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INVARIANT DIFFERENTIAL FORMS ON THE FIRST JET PROLONGATION OF THE COTANGENT BUNDLE

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ABSTRACT. The structure of the differential forms on $J^1(T^*M)$ which are invariant under the natural representation of the gauge algebra of the trivial principal bundle $\pi\colon M \times U(1) \to M$ and the structure of the horizontal forms on $J^1(T^*M)$ which are invariant under the Lie algebra of all infinitesimal automorphisms of $\pi\colon M \times U(1) \to M$ are determined.

1. INTRODUCTION

The main goal of this paper is to determine the structure of gauge forms on $J^1(T^*M)$; that is, the forms which are invariant under the natural representation of the gauge algebra of the trivial principal bundle $\pi\colon M \times U(1) \to M$ on $J^1(T^*M)$. The variational problems defined by such forms are also studied and the structure of the horizontal forms which are not only gauge invariant but also invariant under the Lie algebra $\mathrm{aut}(M \times U(1))$ of all infinitesimal automorphisms is determined as well. Unlike gauge forms, $\mathrm{aut}(M \times U(1))$-invariant horizontal forms do not depend on arbitrary functions and do not provide interesting variational problems. In fact, such forms are isomorphic to $\mathbb{R}[\kappa^* \Omega_2]$, where Ω_2 stands for the canonical 2-form on $\bigwedge^2 T^*(M)$, and $\kappa\colon J^1(T^*M) \to \bigwedge^2 T^*(M)$ is the mapping $\kappa(j^1_x\omega) = d_x\omega$.

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The first motivation for these results is the geometric formulation of Utiyama's theorem ([22]) characterizing gauge invariant Lagrangians, which is nowadays formulated as follows. Let $\pi\colon P \to M$ be an arbitrary principal G-bundle. If we consider the induced action of G on TP, the quotient vector bundle $Q = T(P)/G \to M$ exists, $\Gamma(M, Q)$ is identified to G-invariant vector fields on P, and we have an exact sequence ([1, Theorem 1], [5, §4]), $0 \to L(P) \to Q \to T(M) \to 0$, $L(P)$ being the adjoint bundle, whose splittings are the connections on P so that connections can be identified with the sections of an affine bundle $p\colon \mathcal{C}(P) \to M$ modelled over $T^*(M) \otimes L(P)$ ([5, Definition 4.5]). A Lagrangian density $\mathcal{L}\mathrm{d}q_1 \wedge \ldots \wedge \mathrm{d}q_m$, $\mathcal{L}\colon J^1(\mathcal{C}(P)) \to \mathbb{R}$, is gauge invariant if and only if $\mathcal{L}$ factors by the curvature mapping $\kappa\colon J^1(\mathcal{C}(P)) \to \bigwedge^2 T^*(M) \otimes L(P)$ through a function $\bar{\mathcal{L}}\colon \bigwedge^2 T^*(M) \otimes L(P) \to \mathbb{R}$, which must be invariant under the natural representation of the gauge algebra on the curvature bundle (*cf.* [2], [3], [5], [8]). This result was the starting point for the study of gauge invariance and gauge-natural objects in differential geometry ([5], [6], [7], [21]) and it lies on the basis of the geometric formulations of gauge theories (*e.g.*, see [5], [7], [13], [17]). Because of the importance of Utiyama's result it seems natural to try classifying differential forms of arbitrary degree and not necessarily horizontal (as Lagrangian densities) under the representation of the gauge algebra on the first jet prolongation of the bundle of connections of a principal bundle. In the abelian case, $G = U(1)$, which corresponds to the classical electromagnetism, this problem poses a general question on the geometry of differentiable manifolds, due to the elementary fact that the bundle of connections of the bundle $\pi\colon M \times U(1) \to M$ can be naturally identified with the cotangent bundle of M, thus inducing a Lie algebra representation of $\mathrm{aut}(M \times U(1))$ into the vector fields of $J^1(T^*M)$. In [11] we solved what could be called the 'geometric part' of this problem; *i.e.*, we classified gauge invariant differential forms on $J^0(T^*M) = T^*M$, proving that they are determined by the symplectic form of the cotangent bundle. In the present work we solve the corresponding problem for the forms on the first prolongation bundle thus answering to the original motivation of the problem. As could be expected from their physical meaning, the algebra of gauge invariant forms on $J^1(T^*M)$ is larger than that of T^*M, and, in fact, there are plenty of such forms which are related to the canonical contact differential system on the jet bundle and whose module structure is determined by the curvature mapping. Accordingly, the proofs in the present case involve different ideas and techniques to those of the geometric case, which have been strongly influenced by the approach of [5] in what concerns gauge invariance.

2. The fundamental representation

In this section we define the natural representation of the Lie algebra of all infinitesimal automorphisms of the principal bundle $\pi\colon M \times U(1) \to M$ with structure group $U(1) = \{z \in \mathbb{C}\colon |z| = 1\}$ into the vector fields of $J^1(T^*M)$. If t stands for the angle in $U(1)$, and A is the standard basis of the Lie algebra $\mathfrak{u}(1)$ (*i.e.*, $A \in \mathfrak{u}(1)$ is the element determined by $\mathbb{R} \to U(1)$, $t \mapsto \exp(it)$), then each connection form ω_Γ can be identified with the ordinary one-form ω on M defined by the formula $\omega_\Gamma = (\mathrm{d}t + \pi^*\omega) \otimes A$; in other words, the bundle of connections on $M \times U(1)$ can be identified with the cotangent bundle, *i.e.*, $C(M \times U(1)) \cong T^*(M)$.

2.1. **Infinitesimal automorphisms.** Let us denote by $\mathrm{Aut}(P)$ (respectively by $\mathrm{Gau}(P)$) the group of all automorphisms of a principal G-bundle $\pi\colon P \to M$ (respectively the group of gauge transformations of P). A vector field X on P is G-invariant if and only if its flow Φ_t satisfies $\Phi_t \in \mathrm{Aut}(P)$, $\forall t \in \mathbb{R}$. Because of this we consider the Lie algebra of G-invariant vector fields on P as the "Lie algebra" of $\mathrm{Aut}(P)$ (*cf.* [10, III. §35]), [3, 3.2.9-3.2.17]). Accordingly we write $\mathrm{aut}(P) = \Gamma(M, Q)$. Similarly, we set $\mathrm{gau}(P) = \Gamma(M, L(P))$ and we think of $\mathrm{gau}(P)$ as being the "Lie algebra" of the gauge group.

On $P = M \times U(1)$, the angle in $U(1)$ defines a global one-form $\mathrm{d}t$ and the fundamental vector field A^* associated with the standard basis $A \in \mathfrak{u}(1)$ is the unique π-vertical vector field on $M \times U(1)$ such that $\mathrm{d}t(A^*) = 1$. Note that A^* is $U(1)$-invariant (and hence $A^* \in \mathrm{gau}(M \times U(1))$) since $U(1)$ is abelian. Let $(N; q_1, \dots, q_m)$, $m = \dim M$, be an open coordinate domain in M. Then, a vector field $X \in \mathfrak{X}(N \times U(1))$ is $U(1)$-invariant if and only if it can be written as (*e.g.*, see [11, III.B, Proposition 1–(4)]),

$$X = \sum_{i=1}^{m} f_i(q_1, \dots, q_m) \frac{\partial}{\partial q_i} + g(q_1, \dots, q_m) A^*, \ f_i, g \in C^\infty(N). \tag{2.1}$$

In particular, X belongs to $\mathrm{gau}(N \times U(1))$ if and only if,

$$X = g(q_1, \dots, q_m) A^*, \ g \in C^\infty(N). \tag{2.2}$$

Remark. Decomposition (2.1) has a global meaning. As the bundle is trivial, every $X' \in \mathfrak{X}(M)$ defines a vector field on $M \times U(1)$ and every $X \in \mathrm{aut}(M \times U(1))$ is uniquely decomposed as $X = X' + gA^*$, $X' \in \mathfrak{X}(M)$ being the π-projection of X and $g \in C^\infty(M)$. Hence $\mathrm{aut}(M \times U(1))$ can be identified with the sections of the vector bundle $T(M) \oplus (M \times \mathbb{R})$. Recalling $\mathrm{gau}(M \times U(1))$ is abelian, it follows that $\mathrm{aut}(M \times U(1))$ is the semidirect product of $\mathfrak{X}(M)$ and $\mathrm{gau}(M \times U(1))$ via the adjoint representation; *i.e.*, $\rho\colon \mathfrak{X}(M) \to \mathrm{Der}(\mathrm{gau}(M \times U(1)))$, $\rho(X')(gA^*) = [X', gA^*] = (X'g)A^*$.

2.2. **The action of** $\mathrm{Aut}(M \times U(1))$ **on** $T^*(M)$. Each automorphism Φ of a principal bundle $\pi\colon P \to M$, acts in a natural way on the connections of P (*cf.* [12, II §6]) by pulling-back connection forms, *i.e.*, $(\Phi^{-1})^* \omega_\Gamma = \omega_{\Phi \cdot \Gamma}$, and this action induces an action of $\mathrm{Aut}(P)$ on the bundle of connections $\mathcal{C}(P)$, which we denote by $\tilde{\Phi}\colon \mathcal{C}(P) \to \mathcal{C}(P)$, in such a way that if $p\colon \mathcal{C}(P) \to M$ stands for the bundle projection, then we have $p \circ \tilde{\Phi} = \varphi \circ p$, where $\varphi\colon M \to M$ is the diffeomorphism induced by Φ on the ground manifold. If $X \in \mathrm{aut}(M \times U(1))$, then its flow Φ_t is a family of automorphisms of $M \times U(1)$ and $\tilde{\Phi}_t$ induces a flow on $\mathcal{C}(M \times U(1)) \cong T^*(M)$. Let us denote by $\tilde{X} \in \mathfrak{X}(T^*M)$ the vector field generated by $\tilde{X}$. The local expression of $\tilde{X}$ and the basic properties of the mapping $\mathrm{aut}(M \times U(1)) \to \mathfrak{X}(T^*M)$, $X \mapsto \tilde{X}$, are as follows (for the details see [11]):

1. If X is given by the formula (2.1), then

$$\tilde{X} = \sum_{i=1}^{m} f_i \frac{\partial}{\partial q_i} - \sum_{i=1}^{m} \left(\frac{\partial g}{\partial q_i} + \sum_{h=1}^{m} \frac{\partial f_h}{\partial q_i} p_h \right) \frac{\partial}{\partial p_i}, \tag{2.3}$$

   where (q_i, p_i), $1 \le i \le m$, is the coordinate system induced by $(N; q_1, \dots, q_m)$ in $p^{-1}(N) \subseteq T^*(M)$; *i.e.*, $w = \sum_{i=1}^m p_i(w) \mathrm{d}_x q_i$, $\forall w \in T_x^*(M)$, $x \in N$.
2. For every $X \in \mathrm{aut}(M \times U(1))$, $\tilde{X}$ is p-projectable and its projection is X', the projection of X onto M.
3. $X \mapsto \tilde{X}$ is a Lie algebra homomorphism.

