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STAR-COUNTABLE k-NETWORKS, COMPACT-COUNTABLE k-NETWORKS, AND RELATED RESULTS

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ABSTRACT. In the theory of generalized metric spaces, the notion of k-networks has played an important role. Every locally separable metric space or CW-complex, more generally, every space dominated by locally separable metric spaces has a star-countable k-network. Every Lašnev space, as well as, every space dominated by Lašnev spaces has a σ-compact-finite k-network. We recall that every space has a compact-countable k-network if it has a star-countable k-network, a σ-hereditarily closure preserving k-network, or a σ-compact-finite k-network. We investigate around spaces with a star-countable k-network, or a compact-countable k-network.

1. INTRODUCTION

Let X be a space, and let $\mathcal{P}$ be a collection of subsets of X. We recall that $\mathcal{P}$ is *point-countable* if every $x \in X$ is in at most countably many $P \in \mathcal{P}$. And, $\mathcal{P}$ is star-countable if every $P \in \mathcal{P}$ meets at most countably many other $Q \in \mathcal{P}$. Also, $\mathcal{P}$ is *compact-countable* (resp. *compact-finite*) if every compact subset of X meets at most countable (resp. finitely) many $P \in \mathcal{P}$.

Let X be a space, and $\mathcal{P}$ be a cover of X. Recall that $\mathcal{P}$ is a *k-network* if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$. As is well-known, spaces with a countable (resp. σ-locally finite) k-network are called *$\aleph_0$-spaces* (resp. *$\aleph$-spaces*), For $\aleph$-spaces; spaces with various types of k-networks; and spaces with certain point-countable k-networks, respectively see [6]; [28]; and [4], [7], [24], [26], etc.

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Let X be a space, and $\mathcal{C}$ be a cover of X. Then X *is determined by* $\mathcal{C}$ [4] ($= X$ has the weak topology with respect to $\mathcal{C}$ in the usual sense), if $F \subset X$ is closed in X if and only if $F \cap C$ is closed in C for every $C \in \mathcal{C}$. A space X is a *k-space* (resp. *sequential space*) if it is determined by a cover of compact subsets (resp. compact metric subsets) of X. A space X is a *k_ω-space* [14] if it is determined by a countable cover of compact subsets of X. Also, we recall that X *is dominated by* $\mathcal{C}$ if the union of any subcollection $\mathcal{C}^*$ of $\mathcal{C}$ is closed in X, and any subset of the union is determined by $\mathcal{C}^*$. Every space determined by an increasing countable closed cover $\mathcal{C}$ is dominated by $\mathcal{C}$. As is well-known, every CW-complex is dominated by a cover of compact metric subsets.

We recall the following basic facts around spaces with a star-countable or compact-countable k-network. For these facts, see [5], [11], [12], and [4], etc. More detailed properties for spaces with a star-countable k-network, or a compact-countable k-network, see [5], [11], [12], [18], etc.

Facts: (a) (i) Every space with a locally countable k-network has a star-countable closed k-network.

(ii) Every k-space with a star-countable closed k-network is the topological sum of $\aleph_0$-spaces. Here, a closed k-network is a k-network consisting of closed subsets.

(b) (i) Every closed image of a locally separable metric (resp. metric) space has a star-countable (resp. σ-hereditarily closure preserving) k-network.

(ii) Every space dominated by locally separable metric (resp. metric) subsets has a star-countable (resp. compact-countable) k-network.

(c) (i) Every space with a point-countable base has a compact-countable k-network.

(ii) Every space with a star-countable or σ-hereditarily closure preserving k-network has a σ-compact-finite, and thus, compact-countable k-network.

(d) Every quotient s-image of a space with a point-countable base has a point-countable k-network.

We note that not every Lindelöf, k-space with a compact-countable k-network has a σ-compact finite k-network (indeed, every Lindelöf space with a point-countable base is not metric, thus, it does not have a σ-compact-finite k-network by [8]). Also, note that not every separable, Lindelöf, k-space with a point-countable closed k-network has a compact-countable k-network; see [12].

