{2.7}$$

$$D = \bigcup_{\theta \in (-\frac{\pi}{2j}, \frac{\pi}{2j})} \Omega(\theta) .$$

Clearly $0 \in \partial D$. Using the fact that a domain $\{r < \beta(\phi)\}$ is convex if and only if $\beta^2 + 2(\beta')^2 - \beta\beta'' \geq 0$ it is easy to see that $\Omega(\theta)$ is the intersection of two convex domains, hence it is convex. It follows that D is starlike with respect to 0, hence simply connected. Now $\partial\Omega$ has two tangents at 0, namely the rays $\{z : |\arg z - \theta| = \frac{\pi}{2j}\}$. This implies that $D \subseteq \{z : |\arg z| < \pi/j\}$ and that the rays $\{z : |\arg z| = \pi/j\}$ are tangent to ∂D at 0, so assertion (*i*) of the theorem holds.

Lemma 2.5. *$G \in X(\Omega(\theta), -j\theta)$ and $G(\Omega(\theta)) \supset \overline{\Omega(\theta)} \setminus \{0\}$.*

PROOF. Recall the definition of $X(\Omega, \alpha)$ from section 1, using the conditions (C1), (C2), (C3). By (*i*) and (*ii*) of Lemma 2.4 and by the definition (2.7) of $\Omega(\theta)$ we have that if $z = re^{i\phi} \in \Omega(\theta)$ then $|z| < r_0$ and

$$\begin{aligned} \Re\left[e^{-ij\theta}(G'(z) - 1)\right] &= a(j+1)\Re\left(z^j e^{-ij\theta}\right) + \Re(e^{-ij\theta}h'(z)) \\ &> a(j+1)r^j \cos j(\phi - \theta) - C_0 r^{j+1} > 0 , \end{aligned}$$

so (C2) holds. We now show that $G(\Omega(\theta)) \supset \overline{\Omega(\theta)} \setminus \{0\}$. This is equivalent to showing (since G is conjugated to H by $w = z^{-j}$) that $H(\omega(\theta)) \supset \overline{\omega(\theta)}$, where

$$\omega(\theta) = \Omega(\theta)^{-j} = \left\{ w : \Re\left(we^{ij\theta}\right) > \max\left(\frac{C_0}{a(j+1)} |w|^{\frac{j-1}{j}}, C(\theta) \right) \right\} .$$

It suffices to prove that $H(\partial\omega(\theta)) \cap \overline{\omega(\theta)} = \emptyset$. Let $w \in \partial\omega(\theta)$. Then $|w| \geq C(\theta)$. We have two cases.

Case 1. $\Re\left(we^{ij\theta}\right) = \frac{C_0}{a(j+1)} |w|^{\frac{j-1}{j}}$. Then by (2.5) and by (*iii*) and (*iv*) of Lemma 2.4 we have

$$\begin{aligned} \Re\left(H(w)e^{ij\theta}\right) &< \Re\left(we^{ij\theta}\right) - aj\cos j\theta + C_0|w|^{-1/j} \\ &= \frac{C_0}{a(j+1)}|w|^{\frac{j-1}{j}} - aj\cos j\theta + C_0|w|^{-1/j} \\ &< \frac{C_0}{a(j+1)}|H(w)|^{\frac{j-1}{j}} + \frac{C_0^2}{a(j+1)}|w|^{-1/j} - aj\cos j\theta + C_0|w|^{-1/j} \\ &\leq \frac{C_0}{a(j+1)}|H(w)|^{\frac{j-1}{j}} + C_0\left(1 + \frac{C_0}{a(j+1)}\right)C(\theta)^{-1/j} - aj\cos j\theta \\ &= \frac{C_0}{a(j+1)}|H(w)|^{\frac{j-1}{j}} , \end{aligned}$$

the last equality following from the definition (2.6) of $C(\theta)$. So $H(w)$ is not in $\overline{\omega(\theta)}$.

Case 2. $\Re(we^{ij\theta}) = C(\theta)$. Then by (2.5) and (2.6) we get

$$\begin{aligned}\Re(H(w)e^{ij\theta}) &< \Re(we^{ij\theta}) - aj\cos j\theta + C_0|w|^{-1/j} \\ &\leq C(\theta) - aj\cos j\theta + C_0 C(\theta)^{-1/j} < C(\theta) ,\end{aligned}$$

hence $H(w)$ is not in $\overline{\omega(\theta)}$.

We already proved that $\Omega(\theta)$ is convex, so the proof of the lemma is complete once we notice that $(0, x(\theta)] \subset \Omega(\theta) \cap \mathbb{R}$, for some $x(\theta) > 0$. □

We now return to the proof of the theorem. By Theorem 1.1 there are univalent functions $f_\theta : \Omega(\theta) \to f_\theta(\Omega(\theta))$ and $g_\theta = (f_\theta)^{-1}$ satisfying conclusion (i) of Theorem 1.1. It follows from the proof of Theorem 1.1 and from Lemma 2.3 that for θ, $\theta' \in (-\frac{\pi}{2j}, \frac{\pi}{2j})$ we have $f_\theta = f_{\theta'}$ and $g_\theta = g_{\theta'}$ on $\Omega(\theta) \cap \Omega(\theta') \cap \mathbb{R}$. Since D is simply connected we conclude that there are holomorphic functions f and g defined on D such that $f = f_\theta$ and $g = g_\theta$ on $\Omega(\theta)$ for all $\theta \in (-\frac{\pi}{2j}, \frac{\pi}{2j})$. Now $g(D) = \bigcup_\theta g(\Omega(\theta)) \subseteq \bigcup_\theta \Omega(\theta) = D$ and since $f_\theta \circ g_\theta = id$ and $f_\theta + g_\theta = 2G$ on $\Omega(\theta)$ it follows that $f \circ g = id$ and $f + g = 2G$ on D. Since $(g_\theta)^n \to 0$ on $\Omega(\theta)$ we see that $g^n \to 0$ pointwise on D, and hence locally uniformly on D, by Montel's theorem. Assertion (iii) of the theorem is now proved as in Theorem 1.1.