2.3. **Infinitesimal contact transformations.** Given an arbitrary fibred manifold $p\colon E \to M$, let us denote by $J^1(E)$ the 1-jet bundle of local sections of p, with bundle projections $p_1 : J^1(E) \to M$, $p_1(j_x^1 s) = x$; $p_{10}\colon J^1(E) \to J^0(E) = E$, $p_{10}(j_x^1 s) = s(x)$. The manifold $J^1(E)$ is endowed with a canonical Pfaffian differential system $\mathcal{S}$, locally generated by the contact one-forms $\theta_j = \mathrm{d}y_j - \sum_{i=1}^m y_i^j \mathrm{d}x_i$, $1 \le j \le n$, where $m = \dim E - \dim M$, and $(x_i, y_j; y_i^j)$, $1 \le i \le m$, $1 \le j \le n$, are the coordinates induced on $J^1(E)$ by a fibred coordinate system (x_i, y_j) of $p\colon E \to M$; *i.e.*, $y_i^j(j_x^1 s) = (\partial(y_j \circ s)/\partial x_i)(x)$ ([4], [9], [14], [15]). Given $X \in \mathfrak{X}(E)$, there exists a unique vector field $X_{(1)} \in \mathfrak{X}(J^1(E))$ such that 1) $X_{(1)}$ is p_{10}-projectable onto X, and 2) $X_{(1)}$ leaves invariant the contact system $\mathcal{S}$; *i.e.*,

$L_{X_{(1)}}\mathcal{S} \subseteq \mathcal{S}$. The vector field $X_{(1)}$ is called the *infinitesimal contact transformation* attached to X, and the mapping $\mathfrak{X}(E) \to \mathfrak{X}(J^1(E))$, $X \mapsto X_{(1)}$, is a Lie algebra injection. If X is p-projectable, $X_{(1)}$ coincides with the infinitesimal generator of the usual 1-jet prolongation of the flow of X (*cf.* [15], [18], [19]).

2.4. **Representing** $\operatorname{Aut}(M \times U(1))$ **into** $\mathfrak{X}(J^1(T^*M))$. By composing the representation $\operatorname{aut}(M \times U(1)) \to \mathfrak{X}(T^*M)$ given in §2.2, with the 1-jet prolongation described in §2.3, we obtain the natural representation: $\operatorname{aut}(M \times U(1)) \to \mathfrak{X}(T^*(M)) \to \mathfrak{X}(J^1(T^*M))$, $X \mapsto \tilde{X} \mapsto \tilde{X}_{(1)}$. Formula (2.3) and the general formulas for jet prolongation (*e.g.*, see [19]) yield the following local expression of the natural representation:

$$\text{(2.4)}\quad \tilde{X}_{(1)} = \sum_{i=1}^{m} f_i \frac{\partial}{\partial q_i} - \sum_{i=1}^{m}\left(\frac{\partial g}{\partial q_i} + \sum_{h=1}^{m}\frac{\partial f_h}{\partial q_i} p_h\right)\frac{\partial}{\partial p_i} - \sum_{i,j=1}^{m}\left\{\frac{\partial^2 g}{\partial q_i \partial q_j} + \sum_{h=1}^{m}\left(\frac{\partial^2 f_h}{\partial q_i \partial q_j} p_h + \frac{\partial f_h}{\partial q_i} p_j^h + \frac{\partial f_h}{\partial q_j} p_h^i\right)\right\}\frac{\partial}{\partial p_j^i}.$$

In particular, for $f_i = 0$ we find the 1-prolongation of the gauge representation:

$$\text{(2.5)}\qquad (gA^*\tilde{)}_{(1)} = -\sum_{i=1}^{m}\frac{\partial g}{\partial q_i}\frac{\partial}{\partial p_i} - \sum_{i,j=1}^{m}\frac{\partial^2 g}{\partial q_i \partial q_j}\frac{\partial}{\partial p_j^i}.$$

3. $\mathfrak{g}_{T^*M}$-INVARIANCE

In this section to each manifold M we attach an abelian Lie subalgebra $\mathfrak{g}_{T^*M} \subset \mathfrak{X}(T^*M)$ which generalizes the gauge algebra representation and we study invariance with respect to such a subalgebra as a first step in studying gauge invariance.

3.1. **The gauge algebra of** T^*M. We denote by $\mathfrak{g}_{T^*M}$ and we call it the *gauge algebra* of T^*M, the abelian Lie algebra of all p-vertical vector fields on T^*M which are invariant under translations of the fibres. More precisely: every $w \in T_x^*(M)$ gives rise to a translation $\tau_w \colon T_x^*(M) \to T_x^*(M)$, $\tau_w(w') = w + w'$. A p-vertical vector field $X \in \mathfrak{X}(T^*M)$ belongs to $\mathfrak{g}_{T^*M}$ if and only if its restriction to each fibre is invariant under translations; *i.e.*, $\tau_w \cdot (X_{|p^{-1}(x)}) = X_{|p^{-1}(x)}$, $\forall x \in M$, $\forall w \in T_x^*M$.

Let $(N; q_1, \dots, q_m)$ be an open coordinate domain of M and let (q_i, p_i), $1 \le i \le m$, be the coordinate system induced in $p^{-1}(N)$ (see §2.2-1.). Then, it is not difficult to prove that a vector field $X \in \mathfrak{X}(T^*M)$ belongs to $\mathfrak{g}_{T^*M}$ if and only if, locally, it can be written as

$$(3.1) \qquad X = \sum_{i=1}^{r} g_i(q_1, ..., q_m) \frac{\partial}{\partial p_i}, \; g_i \in C^\infty(N).$$

Hence the mapping which associates $\mathfrak{g}_{p^{-1}(U)}$ to each open subset $U \subseteq M$ is a locally free sheaf of C^∞_M-modules of rank m. A vector field $X \in \mathfrak{g}_{T^*M}$ belongs, locally, to the image of the gauge representation (see (2.3) for $f_i = 0$) if and only if there exists $g \in C^\infty(N)$ such that $g_i = -\partial g/\partial q_i$, $1 \leq i \leq m$, thus explaining our terminology. In fact, we have

Proposition 3.1. *Let $\Omega_1 = \sum_{i=1}^m p_i \mathrm{d}q_i$ be the Liouville form on the cotangent bundle $p\colon T^*M \to M$. Then*

1. *A p-vertical vector field $X \in \mathfrak{X}(T^*M)$ belongs to $\mathfrak{g}_{T^*M}$ if and only if $L_X\Omega_1$ is p-projectable.*
2. *The mapping $\iota\colon \mathfrak{g}_{T^*M} \to \Omega(M)$, $\iota(X) = i_X \mathrm{d}\Omega_1$, is an isomorphism of $C^\infty(M)$-modules.*

Denoting by $Z^1(M)$, $B^1(M)$ the closed and exact 1-forms on M, respectively, we have $\mathrm{gau}(M \times U(1)) = \iota^{-1}(B^1(M))$. *Moreover, if we define* $\mathrm{gau}_{loc}(M \times U(1)) = \iota^{-1}(Z^1(M))$, *then* $\mathrm{gau}_{loc}(M \times U(1)) / \mathrm{gau}(M \times U(1)) \cong H^1(M; \mathbb{R})$.

3.2. **$\mathfrak{g}_{T^*M}$-invariance and contact forms.** A differential n-form on $J^1(T^*M)$, Ω_n, is said to be $\mathfrak{g}_{T^*M}$*-invariant* if for every $X \in \mathfrak{g}_{T^*M}$ we have $L_{X_{(1)}}\Omega_n = 0$, where $X_{(1)}$ stands for the 1-jet prolongation of X (*cf.* §2.3). Let us denote by $\Theta^n(T^*M)$ the set of all $\mathfrak{g}_{T^*M}$-invariant n-forms on $J^1(T^*M)$. It is clear that $\Theta(T^*M) = \bigoplus_{n=0}^\infty \Theta^n(T^*M)$ is a graded $C^\infty(M)$-subalgebra of $\Omega^\cdot(T^*M) = \bigoplus_{n=0}^\infty \Omega^n(T^*M)$. A $\mathfrak{g}_{T^*M}$-invariant form $\Omega_n \in \Theta^n(T^*M)$ is said to be of type $(s, t, n-s-t)$ at a point $\bar{w} \in J^1(T^*M)$ if either $\Omega_n(\bar{w}) = 0$ or $\Omega_n(\bar{w}) \neq 0$ and then $n - s - t$ should be even, say $n - s - t = 2u$, and the two conditions below hold true:

1. There exists a linearly independent system of tangent vectors $X_1^0, ..., X_{s+u}^0 \in T_x(M)$ such that for every local section ω of p with $j_x^1\omega = \bar{w}$ and every system of tangent vectors $X_0, X_1, ..., X_{s+u} \in T_x(M)$ we have

$$i_{(j^1\omega)_* X_1^0} \cdots i_{(j^1\omega)_* X_{s+u}^0} \Omega_n \neq 0, \quad i_{(j^1\omega)_* X_0} i_{(j^1\omega)_* X_1} \cdots i_{(j^1\omega)_* X_{s+u}} \Omega_n = 0.$$

2. There exists a linearly independent system of p_{10}-vertical tangent vectors $Y_1^0, ..., Y_u^0 \in T_{\bar{w}}(J^1T^*M)$ such that for every system of p_{10}-vertical tangent vectors $Y_0, Y_1, ..., Y_u \in T_{\bar{w}}(J^1(T^*M))$ we have

$$i_{Y_1^0} \cdots i_{Y_u^0} \Omega_n \neq 0, \quad i_{Y_0} i_{Y_1} \cdots i_{Y_u} \Omega_n = 0.$$

Note that the above two conditions completely determine the integers s,t and u. We denote by $\Theta^{s,t,2u}(V) \subset \Theta^{s+t+2u}(V)$ the subspace of the forms of type $(s,t,2u)$.