We shall recall other definitions. A space X is an *inner-closed A-space* [16] (or [17]), if whenever $\{A_n : n \in N\}$ is a decreasing sequence of subsets of X such that $\mathrm{cl}(A_n - \{x\}) \ni x$ for each $n \in N$, there exist $B_n \subset A_n$ which are closed in

X, but $\cup\{B_n : n \in N\}$ is not closed in X. If the sets B_n are singletons, then such a space X is called *inner-one* A. Countably bi-quasi-k-spaces in the sense of [15] are inner-one A, and inner-one A-spaces are inner-closed A.

Also, we recall canonical quotient spaces S_α and S_2. For an infinite cardinal number α, S_α is the space obtained from the topological sum of α many convergent sequences by identifying all limit points to a single point ∞. And, $S_2 = (N \times N) \cup N \cup \{\infty\}$ is the space with each point of $N \times N$ isolated. A basic *nbd* of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m,n) : m \geq k\}$. And U is a *nbd* of ∞ if and only if $U \ni \infty$, and U is a *nbd* of all but finitely many $n \in N$. (The space S_ω is called the *sequential fan*, and the space S_2 is the *Aren's space*). We note that neither S_α nor S_2 is an inner-closed A-space.

In this paper, we shall consider around spaces with a star-countable or compact-countable k-network, and also, extend certain theorems on spaces with a star-countable k-network, as well as spaces with a σ-hereditarily closure preserving k-network, to spaces with a compact-countable or point-countable k-network. For example, we obtain the following results.

(A) Every k-space X with a star-countable k-network is metric if and only if X contains no closed copy of the sequential fan S_ω and no the Arens' space S_2.

(B) Every k-space X with a point-countable k-network has a point-countable base if and only if X is an inner-closed A-space.

(C) (i) Let X have a compact-countable k-network. Then X^2 is a k-space if and only if X has a point-countable base, or X is a locally k_ω-space.

(ii) Let X have a point-countable k-network. Then X^ω is a k-space if and only if X has a point-countable base.

All spaces are regular T_1-spaces, and, all maps are continuous and onto.

2. Spaces with a star-countable or point-countable k-network

Not every space with a star-countable closed k-network of singletons is meta-Lindelöf; see [4; Example 9.1]. Here, a space is *meta-Lindelöf* if every open cover has a point-countable open refinement. The first author showed that, among k-spaces, every space with a star-countable k-network is hereditarily meta-Lindelöf. Recently, the following result is obtained by M. Sakai [18].

Theorem 2.1. *Let X be a k-space with a star-countable k-network. Then X is a paracompact σ-space (thus, X is a hereditarily paracompact space).*

We recall that a space X has *countable tightness* if whenever $x \in \mathrm{cl}A$, then $x \in \mathrm{cl}C$ for some countable $C \subset A$. Sequential spaces have countable tightness; see [15], for example.

Lemma 2.2. *Let X be a k-space with a point-countable k-network. Then the following (i) and (ii) hold.*

(i) X is sequential, and thus, X has countable tightness ([4]).

(ii) If X is countably compact, then X is metric ([26]).

Lemma 2.3. *Let X be a k-space with a star-countable k-network $\mathcal{P}$. Then X is a disjoint union of $\{X_\gamma : \gamma \in \Gamma\}$, where each X_γ is the countable union of elements of $\mathcal{P}$. Also, if $x_\gamma \in X_\gamma$ for each $\gamma \in \Gamma$, $D = \{x_\gamma : \gamma \in \Gamma\}$ is discrete and closed in X.*

PROOF. The first half is due to [5; Lemma 1.1]. For the latter part, suppose some subset A of D is not closed in X. Since X is sequential by Lemma 2.2(i), A contains an infinite convergent sequence L. Then, there exists $P_0 \in \mathcal{P}$ which contains an infinite subsequence of L. This shows that P_0 meets infinitely many X_γ's. This is a contradiction. Then, D is discrete and closed in X. $\square$