Remark. The domain D constructed here has maximal aperture at 0, in the sense that G doesn't satisfy condition (C2) on domains with aperture larger than $2\pi/j$. Indeed, if $z = re^{i\phi}$ is such that $|\phi| = \pi/j$ then for any $\alpha \in (-\pi/2, \pi/2)$ we have

$$\Re[e^{i\alpha}(G'(z) - 1)] \leq -a(j+1)r^j \cos\alpha + C_0 r^{j+1} < 0 ,$$

provided that $C_0 r < a(j+1)\cos\alpha$.

Proof of Theorem 1.3. We fix $k \in \{0, ..., j-1\}$ and consider the following rays L_k:

$$L_k = \left\{ r\exp\left(\frac{2k\pi i}{j}\right) : 0 < r < \infty \right\} .$$

We also define, for $\beta \in (0, \pi/j)$,

$$\Omega(k,\beta) = \left\{ re^{i\phi} : 0 < r < \infty, \ \frac{(2k-1)\pi}{j} + \beta < \phi < \frac{2k\pi}{j} + \beta \right\} .$$

We have $L_k \subset \Omega(k,\beta)$ for every β, $\bigcup_{\beta\in(0,\pi/j)} \Omega(k,\beta) = D_k$, where D_k is defined by (1.3). Also, if $\alpha = \pi/2 - j\beta$ then $\alpha \in (-\pi/2, \pi/2)$ and G_j satisfies $\Re[e^{i\alpha}((G_j)'(z) - 1)] > 0$ for $z \in \Omega(k,\beta)$. We claim that $\Omega(k,\beta) \subseteq G_j(\Omega(k,\beta))$, for all β. Indeed, if we write $\partial\Omega(k,\beta) = R_- \cup R_+$, where $\phi_- = (2k-1)\pi/j + \beta$, $\phi_+ = 2k\pi/j + \beta$, $R_- = \{r\exp(i\phi_-)\}$, $R_+ = \{r\exp(i\phi_+)\}$, then $(R_-)^{j+1}$ is a ray lying in the half plane $H_- = \{z : \phi_- - \pi < \arg z < \phi_-\}$, so $G_j(R_-) \subset$

$R_- + (R_-)^{j+1} \subseteq H_-$. Similarly, $G_j(R_+) \subset R_+ + (R_+)^{j+1} \subseteq H_+$, where $H_+ = \{z : \phi_+ < \arg z < \phi_+ + \pi\}$, and the claim is proved.

By Theorem 1.1 there are univalent functions $f_{k,\beta} : \Omega(k,\beta) \to f_{k,\beta}(\Omega(k,\beta))$ and $g_{k,\beta} = (f_{k,\beta})^{-1}$ satisfying conclusion (i) of Theorem 1.1. As the rays L_k are invariant for G_j, it follows from the proof of Theorem 1.1 that they will be invariant for $f_{k,\beta}$ and $g_{k,\beta}$ as well and that all the functions $f_{k,\beta}$ agree on L_k and all the functions $g_{k,\beta}$ agree on L_k. Thus there are holomorphic functions f_k and g_k defined on D_k so that $f_{k,\beta} = f_k$ and $g_{k,\beta} = g_k$ on $\Omega(k,\beta)$, for all β. As in the proof of Theorem 1.2 we have $g_k(D_k) \subseteq D_k$ and $f_k \circ g_k = id$ on D_k. Assuming for a contradiction that $g_k(D_k) = D_k$ (or $f_k(D_k) = D_k$) we get $f_k(D_k) = D_k$ (or $g_k(D_k) = D_k$) and $2G_j(D_k) \subseteq f_k(D_k) + g_k(D_k) = D_k$, so $G_j(D_k) \subseteq D_k$, which is false.

Finally, as $G_j(\exp(2k\pi i/j)z) = \exp(2k\pi i/j)G_j(z)$, $z \in D_0$, it follows easily from the uniqueness part (conclusion (ii)) of Theorem 1.1 that

$$f_k(\exp(2k\pi i/j)z) = \exp(2k\pi i/j)f_0(z)\ ,$$

$$g_k(\exp(2k\pi i/j)z) = \exp(2k\pi i/j)g_0(z)\ ,$$

for $z \in D_0$. The rest of the assertions of Theorem 1.3 follow directly from Theorem 1.1.

Remark. In [F], Theorem 4, the author considers the real Henon map $F(x,y) = (y, -x + 2G(y))$, obtained for $G(x) = x + x^2$. He shows, via real methods, that the functions $g(x)$ and $f(x)$, defined for $x \geq 0$ and whose graphs give the stable/unstable manifolds of F in $\{(x,y) : x \geq 0\}$, are real analytic on $(0,\infty)$ and the radii of convergence of their Taylor series at $x_0 \in (0,\infty)$ are x_0; thus f and g are analytic on the right half plane $\{\Re z > 0\}$. Theorem 1.3 shows that these functions f and g are actually analytic on the slit plane $\mathbb{C} \setminus \{z \leq 0\}$.

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