Theorem 3.2. *Let $(N; q_i, p_i)$, $1 \le i \le m$ be as in §2.2–1., and let p^i_j, $1 \le i \le m$, $1 \le j \le m$, be the coordinate system induced on $J^1(p^{-1}N)$ (cf. §2.3). We set $\theta_i = \mathrm{d}p_i - \sum_{j=1}^m p^i_j \mathrm{d}q_j$, $1 \le i \le m$. Then, the forms $\varphi_{hij} = \mathrm{d}q_{h_1} \wedge \dots \wedge \mathrm{d}q_{h_s} \wedge \theta_{i_1} \wedge \dots \wedge \theta_{i_t} \wedge \mathrm{d}\theta_{j_1} \wedge \dots \wedge \mathrm{d}\theta_{j_u}$, with $h = (h_1, \dots, h_s)$, $1 \le h_1 < \dots < h_s \le m$; $i = (i_1, \dots, i_t)$, $1 \le i_1 < \dots < i_t \le m$; $j = (j_1, \dots, j_u)$, $1 \le j_1 \le \dots \le j_u \le m$; $s + u \le m$, are a basis for $\Theta^{s,t,2u}(p^{-1}N)$ as a $C^\infty(N)$-module. Furthermore, $\Theta^n(T^*M) = \bigoplus_{s+t+2u=n} \Theta^{s,t,2u}(T^*M)$.*

PROOF. If the local expression of a vector field $X \in \mathfrak{g}_{T^*M}$ is as in (3.1), then

$$X_{(1)} = \sum_{i=1}^m g_i \frac{\partial}{\partial p_i} + \sum_{i=1}^m \sum_{j=1}^m \frac{\partial g_i}{\partial q_j} \frac{\partial}{\partial p^i_j}. \tag{3.2}$$

Hence $L_{X_{(1)}} \mathrm{d}q_h = 0$, $L_{X_{(1)}} \theta_i = 0$, $L_{X_{(1)}} \mathrm{d}\theta_i = 0$, and the $C^\infty(N)$-module spanned by all forms in the statement is contained in $\Theta\left(p^{-1}N\right)$. Moreover, it follows from the very definitions that $\varphi_{hij} \in \Theta^{s,t,2u}(p^{-1}N)$. We shall prove that the above forms are linearly independent. Let us denote by I the set of indices (h,i,j) satisfying the conditions of the statement. Let us fix an index $(a,b,c) \in I$. Let $a'_1 < \dots < a'_{m-s}$ be the complementary set of $\{a_1, \dots, a_s\}$ in $\{1, \dots, m\}$. As $s + u \le m$ we have $u \le m - s$. We set $a' = (a'_1, \dots, a'_{m-s})$, $a'' = (a''_1, \dots, a''_u)$, so that $\{a''\} \subseteq \{a'\}$. (For any ordered system $\lambda = (\lambda_1, \dots, \lambda_k)$ we denote by $\{\lambda\}$ the underlying set; *i.e.*, $\{\lambda\} = \{\lambda_1, \dots, \lambda_k\}$.) Assume for some functions $f_{hij} \in C^\infty(N)$ we have $\sum_{(h,i,j)\in I} f_{hij}\varphi_{hij} = 0$. By taking interior products successively with $\partial/\partial p^{c_1}_{a'_1}, \dots, \partial/\partial p^{c_u}_{a'_u}$ in the above equation, we obtain $\sum_{(h,i,c)\in I} f_{hic}\varphi_{hia''} = 0$. Again taking interior products with $\partial/\partial p_1, \dots, \partial/\partial p_t$, we have $\sum_{(h,b,c)\in I} f_{hbc}\, \mathrm{d}q_{h_1} \wedge \dots \wedge \mathrm{d}q_{h_s} \wedge \mathrm{d}q_{a'_1} \wedge \dots \wedge \mathrm{d}q_{a'_u} = 0$. Hence, $f_{hbc} = 0$ whenever $\{h\} \cap \{a''\} = \emptyset$. In particular, $f_{abc} = 0$. Accordingly, we only need to prove that the forms φ_{hij}, $s + t + 2u = n$, span $\Theta^n(p^{-1}N)$. Let $\Omega_n = \sum_{\sigma+t+u=n} f^j_{hik}\, \mathrm{d}q_h \wedge \theta_i \wedge \mathrm{d}p^j_k$ be the local expression of an n-form in the basis $(\mathrm{d}q_\alpha, \theta_\beta, dp^\alpha_\beta)$, with $\mathrm{d}q_h = \mathrm{d}q_{h_1} \wedge \dots \wedge \mathrm{d}q_{h_\sigma}$, $h = (h_1 < \dots < h_\sigma)$; $\theta_i = \theta_{i_1} \wedge \dots \wedge \theta_{i_t}$, $i = (i_1 < \dots < i_t)$; $\mathrm{d}p^j_k = \mathrm{d}p^{j_1}_{k_1} \wedge \dots \wedge \mathrm{d}p^{j_u}_{k_u}$, $(j,k) = \left((j_1,k_1) < \dots < (j_u,k_u)\right)$, lexicographically. By taking X in (3.1) successively equal to $\partial/\partial p_l$, $q_a(\partial/\partial p_b)$ and $q_a q_b(\partial/\partial p_c)$, from (3.2) we deduce that $X_{(1)}$ is equal to $\partial/\partial p_l$, $q_a(\partial/\partial p_b) + \partial/\partial p^b_a$ and $q_a q_b(\partial/\partial p_c) + q_a(\partial/\partial p^c_b) + q_b(\partial/\partial p^c_a)$, respectively. The invariance condition $L_{X_{(1)}}\Omega_n = 0$ yields $\partial f^j_{hik}/\partial p_l = \partial f^j_{hik}/\partial p^b_a = 0$ (hence $f^j_{hik} \in C^\infty(N)$) and $f^j_{hik} = 0$ for $a \notin \{h\}$, $a \in \{k\}$. Hence if $f^j_{hik} \ne 0$, we can write $\{h\} = \{k\} \cup \{l\}$, $\{k\} \cap \{l\} = \emptyset$, $l = (l_1 < \dots < l_s)$, $s = \sigma - u$, and then, $\Omega_n = \sum_{s+t+2u=n} \pm f^j_{hik} \mathrm{d}q_{l_1} \wedge \dots \wedge \mathrm{d}q_{l_s} \wedge \theta_i \wedge \left(\sum_{k_1} \mathrm{d}q_{k_1} \wedge \mathrm{d}p^{j_1}_{k_1}\right) \wedge \dots \wedge \left(\sum_{k_u} \mathrm{d}q_{k_u} \wedge \mathrm{d}p^{j_u}_{k_u}\right)$. □

Remark. A form Ω_n on a fibred manifold $p\colon E \to M$ is *p-horizontal* if $i_X \Omega_n = 0$ for every p-vertical $X \in \mathfrak{X}(E)$. Theorem 3.2 implies that $\mathfrak{g}_{T^*M}$-invariant forms are spanned by contact forms, their exterior differentials and p_1-horizontal forms.

4. gau $(M \times U(1))$-INVARIANCE

Let $p_n\colon \bigwedge^n T^*(M) \to M$ be the bundle projection and let Ω_n be the *canonical n-form* on $\bigwedge^n T^*(M)$; *i.e.*, $\Omega_n(X_1, \dots, X_n) = w_n((p_n)_* X_1, \dots, (p_n)_* X_n)$, where $X_1, \dots, X_n \in T_{w_n}(\bigwedge^n T^* M)$, $w_n \in \bigwedge^n T^*_x(M)$. Note that Ω_1 is none other than Liouville's form on the cotangent bundle as introduced in Proposition 3.1. In the abelian case the geometric formulation of Utiyama's theorem takes the following form (*cf.* [2], [3], [5], [8]):

Proposition 4.1. *There exists an exact sequence of vector bundles over M,*

$$0 \to J^2(M, \mathbb{R})_0 \xrightarrow{\phi} J^1(T^*M) \xrightarrow{\kappa} \bigwedge^2 T^*(M) \to 0,$$

given by $\phi(j^2_x f) = j^1_x(\mathrm{d}f)$, $\kappa(j^1_x \omega) = \mathrm{d}_x \omega$, and we have

1. $\phi_* T_{j^2_x f}\left(J^2(M, \mathbb{R})_0\right) = \left\{\tilde{X}_{(1)}\left(j^1_x(\mathrm{d}f)\right) : X \in \mathrm{gau}(M \times U(1))\right\}$.
2. *A function $f \in C^\infty\left(J^1(T^*M)\right)$ satisfies $\tilde{X}_{(1)} f = 0, \forall X \in \mathrm{gau}(M \times U(1))$ if and only if a function $g \in C^\infty(\bigwedge^2 T^*M)$ exists such that $f = g \circ \kappa$.*
3. *Let p_{ij}, $1 \le i < j \le m$, be the coordinate system induced by $(q_1, \dots, q_m)$ on $\bigwedge^2 T^*M$;* i.e., *for $\omega_2 \in \bigwedge^2 T^*_x(M)$, $\omega_2 = \sum_{i<j} p_{ij}(\omega_2) \mathrm{d}_x q_i \wedge \mathrm{d}_x q_j$. Then, with the same notations as above we have $\kappa^*(p_{ij}) = p^j_i - p^i_j$.*

PROOF. Let (y_α), $|\alpha| \le 2$, be the coordinate system induced on $J^2(M, \mathbb{R})$ by $(q_1, \dots, q_m)$ of M; *i.e.*, $y_\alpha(j^2_x f) = \left(\partial^{|\alpha|} f / \partial q^\alpha\right)(x)$. It follows from the definition of ϕ that $\phi^*(p^i_\alpha) = y_{\alpha+(i)}$, $|\alpha| \le 1$, $1 \le i \le m$. Hence $\phi_*(\partial/\partial y_i) = \partial/\partial p_i$, $\phi_*(\partial/\partial y_{(ij)}) = \partial/\partial p^i_j + \partial/\partial p^j_i$, thus proving 1., taking into account the local expression of the gauge representation in (2.5). As the fibres of κ are connected, part 2. follows from 1. Finally, 3. follows from the local expression of the exterior differential. □

An n-form Ω_n on $J^1(T^*M)$ is said to be gau$(M \times U(1))$-*invariant* (or *gauge invariant*) if $L_{\tilde{X}_{(1)}} \Omega_n = 0$ for all $X \in \mathrm{gau}(M \times U(1))$. As gau$(M \times U(1)) \subset$

$\mathfrak{g}_{T^*M}$, it follows from the very definitions that $\mathfrak{g}_{T^*M}$-invariant forms are gauge invariant. Moreover, it follows from 2. of the previous proposition that every differential form on $\bigwedge^2 T^*(M)$ is gauge invariant. In fact, as a consequence of the results in the next section, we shall see that $\Theta(T^*M)$ and $\kappa^*\Omega^{\cdot}(\bigwedge^2 T^*M)$ span gauge forms. Also note that $\mathrm{gau}(M \times U(1))$-invariance and $\mathrm{gau}_{loc}(M \times U(1))$-invariance are equivalent notions.

5. Local structure of gauge forms

Let us denote by $\mathcal{G}^n_M$ the sheaf of gauge n-forms on M. We set $\mathcal{G}_M = \bigoplus_{n=0}^{\infty} \mathcal{G}^n_M$.

Theorem 5.1. *With the above notations we have*

1. *For $1 \leq n \leq m = \dim M$, $\mathcal{G}^n_M$ is locally generated over $\kappa^* C^\infty_{\bigwedge^2 T^*M}$ by the following forms:*

$$\mathrm{d}q_{h_1} \wedge \ldots \wedge \mathrm{d}q_{h_r} \wedge \theta_{i_1} \wedge \ldots \wedge \theta_{i_s} \wedge \mathrm{d}\theta_{j_1} \wedge \ldots \wedge \mathrm{d}\theta_{j_t} \wedge \mathrm{d}p_{k_1 l_1} \wedge \ldots \wedge \mathrm{d}p_{k_u l_u} \tag{5.1}$$

with $1 \leq h_1 < \ldots < h_r \leq m$, $1 \leq i_1 < \ldots < i_s \leq m$, $1 \leq j_1 \leq \ldots \leq j_t \leq m$, $1 \leq k_\alpha < l_\alpha \leq m$, $1 \leq \alpha \leq u$, $r + s + 2t + u = n$.

2. *For $n > m$, $\mathcal{G}^n_M$ is locally generated over $\kappa^* C^\infty_{\bigwedge^2 T^*M}$ by the forms in formula (5.1) together with the following forms:*

$$\mathrm{d}q_1 \wedge \ldots \wedge \mathrm{d}q_m \wedge \mathrm{d}p_{i_1} \wedge \ldots \wedge \mathrm{d}p_{i_s} \wedge \mathrm{d}p^{j_1}_{k_1} \wedge \ldots \wedge \mathrm{d}p^{j_t}_{k_t}, \; s + t = n - m. \tag{5.2}$$

For every $n > 0$, $\mathcal{G}^n_M$ is a locally free sheaf of $\kappa^* C^\infty_{\bigwedge^2 T^*M}$-modules of finite rank.