Corollary 2.4. *[18] Every separable k-space X with a star-countable k-network is an $\aleph_0$-space.*

PROOF. Every separable paracompact space in Lindelöf. Then X is Lindelöf by Theorem 2.1. Thus, X is an $\aleph_0$-space in view of Lemma 2.3 $\square$

We recall that a space X is *strongly Fréchet [19]* (= countable bi-sequential in the sense of [15], if whenever $\{A_n : n \in N\}$ is a decreasing sequence of subsets of X with $\mathrm{cl}(A_n - \{x\}) \ni x$, there exists a sequence $\{x_n : n \in N\}$ with $x_n \in A_n$ which converges to the point x. If the sets A_n are the same, then such a space is called *Fréchet.* First countable spaces are strongly Fréchet.

Lemma 2.5. *Let X be a strongly Fréchet space. Then the following (i) and (ii) hold.*

(i) If X has a star-countable k-network, then X is locally separable metric ([5]).

(ii) If X has a σ-compact-finite k-network, then X is metric [(7)].

Lemma 2.6. *[23]. Let X be a sequential space in which every point is a G_δ-set. Then X is strongly Fréchet if and only if X contains no closed copy of S_ω, and no S_2.*

Theorem 2.7. *Let X be a k-space with a star-countable k-network. Then X is a locally separable metric space if and only if X contains no closed copy of S_ω, and no S_2.*

PROOF. The "only if" part is obvious, so we show the "if" part holds. In view of Lemma 2.5(i), it is sufficient to show that X is strongly Fréchet. To show this, for $x \in X$, let $\{A_n : n \in N\}$ be a decreasing sequence of subsets of X with $\mathrm{cl}(A_n - \{x\}) \ni x$. Since X has countable tightness by Lemma 2.2(i), there exists a countable subset $C_n \subset A_n - \{x\}$ with $x \in \mathrm{cl}C_n$ for each $n \in N$. Let $C = \cup\{C_n : n \in N\}$. Then, $S = \mathrm{cl}C$ is a separable k-space with a star-countable k-network. Then S is an $\aleph_0$-space by Corollary 2.4. Hence, each point of S is a G_δ-set in S. But, S is sequential by Lemma 2.2(i), and, S contains no closed copy of S_ω, and no S_2. Then S is strongly Fréchet by Lemma 2.6. Let $B_n = \cup\{C_m : m \geq n\}$ for each $n \in N$. Then $\{B_n : n \in N\}$ is a decreasing sequence of subsets of S with $\mathrm{cl}(B_n - \{x\}) \ni x$. Since S is strongly Fréchet, there exists a sequence $\{x_n : n \in N\}$ with $x_n \in B_n$ which converges to the point x. Here, we can assume that $x_n \in A_n$ for each $n \in N$. This shows that X is strongly Fréchet. □

The following Corollary due to [29] hold by Fact (b)(ii) and Theorem 2.7.

Corollary 2.8. *Let X be dominated by locally separable metric subsets. Then X is locally separable metric if and only if X contains no closed copy of S_ω, and no S_2.*

We recall that a space X is ω_1-*compact* if any subset of cardinality ω_1 has an accumulation point in X.

Lemma 2.9. *Let X be a k-space, and let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in N\}$ be a σ-compact finite collection in X. Then, any ω_1-compact subset A of X meets only countably many elements of $\mathcal{P}$.*

PROOF. Suppose that A meets uncountably many elements of $\mathcal{P}$. Then A meets uncountably many elements of some $\mathcal{P}_n$. Since $\mathcal{P}$ is point-finite, there exist uncountably many points $x_\alpha \in A$, and uncountably many distinct elements $P_\alpha \in \mathcal{P}_n$ with $x_\alpha \in P_\alpha (\alpha < \omega_1)$. Since A is ω_1-compact, $B = \{x_\alpha : \alpha < \omega_1\}$ has an accumulation point in X. But X is a k-space. Then there exists a compact subset C of X containing infinitely many points in B. Then the compact set C meets infinitely many elements $P_\alpha \in \mathcal{P}_n$. Thus $\mathcal{P}_n$ is not compact-finite. This is a contradiction. Thus A meets only countably many elements of $\mathcal{P}$. □

Theorem 2.10. *Let X be a k-space. Then the following (1) and (2) hold.*

(1) X has a countable (resp. locally countable) k-network if and only if X is an ω_1-compact (resp. locally ω_1-compact) space with a σ-compact-finite k-network.