Proof. Let us consider the local expression of an n-form, $1 \leq n \leq m$,

$$\begin{aligned}
\Omega_n = {} & \frac{1}{n!} f_{i_1 \ldots i_n} \mathrm{d}q^{i_1} \wedge \ldots \wedge \mathrm{d}q^{i_n} + \frac{1}{n!} f^{j_1 \ldots j_n} \theta_{j_1} \wedge \ldots \wedge \theta_{j_n} + \\
& \frac{1}{n!} f^{h_1 \ldots h_n}_{k_1 \ldots k_n} \mathrm{d}p^{k_1}_{h_1} \wedge \ldots \wedge \mathrm{d}p^{k_n}_{h_n} + \\
& \sum_{s+t=n} \frac{1}{s!t!} f^{h_1 \ldots h_t, j_1 \ldots j_s}_{k_1 \ldots k_t} \theta_{j_1} \wedge \ldots \wedge \theta_{j_s} \wedge \mathrm{d}p^{k_1}_{h_1} \wedge \ldots \wedge \mathrm{d}p^{k_t}_{h_t} + \\
& \sum_{r+s=n} \frac{1}{r!s!} f^{j_1 \ldots j_s}_{i_1 \ldots i_r} \mathrm{d}q^{i_1} \wedge \ldots \wedge \mathrm{d}q^{i_r} \wedge \theta_{j_1} \wedge \ldots \wedge \theta_{j_s} + \\
& \sum_{r+t=n} \frac{1}{r!t!} f^{h_1 \ldots h_t}_{k_1 \ldots k_t, i_1 \ldots i_r} \mathrm{d}q^{i_1} \wedge \ldots \wedge \mathrm{d}q^{i_r} \wedge \mathrm{d}p^{k_1}_{h_1} \wedge \ldots \wedge \mathrm{d}p^{k_t}_{h_t} + \\
& \sum_{r+s+t=n} \frac{1}{r!s!t!} f^{j_1 \ldots j_s, h_1 \ldots h_t}_{i_1 \ldots i_r, k_1 \ldots k_t} \mathrm{d}q^{i_1} \wedge \ldots \wedge \mathrm{d}q^{i_r} \wedge \theta_{j_1} \wedge \ldots \wedge \theta_{j_s} \\
& \wedge \mathrm{d}p^{k_1}_{h_1} \wedge \ldots \wedge \mathrm{d}p^{k_t}_{h_t},
\end{aligned} \tag{5.3}$$

where the coefficients are functions on $J^1(T^*M)$, skew-symmetric separately with respect to the indices $i_1,...,i_r$; $j_1,...,j_s$ and $(h_1,k_1),...,(h_t,k_t)$, and we use Einstein's convention for repeated indices.
By imposing $L_{\tilde{X}_{(1)}}\Omega_n = 0$, $\forall X = g(q_1,...,q_m)A^*$ (*cf.* (2.2)) and using (2.5), a direct calculation shows that Ω_n is gauge invariant if and only if all coefficients f's in (5.3) are defined on $\bigwedge^2 T^*(M)$ and satisfy

$$f^{h_1...h_n}_{k_1...k_n}\left(\partial^3 g/\partial q_{h_1}\partial q_{k_1}\partial q_i\right) = 0, \tag{5.4}$$

$$f^{j_1...j_s,h_1...h_t}_{k_1...k_t}\left(\partial^3 g/\partial q_{h_1}\partial q_{k_1}\partial q_i\right) = 0,\ s+t=n, \tag{5.5}$$

$$\sum_\sigma \epsilon(\sigma) f^{h_1...h_t}_{i_1...i_{r-1},k_1...k_t}\left(\partial^3 g/\partial q_{h_1}\partial q_{k_1}\partial q_{i_r}\right) = 0,\ r+t=n+1,\ 2\le r\le n, \tag{5.6}$$

$$\sum_\sigma \epsilon(\sigma) f^{j_1...j_s,h_1...h_t}_{i_1...i_{r-1},k_1...k_t}\frac{\partial^3 g}{\partial q_{h_1}\partial q_{k_1}\partial q_{i_r}} = 0, \quad r+s+t=n+1,\ 2\le r\le n, \tag{5.7}$$

$\epsilon(\sigma)$ being the signature of a permutation σ of $(i_1,...,i_r)$. Thus in (5.3) the terms $f_{i_1...i_n}dq^{i_1}\wedge...\wedge dq^{i_n}$, $f^{j_1...j_n}\theta_{j_1}\wedge...\wedge\theta_{j_n}$ and $f^{j_1...j_s}_{i_1...i_r}dq^{i_1}\wedge...\wedge dq^{i_r}\wedge\theta_{j_1}\wedge...\wedge\theta_{j_s}$ are as in the statement. We proceed to calculate the other terms in (5.3) showing that they are expressed by means of the forms in (5.1). To do this we first note that from (5.4) we obtain $f^{h_1...h_n}_{k_1...k_n} + f^{k_1...h_n}_{h_1...k_n} = ... = f^{h_1...h_n}_{k_1...k_n} + f^{h_1...k_n}_{k_1...h_n} = 0$.
Thus, we have $f^{h_1...h_n}_{k_1...k_n}dp^{k_1}_{h_1}\wedge...\wedge dp^{k_n}_{h_n} = 2^{-n}f^{(h_1k_1)...(h_nk_n)}dp_{h_1k_1}\wedge...\wedge dp_{h_nk_n}$, where we set $f^{...h...}_{...k...} = f^{...(hk)...}$, and we write $p_{ij} = p^j_i - p^i_j$, instead of $\kappa^*(p_{ij}) = p^j_i - p^i_j$ (*cf.* Proposition 4.1–3.). Similarly, using (5.5) we obtain $f^{j_1...j_s,h_1...h_t}_{k_1...k_t}\theta_{j_1}\wedge...\wedge\theta_{j_s}\wedge dp^{k_1}_{h_1}\wedge...\wedge dp^{k_t}_{h_t} = 2^{-t}f^{j_1...j_s,(h_1k_1)...(h_tk_t)}\theta_{j_1}\wedge...\wedge\theta_{j_s}\wedge dp_{h_1k_1}\wedge...\wedge dp_{h_tk_t}$.
Next, we examine (5.6). First, for $r=2$ it becomes

$$f^{(h_1k_1)...(h_{n-1}k_{n-1})}_{i_1}\frac{\partial^3 g}{\partial q_{h_1}\partial q_{k_1}\partial q_{i_2}} - f^{(h_1k_1)...(h_{n-1}k_{n-1})}_{i_2}\frac{\partial^3 g}{\partial q_{h_1}\partial q_{k_1}\partial q_{i_1}} = 0.$$

As g is an arbitrary function, we obtain

$$\left\{\begin{array}{l} f^{(h_1k_1)...(h_{n-1}k_{n-1})}_{i_1} + f^{(k_1h_1)...(h_{n-1}k_{n-1})}_{i_1} = 0, \\ \text{for } h_1\neq i_1, k_1\neq i_1, \\ f^{(i_1k_1)...(h_{n-1}k_{n-1})}_{i_1} + f^{(k_1i_1)...(h_{n-1}k_{n-1})}_{i_1} = f^{(k_1k_1)...(h_{n-1}k_{n-1})}_{k_1}, \\ \text{for } k_1\neq i_1. \end{array}\right. \tag{5.8}$$

Then, by using (5.8) we obtain

$$(5.9)\quad f_{i_1}^{(h_1k_1)\dots(h_{n-1}k_{n-1})}\mathrm{d}q^{i_1}\wedge \mathrm{d}p_{h_1}^{k_1}\wedge\dots\wedge \mathrm{d}p_{h_{n-1}}^{k_{n-1}} = \sum_{\{h_1,k_1\}\cap\{i_1\}=\emptyset} \frac{1}{2} f_{i_1,k_2\dots k_{n-1}}^{(h_1k_1)h_2\dots h_{n-1}}\mathrm{d}q^{i_1}\wedge \mathrm{d}p_{h_1k_1}\wedge \mathrm{d}p_{h_2}^{k_2}\wedge\dots\wedge \mathrm{d}p_{h_{n-1}}^{k_{n-1}} + f_{i_1,k_2\dots k_{n-1}}^{(k_1i_1)h_2\dots h_{n-1}}\mathrm{d}q^{i_1}\wedge \mathrm{d}p_{k_1i_1}\wedge \mathrm{d}p_{h_2}^{k_2}\wedge\dots\wedge \mathrm{d}p_{h_{n-1}}^{k_{n-1}} + f_{k_1,k_2\dots k_{n-1}}^{(k_1k_1)h_2\dots h_{n-1}}\mathrm{d}\theta_{k_1}\wedge \mathrm{d}p_{h_2}^{k_2}\wedge\dots\wedge \mathrm{d}p_{h_{n-1}}^{k_{n-1}}.$$

As $f_{i_1}^{(h_1k_1)\dots(h_{n-1}k_{n-1})}$ is skew-symmetric with respect to $(h_1k_1),\dots,(h_{n-1}k_{n-1})$ we deduce that (5.8) holds $\forall(h_tk_t)$, $2\le t\le n-1$. Thus, by using (5.9) we finally conclude that $f_{i_1}^{(h_1k_1)\dots(h_{n-1}k_{n-1})}\mathrm{d}q^{i_1}\wedge \mathrm{d}p_{h_1}^{k_1}\wedge\dots\wedge \mathrm{d}p_{h_{n-1}}^{k_{n-1}}$ is expressed by means of $\mathrm{d}q_{i_1}\wedge \mathrm{d}p_{h_1k_1}\wedge\dots\wedge \mathrm{d}p_{h_{n-1}k_{n-1}}$ and $\mathrm{d}\theta_{j_1}\wedge\dots\wedge \mathrm{d}\theta_{j_s}\wedge \mathrm{d}p_{h_1k_1}\wedge\dots\wedge \mathrm{d}p_{h_tk_t}$, $2s+t=n$. For $r>2$ the sum in (5.6) becomes

$$f_{i_1\dots i_{r-1}}^{(h_1k_1)\dots(h_tk_t)}\left(\partial^3 g/\partial q_{h_1}\partial q_{k_1}\partial q_{i_r}\right) - f_{i_1\dots i_{r-2}i_r}^{(h_1k_1)\dots(h_tk_t)}\left(\partial^3 g/\partial q_{h_1}\partial q_{k_1}\partial q_{i_{r-1}}\right) - \dots - f_{i_1i_ri_3\dots i_{r-1}}^{(h_1k_1)\dots(h_tk_t)}\left(\partial^3 g/\partial q_{h_1}\partial q_{k_1}\partial q_{i_2}\right) - f_{i_ri_2i_3\dots i_{r-1}}^{(h_1k_1)\dots(h_tk_t)}\left(\partial^3 g/\partial q_{h_1}\partial q_{k_1}\partial q_{i_1}\right) = 0,$$

which implies

$$(5.10)\qquad f_{i_1\dots i_{r-1}}^{(h_1k_1)\dots(h_tk_t)} + f_{i_1\dots i_{r-1}}^{(k_1h_1)\dots(h_tk_t)} = 0, \quad \text{for } \{h_1,k_1\}\cap\{i_1,\dots,i_{r-1}\}=\emptyset,$$

$$(5.11)\qquad f_{i_1\dots i_{r-1}}^{(i_1k_1)(h_2k_2)\dots(h_tk_t)} + f_{i_1\dots i_{r-1}}^{(k_1i_1)(h_2k_2)\dots(h_tk_t)} = f_{k_1}^{(k_1k_1)(h_2k_2)\dots(h_tk_t)}, \quad \text{for } k_1\notin\{i_1,\dots,i_{r-1}\},$$

$$(5.12)\qquad f_{i_1i_2i_3\dots i_{r-1}}^{(i_1i_2)(h_2k_2)\dots(h_tk_t)} + f_{i_1i_2i_3\dots i_{r-1}}^{(i_2i_1)(h_2k_2)\dots(h_tk_t)} + f_{i_2i_ri_3\dots i_{r-1}}^{(i_2i_r)(h_2k_2)\dots(h_tk_t)} + f_{i_2i_ri_3\dots i_{r-1}}^{(i_ri_2)(h_2k_2)\dots(h_tk_t)} + f_{i_ri_1i_3\dots i_{r-1}}^{(i_ri_1)(h_2k_2)\dots(h_tk_t)} + f_{i_ri_1i_3\dots i_{r-1}}^{(i_1i_r)(h_2k_2)\dots(h_tk_t)} = 0, \quad \text{for } 3\le r\le n-1.$$

Further, we decompose the term we are interested in as follows:

$$(5.13)\quad f^{(h_1k_1)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p^{k_1}_{h_1}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}=$$
$$\sum_{\{h_1,k_1\}\cap\{i_1,\dots,i_{r-1}\}=\emptyset} f^{(h_1k_1)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p^{k_1}_{h_1}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+$$
$$(r-1)\sum_{k_1\notin\{i_1,\dots,i_{r-1}\}} f^{(i_1k_1)(h_2k_2)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}$$
$$\wedge\mathrm{d}p^{k_1}_{i_1}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+$$
$$(r-1)\sum_{k_1\notin\{i_1,\dots,i_{r-1}\}} f^{(k_1i_1)(h_2k_2)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}$$
$$\wedge\mathrm{d}p^{i_1}_{k_1}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+$$
$$(r-1)f^{(i_1i_1)(h_2k_2)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p^{i_1}_{i_1}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+$$
$$\binom{r-1}{2}\left\{f^{(i_1i_2)(h_2k_2)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p^{i_2}_{i_1}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+\right.$$
$$\left.f^{(i_2i_1)(h_2k_2)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p^{i_1}_{i_2}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}\right\}.$$

By using (5.10) we obtain

$$(5.14)\quad \sum_{\{h_1,k_1\}\cap\{i_1,\dots,i_{r-1}\}=\emptyset} f^{(h_1k_1)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p^{k_1}_{h_1}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}=$$
$$\sum_{\{h_1,k_1\}\cap\{i_1,\dots,i_{r-1}\}=\emptyset}\frac{1}{2}f^{(h_1k_1)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}$$
$$\wedge\mathrm{d}p_{h_1k_1}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}.$$

Let us denote by F the sum in the first brackets of (5.13). From (5.11) we obtain

$$(5.15)\quad F=f^{(k_1i_1)(h_2k_2)\dots(h_tk_t)}_{i_1\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p_{k_1i_1}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+$$
$$(-1)^r f^{(k_1k_1)(h_2k_2)\dots(h_tk_t)}_{k_1i_2\dots i_{r-1}}\mathrm{d}q^{i_2}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}\theta_{k_1}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}.$$

Similarly, let us denote by G the sum in the last brackets of (5.13). Thus,

$$(5.16)\quad G=f^{(i_1i_2)(h_2k_2)\dots(h_tk_t)}_{i_1i_2i_3\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p_{i_1i_2}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+$$
$$\left(f^{(i_1i_2)(h_2k_2)\dots(h_tk_t)}_{i_1i_2i_3\dots i_{r-1}}+f^{(i_2i_1)(h_2k_2)\dots(h_tk_t)}_{i_1i_2i_3\dots i_{r-1}}\right)\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}$$
$$\wedge\mathrm{d}p^{i_1}_{i_2}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}.$$

By induction on m the following lemma is easily checked:

Lemma 5.2. *Let (A_{ij}) be a $m\times m$ skew-symmetric matrix of differentiable functions satisfying $A_{ij}+A_{jk}+A_{ki}=0$, $i,j,k\in[1,m]$. Set $S_m=\sum_{i<j}A_{ij}\mathrm{d}q_i\wedge\mathrm{d}q_j\wedge\mathrm{d}p^i_j$, $m\geq 2$. Then,*

$$S_m=A_{1m}\mathrm{d}q_1\wedge\mathrm{d}\theta_1+\sum_{j=2}^{m}\left\{A_{jm}\mathrm{d}q_j\wedge\left(\sum_{i=1}^{j-1}\mathrm{d}q_i\wedge\mathrm{d}p^i_j+\mathrm{d}\theta_j\right)\right\}.$$

Let us apply the lemma to $A^{(h_2k_2)\dots(h_tk_t)}_{i_1i_2i_3\dots i_{r-1}}=f^{(i_1i_2)(h_2k_2)\dots(h_tk_t)}_{i_1i_2i_3\dots i_{r-1}}+f^{(i_2i_1)(h_2k_2)\dots(h_tk_t)}_{i_1i_2i_3\dots i_{r-1}}$, taking into account that for any set of indices $\{i_3,\dots,i_{r-1}\}$ and $\{(h_2k_2),\dots,(h_tk_t)\}$ the $m\times m$ matrix $\left(A^{(h_2k_2)\dots(h_tk_t)}_{i_1i_2i_3\dots i_{r-1}}\right)$ (whose entries are i_1,i_2) is skew-symmetric and it satisfies the relations in the lemma, as follows from (5.12). Hence each sum $S^{(h_2k_2)\dots(h_tk_t)}_{mi_3\dots i_{r-1}}=A^{(h_2k_2)\dots(h_tk_t)}_{i_1i_2i_3\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\mathrm{d}q^{i_2}\wedge\mathrm{d}p^{i_1}_{i_2}$, which appears in (5.16) is expressed as in the lemma. Thus, (5.16) becomes

$$\begin{aligned}(5.17)\quad G\ =\ & f^{(i_1i_2)(h_2k_2)\dots(h_tk_t)}_{i_1i_2\dots i_{r-1}}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p_{i_1i_2}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+\\ &(-1)^{r-1}A^{(h_2k_2)\dots(h_tk_t)}_{1mi_3\dots i_{r-1}}\mathrm{d}q_1\wedge\mathrm{d}\theta_1\wedge\mathrm{d}q^{i_3}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\\ &\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+\\ &(-1)^{r-1}\sum_{j=2}^{m}\left(A^{(h_2k_2)\dots(h_tk_t)}_{jmi_3\dots i_{r-1}}\mathrm{d}q_j\wedge\left(\sum_{i=1}^{j-1}\mathrm{d}q_i\wedge\mathrm{d}p_{ji}+\mathrm{d}\theta_j\right)\right)\\ &\wedge\mathrm{d}q^{i_3}\wedge\dots\wedge\mathrm{d}q^{i_{r-1}}\wedge\mathrm{d}p^{k_2}_{h_2}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}.\end{aligned}$$

As $f^{(h_1k_1)\dots(h_tk_t)}_{i_1\dots i_{r-1}}$ is skew-symmetric with respect to the upper indices, the calculations performed with respect to (h_1k_1) in order to obtain (5.14), (5.15) and (5.17) can be repeated for any other of the indices $(h_2k_2),\dots,(h_tk_t)$. Thus, the right hand side of (5.13) is finally expressed as a linear combination with coefficients in $C^\infty_{\bigwedge^2T^*M}$ of the differential forms in (5.1). The above proof still works for the term $f^{j_1\dots j_s,h_1\dots h_t}_{i_1\dots i_r,k_1\dots k_t}\mathrm{d}q^{i_1}\wedge\dots\wedge\mathrm{d}q^{i_r}\wedge\theta_{j_1}\wedge\dots\wedge\theta_{j_s}\wedge\mathrm{d}p^{k_1}_{h_1}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}$, since conditions (5.6) and (5.7) are the same for both types of coefficients.

Now, let us assume $n>m$. Thus, Ω_n is locally expressed as follows:

$$\begin{aligned}\Omega_n\ =\ & \frac{1}{t!}f^{h_1\dots h_t}_{k_1\dots k_t}\mathrm{d}q_1\wedge\dots\wedge\mathrm{d}q_m\wedge\mathrm{d}p^{k_1}_{h_1}\wedge\dots\wedge\mathrm{d}p^{k_t}_{h_t}+\\ &\sum_{a+b+m=n}\frac{1}{a!b!}f^{j_1\dots j_a,h_1\dots h_b}_{k_1\dots k_b}\mathrm{d}q_1\wedge\dots\wedge\mathrm{d}q_m\wedge\mathrm{d}p_{j_1}\wedge\dots\wedge\mathrm{d}p_{j_a}\\ &\wedge\mathrm{d}p^{k_1}_{h_1}\wedge\dots\wedge\mathrm{d}p^{k_b}_{h_b}+\Omega'_n,\end{aligned}$$

$$
\begin{aligned}
\Omega'_n = {} & \frac{1}{n!} f^{h_1\ldots h_n}_{k_1\ldots k_n} \mathrm{d}p^{k_1}_{h_1} \wedge \ldots \wedge \mathrm{d}p^{k_n}_{h_n} + \sum_{r+s=n, r<m} \frac{1}{r!s!} f^{j_1\ldots j_s}_{i_1\ldots i_r} \mathrm{d}q^{i_1} \wedge \ldots \wedge \mathrm{d}q^{i_r} \\
& \wedge \theta_{j_1} \wedge \ldots \wedge \theta_{j_s} + \\
& \sum_{s+t=n} \frac{1}{s!t!} f^{h_1\ldots h_t, j_1\ldots j_s}_{k_1\ldots k_t} \theta_{j_1} \wedge \ldots \wedge \theta_{j_s} \wedge \mathrm{d}p^{k_1}_{h_1} \wedge \ldots \wedge \mathrm{d}p^{k_t}_{h_t} + \\
& \sum_{r+t=n, r<m} \frac{1}{r!t!} f^{h_1\ldots h_t}_{i_1\ldots i_r, k_1\ldots k_t} \mathrm{d}q^{i_1} \wedge \ldots \wedge \mathrm{d}q^{i_r} \wedge \mathrm{d}p^{k_1}_{h_1} \wedge \ldots \wedge \mathrm{d}p^{k_t}_{h_t} + \\
& \sum_{r+s+t=n, r<m} \frac{1}{r!s!t!} f^{j_1\ldots j_s, h_1\ldots h_t}_{i_1\ldots i_r, k_1\ldots k_t} \mathrm{d}q^{i_1} \wedge \ldots \wedge \mathrm{d}q^{i_r} \wedge \theta_{j_1} \wedge \ldots \wedge \theta_{j_s} \\
& \wedge \mathrm{d}p^{k_1}_{h_1} \wedge \ldots \wedge \mathrm{d}p^{k_t}_{h_t}.
\end{aligned}
$$