(2) X has a star-countable k-network if and only if X has a σ-compact-finite k-network of ω_1-compact subsets.

PROOF. The "if" parts in (1) and (2) hold by Lemma 2.9. The "only if" parts hold by Facts (a)(i) and (c)(ii). □

We note that every Lindelöf space X with a point-countable base (thus, X is a k-space with a compact-countable k-network by Fact (c)(i)) need not be a locally $\aleph_0$-space in view of [4; Example 9.4]. But, for k-spaces with a σ-compact-finite k-network, the following holds by means of Fact (a).

Corollary 2.11. *Every ω_1-compact (resp. locally ω_1-compact) k-space with a σ-compact-finite k-network is an $\aleph_0$-space (resp. the topological sum of $\aleph_0$-spaces).*

The following is due to [10], and the latter part is similarly shown.

Lemma 2.12. *Let $\mathcal{P}$ be a point-countable k-network for a k-space X which is closed under finite intersections. If every first countable, closed subset of X is locally compact, then $\{P \in \mathcal{P} : clP$ is compact in $X\}$ is a k-network for X. Also, it is possible to replace "compact" by "ω_1-compact" in the above.*

We note that every k-space with a star-countable *closed* k-network is the topological sum of $\aleph_0$-spaces, but, it is impossible to omit the closedness of the k-network; see [5]. Let us consider conditions for a k-space with a star-countable k-network to be the topological sum of $\aleph_0$-spaces. We shall say that a space is a *pre-Lindelöf S_{ω_1}-space* if it admits a closed, and Lindelöf (i.e., each fiber is Lindelöf) map onto S_{ω_1}.

Theorem 2.13. *Let X be a k-space with a star-countable k-network. Then the following (i)-(v) are equivalent. Also, (vi) $\to$ (v) holds, and, under (CH), the converse holds.*

(i) *X has a σ-compact finite closed k-network.*
(ii) *X has a σ-locally countable k-network.*
(iii) *X is locally separable (equivalently, locally ω_1-compact).*
(iv) *X contains no (closed) pre-Lindelöf S_{ω_1}-subspaces.*
(v) *X is the topological sum of $\aleph_0$-spaces.*
(vi) *The character $\chi(X) \leq \omega_1$.*