Note that the above proof also works for Ω'_n. Hence we conclude that Ω'_n is gauge invariant if and only if Ω'_n is a linear combination with coefficients in $\bigwedge^2 T^*(M)$ of the forms in (5.1) for $r < m$. Furthermore, it is easy to see that Ω_n is gauge invariant if and only if Ω'_n is gauge invariant and all $f^{h_1\ldots h_t}_{k_1\ldots k_t}$, $f^{j_1\ldots j_a, h_1\ldots h_b}_{k_1\ldots k_b}$ belong to $C^\infty_{\bigwedge^2 T^*M}$. Moreover, although in the general case the differential forms in (5.1) and (5.2) are not linearly independent over $\kappa^* C^\infty(\bigwedge^2 T^*M)$, as they satisfy the following relationship: $\sum_{i=1}^m \mathrm{d}q_i \wedge \theta_i + p^*_{10}(\mathrm{d}\Omega_1) = \kappa^*\Omega_2 = \sum_{i<j} \kappa^*(p_{ij}) \mathrm{d}q_i \wedge \mathrm{d}q_j$, where Ω_n stands for the canonical form on $\bigwedge^n T^*(M)$ (*cf.* §4), their matrix in the basis induced by $(\mathrm{d}q_h, \theta_i, \mathrm{d}p^k_j)$ has constant entries, thus showing that the rank of the system of forms (5.1) and (5.2) is locally constant. This proves the last part of the statement and completes the proof. □

Remark. The general formula for the rank of $\mathcal{G}^n_M$ seems to be too complicated to be written down explicitly; for example, for $m \geq 2$, we have $\mathrm{rk}\mathcal{G}^1_M = m(m+3)/2$, $\mathrm{rk}\mathcal{G}^2_M = m(m+1)(m^2+5m+2)/8$, for $m \geq 3$, $\mathrm{rk}\mathcal{G}^3_M = m(m+1)(m+2)(m^3+4m^2-5m+12)/48$. The calculation of the rank for $n > m$ is much more difficult; for example the ranks of $\mathcal{G}^n_M$, $n \geq 3$, for a surface ($\dim M = m = 2$) are $\mathrm{rk}\mathcal{G}^3_M = 17$, $\mathrm{rk}\mathcal{G}^4_M = 17$, $\mathrm{rk}\mathcal{G}^5_M = 22$, $\mathrm{rk}\mathcal{G}^6_M = 15$, $\mathrm{rk}\mathcal{G}^7_M = 6$, $\mathrm{rk}\mathcal{G}^8_M = 1$, $\mathrm{rk}\mathcal{G}^n_M = 0$, $\forall n \geq 9$.

6. Aut $(M \times U(1))$-INVARIANT HORIZONTAL FORMS

A differential n-form Ω_n on $J^1(T^*M)$ is said to be aut $(M \times U(1))$*-invariant* if $L_{\tilde{X}_{(1)}}\Omega_n = 0$ for every $X \in$ aut $(M \times U(1))$.

Theorem 6.1. *The* $\mathbb{R}$*-algebra of* aut $(M \times U(1))$*-invariant horizontal forms on* $J^1(T^*M)$ *is* $\mathbb{R}[\kappa^*\Omega_2]$, *where* $\kappa : J^1(T^*M) \to \bigwedge^2 T^*(M)$ *is as in* Proposition 4.1 *and* Ω_2 *stands for the canonical 2-form on* $\bigwedge^2 T^*(M)$ (*cf.* §4).

PROOF. We have $\Omega_2 = \sum_{i<j} p_{ij} dq_i \wedge dq_j = \frac{1}{2}\sum_{i,j} p_{ij} dq_i \wedge dq_j$. From this local expression and (2.4) it is not difficult to see that every differential form in $\mathbb{R}[\kappa^* \Omega_2]$ is aut $(M \times U(1))$-invariant. Conversely, let us assume that $\Omega_n = \sum_{i_1,\dots,i_n=1}^m \frac{1}{n!} f_{i_1\dots i_n} dq_{i_1} \wedge \dots \wedge dq_{i_n} = \frac{1}{n!} f_{i_1\dots i_n} dq^{i_1} \wedge \dots \wedge dq^{i_n}$ is an aut $(M \times U(1))$-invariant horizontal n-form on $J^1(T^*M)$, where $f_{i_1\dots i_n}$ is skew-symmetric in the indices $i_1, \dots, i_n$. In order to prove that $\Omega_n \in \mathbb{R}[\kappa^* \Omega_2]$, we obtain some formulas for calculation of determinants of square matrices whose entries are p_{ij}. First, we denote by $\omega_{i_1\dots i_n}$ the coefficient of $dq^{i_1} \wedge \dots \wedge dq^{i_n}$ in Ω_2^r, where $n = 2r$. Note that we have $\omega_{i_1\dots i_n} = \epsilon_{i_2} p_{i_1 i_2} \omega_{i_3\dots i_n} + \dots + \epsilon_{i_n} p_{i_1 i_n} \omega_{i_2\dots i_{n-1}}$, where the ω's in the right hand side are some coefficients in Ω_2^{r-1} and ϵ_{i_k}, $2 \le k \le n$, is the sign of the permutation $(i_1, i_k, i_2, \dots, i_{k-1}, i_{k+1}, \dots, i_n)$. Then by induction on n we obtain

Lemma 6.2. *Let $n \geq 2$ be a natural number. With the above notations we have*

$$\Delta_{i_1\dots i_n} = \begin{vmatrix} p_{i_1 i_2} & p_{i_1 i_3} & p_{i_1 i_4} & \cdots & p_{i_1 i_n} \\ p_{i_3 i_2} & 0 & p_{i_3 i_4} & \cdots & p_{i_3 i_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{i_n i_2} & p_{i_n i_3} & p_{i_n i_4} & \cdots & 0 \end{vmatrix} = \begin{cases} \omega_{i_1 i_2\dots i_n} \cdot \omega_{i_3 i_4\dots i_n}, & \text{if } n = 2r, \\ \omega_{i_2 i_3 i_4\dots i_n} \cdot \omega_{i_1 i_3 i_4\dots i_n}, & \text{if } n = 2r+1. \end{cases} \tag{6.1}$$

As Ω_n is aut $(M \times U(1))$-invariant, it is also gauge invariant and according to Theorem 5.1 the functions $f_{i_1\dots i_n}$ are defined on $\bigwedge^2 T^*(M)$. In addition, the invariance condition when applied to a vector field $X \in \mathfrak{X}(M)$, $X = f^i(\partial/\partial q_i)$, yields

$$\left\{ f^t \frac{\partial f_{i_1\dots i_n}}{\partial q^t} - p_{tk} \frac{\partial f^t}{\partial q^h} \frac{\partial f_{i_1\dots i_n}}{\partial p_h^k} \right\} dq^{i_1} \wedge \dots \wedge dq^{i_n} + \\ f_{i_1\dots i_n} \left\{ df^{i_1} \wedge dq^{i_2} \wedge \dots \wedge dq^{i_n} + \dots + dq^{i_1} \wedge \dots \wedge df^{i_n} \right\} = 0. \tag{6.2}$$

Take $f^i = a_i \in \mathbb{R}$, $1 \le i \le m$, to conclude that $f_{i_1\dots i_n}$ only depends on p_{ij}. Thus (6.2) becomes

$$\frac{\partial f^t}{\partial q_{i_1}} f_{t i_2\dots i_n} - \frac{\partial f^t}{\partial q_{i_2}} f_{t i_1 i_3\dots i_n} + \dots + \\ (-1)^{n-1} \frac{\partial f^t}{\partial q_{i_n}} f_{t i_1\dots i_{n-1}} - p_{tk} \frac{\partial f^t}{\partial q^h} \frac{\partial f_{i_1\dots i_n}}{\partial p_h^k} = 0. \tag{6.3}$$

Without loss of generality we may consider from now onwards the permutation $(1,...,n)$ for $(i_1,...,i_n)$. Then (6.3) is arranged as follows:

$$(6.4)\quad \frac{\partial f^t}{\partial q_1}\left\{f_{t2\ldots n}-p_{tk}\frac{\partial f_{1\ldots n}}{\partial p_1^k}\right\}+\ldots+$$
$$\frac{\partial f^t}{\partial q_n}\left\{(-1)^{n-1}f_{ti_1\ldots i_{n-1}}-p_{tk}\frac{\partial f_{1\ldots n}}{\partial p_n^k}\right\}-\sum_{h=n+1}^{m}\left\{\frac{\partial f^t}{\partial q^h}p_{tk}\frac{\partial f_{1\ldots n}}{\partial p_h^k}\right\}=0.$$

For $h,k\in[1,m]$, $h\neq k$, set $X^{h,k}=\partial f_{1\ldots n}/\partial p_h^k$. Let us fix $h\in\{n+1,...,m\}$. From (6.4) we obtain the homogeneous linear system with m equations and $m-1$ unknowns,

$$(6.5)\qquad \sum_{k\neq h} p_{tk}\,X^{h,k}=0\,,\quad 1\le t\le m.$$

Suppose m is odd. Thus we can consider the non-null determinant

$$\delta=\begin{vmatrix} 0 & p_{12} & \cdots & p_{1,h-1} & p_{1,h+1} & \cdots & p_{1m}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ p_{h-1,1} & p_{h-1,2} & \cdots & 0 & p_{h-1,h+1} & \cdots & p_{h-1,m}\\ p_{h+1,1} & p_{h+1,2} & \cdots & p_{h+1,h-1} & 0 & \cdots & p_{h+1,m}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ p_{m1} & p_{m2} & \cdots & p_{m,h-1} & p_{m,h+1} & \cdots & 0\end{vmatrix}$$
$$=\left(\omega_{12\ldots(h-1)(h+1)\ldots m}\right)^2.$$

In case m is even we consider the determinant

$$\delta'=\begin{vmatrix} p_{h1} & p_{h2} & \cdots & p_{h,h-1} & p_{h,h+1} & \cdots & p_{hm}\\ p_{21} & 0 & \cdots & p_{2,h-1} & p_{2,h+1} & \cdots & p_{2m}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ p_{h-1,1} & p_{h-1,2} & \cdots & 0 & p_{h-1,h+1} & \cdots & p_{h-1,m}\\ p_{h+1,1} & p_{h+1,2} & \cdots & p_{h+1,h-1} & 0 & \cdots & p_{h+1,m}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ p_{m1} & p_{m2} & \cdots & p_{m,h-1} & p_{m,h+1} & \cdots & 0\end{vmatrix}.$$

According to Lemma 6.2, we have $\delta'=\omega_{h12\ldots(h-1)(h+1)\ldots m}\cdot\omega_{23\ldots(h-1)(h+1)\ldots m}\neq 0$. Hence in both cases (6.5) only has the trivial solution. Thus the functions we are looking for only depend on p_{hk} with $1\le h,k\le n$. From (6.4) we obtain the system

$$f_{12\dots n} - p_{1k}X^{1,k} = 0,\ p_{2k}X^{1,k} = 0, \dots,\ p_{nk}X^{1,k} = 0,\ 2 \le k \le n. \tag{6.6}$$

Two cases have to be examined:

Case 1: $n = 2r + 1$. Then the homogeneous linear system whose equations are the last $n-1$ ones in (6.6) only has the trivial solution. Hence from the first equation in (6.6) it follows $f_{1\dots n} = 0$. Thus, *there exist no* aut $(M \times U(1))$-*invariant horizontal forms of odd degree in* $J^1\,(T^*M)$.

Case 2: $n = 2r$. There exist non-trivial solutions $\{X^{1,2}, \dots, X^{1,n}\}$, given by

$$\frac{X^{1,2}}{\Delta^{12}} = \dots = \frac{X^{1,n}}{\Delta^{1n}} = K_{12\dots n}(p_{12}, \dots, p_{ij}, \dots, p_{n-1,n}), \tag{6.7}$$

where

$$\Delta^{12} = \begin{vmatrix} 0 & p_{34} & \cdots & p_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n3} & p_{n4} & \cdots & 0 \end{vmatrix}, \Delta^{13} = \begin{vmatrix} p_{23} & p_{34} & \cdots & p_{3n} \\ p_{24} & 0 & \cdots & p_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{2n} & p_{n4} & \cdots & 0 \end{vmatrix}, \dots,$$

$$\Delta^{1n} = \begin{vmatrix} p_{2n} & p_{n3} & \cdots & p_{n,n-1} \\ p_{23} & 0 & \cdots & p_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{2,n-1} & p_{n-1,3} & \cdots & 0 \end{vmatrix}$$

and $K_{12\dots n}$ is a differentiable function which will be determined by using the other systems obtained from (6.4). We first replace $X^{1,2}, \dots, X^{1,n}$ from (6.7) in the first equation of (6.6) and obtain

$$f_{1\dots n} = K_{12\dots n}(p_{12}, \dots, p_{ij}, \dots, p_{n-1,n})\Delta_{12\dots n}. \tag{6.8}$$

Further we consider the coefficients of $\partial f^t/\partial q_2$ in (6.4) and we obtain the system $f_{1\dots n} - p_{2k}X^{2,k} = 0$, $p_{1k}X^{2,k} = 0$, $p_{3k}X^{2,k} = 0, \dots, p_{nk}X^{2,k} = 0$, $k \in \{1, 3, 4, \dots, n\}$. From the last $m-1$ equations it follows: $X^{1,2}/\Delta^{12} = X^{2,3}/\Delta^{23} = \dots = X^{2,n}/\Delta^{2n} = K_{1\dots n}$, where we have set

$$\Delta^{23} = \begin{vmatrix} p_{31} & p_{34} & \cdots & p_{3n} \\ p_{41} & 0 & \cdots & p_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n4} & \cdots & 0 \end{vmatrix}, \dots, \Delta^{2n} = \begin{vmatrix} 0 & p_{34} & \cdots & p_{3,n-1} & p_{31} \\ p_{43} & 0 & \cdots & p_{4,n-1} & p_{41} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n3} & p_{n4} & \cdots & p_{n,n-1} & p_{n1} \end{vmatrix}.$$

Next, denote $Y^{i,j} = \partial K_{1\dots n}/\partial p_i^j$. Then taking into account that Δ^{1k} and Δ^{2k} do not depend on $\{p_{12}, \dots, p_{1n}\}$ and $\{p_{21}, \dots, p_{2n}\}$, respectively, and by using

$\partial X^{h,k}/\partial p_i^j = \partial X^{i,j}/\partial p_h^k$, we obtain

$$\frac{Y^{1,2}}{\Delta^{12}} = \frac{Y^{1,3}}{\Delta^{13}} = \ldots = \frac{Y^{1,n}}{\Delta^{1n}} = \frac{Y^{2,3}}{\Delta^{23}} = \ldots = \frac{Y^{2,n}}{\Delta^{2n}}. \tag{6.9}$$

From the system obtained by vanishing the coefficient of $\partial f^t/\partial q_3$ in (6.4) we consider the equations

$$p_{4k}X^{3,k} = 0, \ldots, \; p_{nk}X^{3,k} = 0, \; k \in \{1,2,4,5,\ldots,n\}. \tag{6.10}$$

Then by using (6.8) and Lemma 6.2, by direct calculations we obtain

$$\left\{\begin{array}{ll} X^{3,1} = & -K_{1\ldots n}\omega_{3456\ldots n}\cdot\omega_{4256\ldots n}, \\ X^{3,2} = & -K_{1\ldots n}\omega_{3456\ldots n}\cdot\omega_{1456\ldots n}, \\ X^{3,4} = & Y^{3,4}\omega_{12\ldots n}\cdot\omega_{34\ldots n}+ \\ & K_{1\ldots n}\left(\omega_{1256\ldots n}\cdot\omega_{3456\ldots n}+\omega_{12\ldots n}\cdot\omega_{56\ldots n}\right), \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X^{3,n} = & Y^{3,n}\omega_{12\ldots n}\cdot\omega_{34\ldots n}+ \\ & K_{1\ldots n}\left(\omega_{1245\ldots n-1}\cdot\omega_{3456\ldots n}+\omega_{12\ldots n}\cdot\omega_{456\ldots n-1}\right). \end{array}\right. \tag{6.11}$$

Replace $X^{3,k}$ from (6.11) in (6.10) obtaining the system $p_{4k}Y^{3,k} = 0, \ldots, p_{nk}Y^{3,k} = 0, 4 \le k \le n$, with non-trivial solutions

$$Y^{3,4} = \frac{\omega_{56\ldots n}}{\omega_{456\ldots n-1}}Y^{3,n}, \ldots, \; Y^{3,n-1} = \frac{\omega_{45\ldots(n-2)n}}{\omega_{(n-1)45\ldots n-2}}Y^{3,n}. \tag{6.12}$$

The equation obtained by vanishing the coefficient of $\partial f^1/\partial q_3$ in (6.4) is

$$p_{1k}X^{3,k} = 0, \; k \in \{1,2,4,5,\ldots,n\}. \tag{6.13}$$

By using (6.11) and (6.12) in (6.13) we obtain

$$\left\{\begin{array}{ll} Y^{3,4} = & -\dfrac{K_{1\ldots n}}{\omega_{345\ldots n}}\mathrm{sign}\,(3,4,5,6,\ldots,n)\,\omega_{56\ldots n}, \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Y^{3,n} = & -\dfrac{K_{1\ldots n}}{\omega_{345\ldots n}}\mathrm{sign}\,(3,n,4,5,\ldots,n-1)\,\omega_{45\ldots n-1}. \end{array}\right. \tag{6.14}$$

By induction on n, and by using the next homogeneous linear systems obtained from (6.4) by vanishing the coefficients of $\partial f^t/\partial q_h$, $4 \le h \le n$, we get in general

$$\left\{\begin{array}{ll} Y^{h,h+1} = & -\dfrac{K_{1\ldots n}}{\omega_{34\ldots n}}\mathrm{sign}\,(h,h+1,3,4,\ldots,\widehat{h},\widehat{h+1},\ldots,n)\cdot \\ & \omega_{34\ldots\widehat{h}\,\widehat{h+1}\ldots n}, \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Y^{h,n} = & -\dfrac{K_{1\ldots n}}{\omega_{34\ldots n}}\mathrm{sign}\,(h,n,3,4,\ldots,\widehat{h},\ldots,\widehat{n})\omega_{34\ldots\widehat{h}\ldots\widehat{n}}, \end{array}\right. \tag{6.15}$$

where the circumflex over a term means that it is to be omitted. Moreover, by using (6.8) and the first equation in (6.11) we obtain $X^{3,1} = Y^{3,1}\omega_{1\dots n} \cdot \omega_{34\dots n} + K_{1\dots n}\omega_{245\dots n} \cdot \omega_{34\dots n} = -K_{1\dots n}\omega_{34\dots n} \cdot \omega_{425\dots n}$. Hence $Y^{3,1} = 0$. Then from (6.9) it follows

$$Y^{1,k} = Y^{2,k} = 0, \ 1 \le k \le n. \tag{6.16}$$

Finally, taking into account that $K_{1\dots n}$ only depends on p_{ij}, $1 \le i,j \le n$, and by using (6.14)-(6.16) we obtain $\mathrm{d}K_{1\dots n} = -(K_{1\dots n}/\omega_{34\dots n})\,\mathrm{d}(\omega_{34\dots n})$; that is, $K_{1\dots n} = \alpha/\omega_{34\dots n}$, $\alpha \in \mathbb{R}$. Replace $K_{1\dots n}$ in (6.8) and obtain $f_{1\dots n} = \alpha\,\omega_{1\dots n}$, which finishes the proof of the theorem. □

Remark. For even dimensions, $m = 2r$, we have $\Omega_2^r = \mathrm{Pfaffian}(p_{ij})\,\mathrm{d}q_1 \wedge \dots \wedge \mathrm{d}q_m$.

7. Variational problems defined by invariant forms

Let $p\colon E \to M$ be a fibred manifold. The *horizontal part* of a differential n-form Ω_n on $J^r(E)$ is the n-form $h(\Omega_n)$ on $J^{r+1}(E)$ such that $h(\Omega_n)(X_1, \dots, X_n) = \Omega_n\big((j^r s)_*(p_{r+1})_*X_1, \dots, (j^r s)_*(p_{r+1})_*X_n\big)$, with $X_1, \dots, X_n \in T_{j_x^{r+1}s}(J^{r+1}E)$ and every local section s of p ([14]).

Assume M is oriented by a volume form v_m. Each form Ω_m on $J^r(T^*M)$ of degree $m = \dim M$ gives rise to a functional on the space of 1-forms on M with compact support, $\mathfrak{L}\colon \Gamma_c(M, T^*M) \to \mathbb{R}$, $\mathfrak{L}(\omega) = \int_M (j^r\omega)^*\Omega_m$. The *first variation* of $\mathfrak{L}$ at a point $\omega \in \Gamma(M, T^*M)$ is the linear functional $\delta_\omega\mathfrak{L}\colon \mathfrak{g}^c_{T^*M} \to \mathbb{R}$, $\delta_\omega\mathfrak{L}(X) = \int_M (j^r\omega)^*\big(L_{X_{(r)}}\Omega_m\big)$, $\mathfrak{g}^c_{T^*M}$ being the subspace of vector fields $X \in \mathfrak{g}_{T^*M}$ whose support has compact image on M, and $X_{(r)}$ is the r-jet prolongation of X. A linear form ω is said to be an *extremal* of $\mathfrak{L}$ if $\delta_\omega\mathfrak{L} = 0$. Two m-forms Ω_m, Ω'_m on $J^r(T^*M)$ are *equivalent* (*cf.* [18]) if for every 1-form ω on M, $\delta_\omega\mathfrak{L} = \delta_\omega\mathfrak{L}'$. Obviously, if ω_m and ω'_m are equivalent, $\mathfrak{L}$ and $\mathfrak{L}'$ have the same extremals. The form Ω_m is said to be *variationally trivial* if it is equivalent to zero; or equivalently, if every 1-form on M is an extremal of its functional. Two forms of different orders, say Ω_m on $J^r(T^*M)$, Ω'_m on $J^{r'}(T^*M)$, and $r' > r$, are said to be equivalent if $p^*_{r',r}\Omega_m$ and Ω'_m are equivalent. We are interested in the variational problems defined by gauge invariant m-forms. Such forms admit a large subspace of symmetries (precisely $\mathrm{gau}(M \times U(1)) \subset \mathfrak{g}_{T^*M}$) but not so large to be variationally trivial.