PROOF. It is obvious that (v) implies (i), (ii), and (iii). We show that (i) or (ii) implies (v). Since X has a star-countable k-network, every first countable closed subset of X is locally ω_1-compact by Lemma 2.5(i). Then, when (i) holds, X has a σ-compact-finite closed k-network of ω_1-compact subsets by Lemma 2.12. Then, X is a k-space with a star-countable closed k-network by Theorem 2.10. Thus, X is the topological sum of $\aleph_0$-spaces by Fact (a)(ii). Similarly, when (ii) holds, X has a σ-locally countable closed k-network of ω_1-compact subsets, but every ω_1-compact subset is Lindelöf by means of Lemma 2.3, thus, X has also a star-countable closed k-network. Thus, (i) or (ii) implies (v). We show that the implications (iii) $\to$ (v), and (vi) $\to$ (iii) hold. Let $\mathcal{P}$ be a star-countable k-network for X. Suppose that (iii) holds, and $x \in X$ has a *nbd* V which is separable. Since V is a separable k-space with a star-countable k-network, V is Lindelöf by Corollary 2.4. Then, in view of Lemma 2.3, it follows that V meets only countably many elements of $\mathcal{P}$. This shows $\mathcal{P}$ is locally countable. Then, X has a locally countable k-network. Thus, (v) holds. Next, suppose that (vi) holds, and $x \in X$ has a local base $\{V_\beta : \beta < \omega_1\}$ in X. Let X be the disjoint union of $\{X_\alpha : \alpha \in A\}$ as in Lemma 2.3. Then, there exists some V_β sucht hat V_β meets only countably many X_α. Otherwise, by induction, there exists a subset $D = \{x_\beta : \beta < \omega_1\}$ of X such that $x_\beta \in V_\beta \cap X_{\alpha(\beta)}$, where $x_\beta \neq x$, and the $X_{\alpha(\beta)}$ are disjoint. But, D is discrete and closed in X by Lemma 2.3. This is a contradiction. Thus, some V_β meets only countably many X_α. Since each X_α is separable, V_β is separable. This shows X is locally separable. Thus (vi) $\to$ (iii) holds. We show that (v) $\to$ (iv), and (iv) $\to$ (iii) hold. To show that (v) $\to$ (iv) holds, suppose that X contains a (closed) pre-Lindelöf S_{ω_1}-subspace F. Since F is locally Lindelöf, S_{ω_1} must be locally Lindelöf. But, S_{ω_1} is not locally Lindelöf. This is a contradiction. Thus, (v) $\to$ (iv) holds. To show that (iv) $\Leftrightarrow$ (iii) holds, let X be the disjoint union of $\{X_\alpha : \alpha \in A\}$ as in Lemma 2.3 let $\mathcal{P}$ be a star-countable k-network for X. Let $\mathcal{P}(X_\alpha) = \{P \in \mathcal{P} : P$ contains a sequence converging to a point in $X_\alpha\}$. Then $|\mathcal{P}(X_\alpha)| \leq \omega$. Otherwise, there exist $P_\beta \in \mathcal{P}(X_\alpha)(\beta < \omega_1)$ such that $P_\beta \subset X_{\alpha(\beta)}$ with the $X_{\alpha(\beta)}$ distinct. For each $\beta < \omega_1$, let L_β be a sequence in P_β converging to a point $x_\beta \in X_\alpha$. Since $H = \mathrm{cl} X_\alpha$ is separable, it is an $\aleph_0$-space by Corollary 2.4. Then, in view of Lemma 2.3, we can assume that each L_β is disjoint from H, and $S = \cup\{L_\beta : \beta < \omega_1\} \cup H$ is a closed subset of X. Hence S is sequential, thus, S is a pre-Lndelöf S_{ω_1}-space which is closed in X, because, the quotient space obtained by identifying the Lindelöf closed subset X_α to a single point is the space S_{ω_1} by Lemma 2.3. This is a contradiction. Thus, $|\mathcal{P}(X_\alpha)| \leq \omega$. Let $V_1 = \cup\mathcal{P}(X_\alpha)$. Then V_1 is separable. Let $\mathcal{P}(V_1) = \{P \in \mathcal{P} : P$ contains a sequence converging to a point in $V_1\}$. Then, by the same way, $|\mathcal{P}(V_1)| \leq \omega$. Let $V_2 = \cup\mathcal{P}(V_1)$. Then V_2 is separable. In this way, we can get separable subsets $V_n (n \in N)$. Let $V = \cup\{V_n : n \in N\}$. Then, V is a separable subset of X with $V \ni x$. To show V is open in X, suppose that V is not in X. Since X is sequential by Lemma 2.2(i), there exists a sequence L in $X - V$, but L has a limit point y in V. Let $y \in V_n$ for some $n \in N$. Since $\mathcal{P}$ is a k-network, for some $P \in \mathcal{P}$, P contains a subsequence M of L. Thus $P \in \mathcal{P}(V_n)$. Hence M is contained in V. This is a contradiction. Thus V is a *nbd* of x, which is separable. Hence X is locally separable. Thus, (iv) $\to$ (iii) holds. Finally, the implication (v) $\to$ (vi) holds under (CH), because $\chi(Y) \leq 2^\omega$ for any separable space Y. That completes the proof. □

Remark. (a) The implication (v) $\to$ (vi) doesn't hold without (CH). Indeed, S_ω is a k-and-$\aleph_0$-space, but, as it is well-known, $\chi(S_\omega) = 2^\omega > \omega_1$ under (MA +]CH).