Theorem 7.1. *With the above hypotheses and notations we have*

1. *Every m-form is equivalent to its horizontal part.*
2. *All $\mathfrak{g}_{T^*M}$-invariant m-forms are variationally trivial.*

3. *Given a gauge invariant m-form Ω_m, an m-form $\bar{\Omega}_m$ exists on $\bigwedge^2 T^*(M)$ such that Ω_m and $\kappa^*\bar{\Omega}_m$ are equivalent.*
4. *If $\bar{\Omega}_m$ is an m-form on $\bigwedge^2 T^*(M)$, then $h(\kappa^*\bar{\Omega}_m)$ projects onto $J^1(T^*M)$ if and only if $\bar{\Omega}_m$ is horizontal.*
5. *Let $\Omega_m = \mathcal{L}\, v_m$ be a gauge invariant horizontal m-form. If ω is an extremal of the functional $\mathfrak{L}$ defined by Ω_m, then for every closed 1-form η on M, $\omega + \eta$ is also an extremal of $\mathfrak{L}$.*

PROOF. 1. Let f, Ω_n, $\Omega'_{n'}$ be arbitrary differentiable functions and forms, respectively on $J^r(T^*M)$. We have $h(\Omega_n \wedge \Omega'_{n'}) = h(\Omega_n) \wedge h(\Omega'_{n'})$, $h(\mathrm{d}f) = \mathrm{d}f - \sum_{i=1}^m \sum_{|\alpha|=0}^r (\partial f/\partial p_\alpha^i)\, \theta_\alpha^i$, where $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$, $|\alpha| = \alpha_1 + ... + \alpha_m$, $\theta_\alpha^i = \mathrm{d}p_\alpha^i - \sum_{j=1}^m p_{\alpha+(j)}^i \mathrm{d}q_j$, (j) being the multi-index $(j)_i = \delta_{ij}$, $1 \le i \le m$, and (p_α^i) are the coordinates induced on $J^r(T^*M)$; *i.e.*, $(p_\alpha^i)(j_x^r\omega) = \left(\partial^{|\alpha|}(p_i \circ \omega)/\partial q^\alpha\right)(x)$. If X is given by (2.2), then $X_{(r+1)} = \sum_{i=1}^m \sum_{|\alpha|=0}^{r+1} \left(\partial^{|\alpha|} g_i/\partial q^\alpha\right)(\partial/\partial p_\alpha^i)$. Hence $L_{X_{(r+1)}}\theta_\alpha^i = 0$, and for every 1-form ω we thus have $(j^{r+1}\omega)^*\left(L_{X_{(r+1)}} h(\mathrm{d}f)\right) = (j^r\omega)^*\left(L_{X_{(r)}}(\mathrm{d}f)\right)$. Part 2. follows from the definitions and 3. follows from 2. taking into account (5.1). In order to prove 4., locally we set $\bar{\Omega}_m = \sum_{r=0}^m \sum_{h,i,j} f_{hij}\, \mathrm{d}q_h \wedge \mathrm{d}p_{ij}$, where $h = (h_1, ..., h_{m-r})$, $1 \le h_1 < ... < h_{m-r} \le m$, $\mathrm{d}q_h = \mathrm{d}q_{h_1} \wedge ... \wedge \mathrm{d}q_{h_{m-r}}$, and $i = (i_1, ..., i_r)$, $j = (j_1, ..., j_r)$, $1 \le i_k < j_k \le m$, $1 \le k \le r$, $(i_1, j_1) \prec ... \prec (i_r, j_r)$ ($\prec$ stands for lexicographic order), $\mathrm{d}p_{ij} = \mathrm{d}p_{i_1 j_1} \wedge ... \wedge \mathrm{d}p_{i_r j_r}$. Let $k_1 < ... < k_r$ be the complement of h; *i.e.*, $\{k_1, ..., k_r\} = \{1, ..., m\} - \{h_1, ..., h_{m-r}\}$, let ϵ_h be the signature of the permutation $(1, ..., m) \mapsto (h_1, ..., h_{m-r}, k_1, ..., k_r)$, and let Π_h be the group of permutations of the set $\{k_1, ..., k_r\}$. Then $h(\kappa^*\bar{\Omega}_m) = \mathcal{L}\mathrm{d}q_1 \wedge ... \wedge \mathrm{d}q_m$, $\mathcal{L} = \sum_{r=0}^m \sum_{h,i,j} \sum_{\sigma \in \Pi_h} \epsilon_h \epsilon(\sigma)(f_{hij} \circ \kappa)\left(p_{(i_1\sigma(k_1))}^{j_1} - p_{(j_1\sigma(k_1))}^{i_1}\right) \cdots \left(p_{(i_r\sigma(k_r))}^{j_r} - p_{(j_r\sigma(k_r))}^{i_r}\right)$, with $(ij) = (i) + (j)$. Hence $\mathcal{L} \in C^\infty(J^1(T^*M))$ if and only if $\forall r > 0$, $f_{hij} = 0$. As the Lagrangian density in 5. is gauge invariant we have $\mathcal{L} = \bar{\mathcal{L}} \circ \kappa$, $\bar{\mathcal{L}} \in C^\infty(\bigwedge^2 T^*M)$. Hence $\left(X_{(1)}\mathcal{L}\right)(j_x^1\omega) = \sum_{i<j}\left(\partial g_j/\partial q_i - \partial g_i/\partial q_j\right)(x)\left(\partial\bar{\mathcal{L}}/\partial p_{ij}\right)(\mathrm{d}_x\omega)$. □

Let us consider a pseudo-Riemannian metric g on M with volume form v_g. For every $x \in M$, $O_x(g)$ stands for the orthogonal group of the scalar product induced by g on the tangent space at x; *i.e.*, $O_x(g)$ is the group of $\mathbb{R}$-linear mappings $A\colon T_x(M) \to T_x(M)$ such that $g(A(X), A(Y)) = g(X, Y)$, $\forall X, Y \in T_x(M)$. The group $O_x(g)$ acts (on the right) on $\bigwedge^n T_x^*(M)$ by setting $(\Omega_n \cdot A)(X_1, ..., X_n) = \Omega_n(A(X_1), ..., A(X_n))$. A gauge invariant horizontal m-form $\Omega_m = \mathcal{L}v_g$ is said to be *g-invariant* if in the decomposition $\mathcal{L} = \bar{\mathcal{L}} \circ \kappa$, the function $\bar{\mathcal{L}}\colon \bigwedge^2 T^*(M) \to \mathbb{R}$ is invariant under the action of the groups $O_x(g)$; *i.e.*, if for every $x \in M$,

$A \in O_x(g)$, $\omega_2 \in \bigwedge^2 T_x^*(M)$, $\bar{\mathcal{L}}(\omega_2 \cdot A) = \bar{\mathcal{L}}(\omega_2)$. We set $m' = m/2$ if m is even, $m' = (m-1)/2$ if m is odd. We have m' natural g-invariant functions $\bar{\mathcal{L}}_k \colon \bigwedge^2 T^*(M) \to \mathbb{R}$, $1 \le k \le m'$, given by $\bar{\mathcal{L}}_k(\omega_2) = g^{(2k)}(\omega_2^k, \omega_2^k)$, where $g^{(k)}$ is the pseudo-Riemannian metric induced by g on $\bigwedge^k T^*(M)$; *i.e.*, $g^{(1)}(w, w') = g(\phi^{-1}w, \phi^{-1}w')$, $\forall w, w' \in T_x^*(M)$, where $\phi \colon T_x(M) \to T_x^*(M)$ is the polarity $\phi(X)(Y) = g(X,Y)$, $X, Y \in T_x(M)$, and for $k > 1$ and every $w_1, \dots, w_k, w_1', \dots, w_k' \in T_x^*(M)$, $g^{(k)}(w_1 \wedge \dots \wedge w_k, w_1' \wedge \dots \wedge w_k') = \det\left(g^{(1)}(w_i, w_j')\right)$.

Theorem 7.2. *A differentiable function $\bar{\mathcal{L}} \colon \bigwedge^2 T^*(M) \to \mathbb{R}$ is g-invariant if and only if there exists a differentiable function $F \colon M \times \mathbb{R}^{m'} \to \mathbb{R}$ such that for every $\omega_2 \in \bigwedge^2 T_x^*(M)$, $\bar{\mathcal{L}}(\omega_2) = F\left(x, \bar{\mathcal{L}}_1(\omega_2), \dots, \bar{\mathcal{L}}_{m'}(\omega_2)\right)$.*

Proof. The action of $O_x(g)$ on $\bigwedge^2 T_x^*(M)$ is isomorphic to that of $O(s, m-s)$ on $\bigwedge^2 (\mathbb{R}^m)^*$, where s is the signature of g. From classical invariant theory (*e.g.*, see [20, Chapter 13, §8, 38]) we know that $\bar{\mathcal{L}}_1, \dots, \bar{\mathcal{L}}_{m'}$ is a basis for the ring of polynomial invariants; *i.e.*, $\mathcal{S}^{\cdot}\left(\bigwedge^2 T_x(M)\right)^{O_x(g)} = \mathbb{R}\left[\left(\bar{\mathcal{L}}_1\right)_x, \dots, \left(\bar{\mathcal{L}}_{m'}\right)_x\right]$, where $\mathcal{S}^{\cdot}$ stands for the graded symmetric algebra and we use that for a finite-dimensional real vector space V, the ring of polynomials over V^* can be identified with $\mathcal{S}^{\cdot}(V)$. From [16] we thus conclude that every invariant differentiable function $\bar{\mathcal{L}}_x \colon \bigwedge^2 T_x^*(M) \to \mathbb{R}$ can be written as $\bar{\mathcal{L}}_x = F_x \circ \left(\left(\bar{\mathcal{L}}_1\right)_x, \dots, \left(\bar{\mathcal{L}}_{m'}\right)_x\right)$, for certain differentiable function F_x, which can be taken to depend smoothly on $x \in M$, thus finishing the proof. □

Remark. Maxwell's equations correspond to the Lagrangian $\mathcal{L} = \bar{\mathcal{L}}_1 \circ \kappa$, for $m = 4$.

Remark. For $m = 2r$ the unique (up to scalar factors) $\operatorname{aut}(M \times U(1))$-invariant horizontal m-form on $J^1(T^*M)$ is $\Omega_m = \kappa^* \Omega_2^r$, which is variationally trivial as $L_{X_{(1)}} \Omega_m = r \mathrm{d}\left(i_X p_{10}^* \mathrm{d}\Omega_1\right) \wedge \kappa^*\left(\Omega_2^{r-1}\right)$, $\forall X \in \mathfrak{g}_{T^*M}^c$. Hence $\left(j^1\omega\right)^*\left(L_{X_{(1)}} \Omega_m\right)$ $r \mathrm{d}\left(\left(j^1\omega\right)^*\left(i_X p_{10}^* \mathrm{d}\Omega_1\right) \wedge \left(\mathrm{d}\omega\right)^{r-1}\right)$, and we can apply Stokes' theorem.

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