(2) in (v), it is impossible to replace "a pre-Lindelöf S_{ω_1}-space" by "S_{ω_1}". Indeed, let I be the closed unit interval, and for each $x \in I$, let L_x be a sequence converging to the point x such that $L_x \cap I = \emptyset$, and the L_x are disjoint. Let X be a space dominated by a cover $\{I \cup I_\chi : x \in I\}$. Then, X is a k-space with a star-countable k-network by Fact (b)(ii), and X contains no copy of S_{ω_1}, by X is not locally separable.

(3) In [13], it is shown that, for a k-space X with a star-countable k-network (more generally, σ-compact-finite k-network), X contains no closed copy of S_{ω_1} if and only if X has a point-countable cs^*-network in the sense of [2] (or [7]).

Corollary 2.14. *Let X be a space dominated by locally separable metric subsets (resp. X be a CW-complex). Suppose that X satisfies one of (i), (ii), (iii), (iv), and (vi) in Theorem 2.13. Then X is the topological sum of $\aleph_0$-spaces. (resp. X is the topological sum of countable CW-complex).*

PROOF. This holds by Fact (b)(ii) and Theorem 2.13. For the parenthetic part, let X be a CW-complex with cells $\{e_\lambda : \lambda\}$. Then, any ω_1-compact subset of X meets only countably many e_λ. Thus, X is the topological sum of countable CW-complexes. □

Inner-closed A spaces are one of fairly generalized spaces; indeed, the class of inner-closed spaces contains a class of countable bi-quasi-k-spaces in the sense of [15]. While, k-spaces with a point-countable k-network are the most general spaces in the theory of k-networks; indeed, the class of these spaces contains classes of Lašnev spaces (i.e., closed images of metric spaces), CW-complexes, and spaces which are quotient s-images of metric spaces, etc. In view of the above, the following theorem is useful.

Theorem 2.15. *Let X be a k-space with a point-countable k-network. Then X has a point-countable base if and only if X is an inner-closed A-space.*

PROOF. The "only if" part is obvious, so we show that "if" part holds. We recall that every k-space with a point-countable cover $\mathcal{P}$ satisfying (*) below has a point-countable base [1]. Then it is sufficient to show that, for a point-countable k-network $\mathcal{P}$ for X, $\mathcal{P}$ satisfies property (*).

(*) If $\chi \in V$ with V open in X, then $x \in \mathrm{Int}(\cup\mathcal{F}) \subset (\cup\mathcal{F}) \subset V$ for some finite $\mathcal{F} \subset \mathcal{P}$.

To show that (*) hold, suppose not. Let $\mathcal{U} = \{P \in \mathcal{P} : P \subset V\}$, and let $\mathcal{U}_1 = \{P \in \mathcal{U} : x \in P\} = \{P_{1i} : i \in N\}$. Let W be a *nbd* of x with $\mathrm{cl}W \subset V$. Now, let $C_0 = \{x\}$. Since X has countable tightness by Lemma 2.2(i), and $x \in \mathrm{cl}(X - P_{1i})$, there exists a countable $C_1 \subset X - P_{1i}$ such that $x \in \mathrm{cl}C_1$ and $C_1 \subset W$. Let $\mathcal{U}_2 = \{P \in \mathcal{U} : P \cap C_1 \neq \emptyset\} = \{P_{2i} : i \in N\}$. Since $x \in \mathrm{cl}(X - (P_{11} \cup P_{12} \cup P_{21} \cup P_{22}))$, there exists a countable $C_2 \subset X - (P_{11} \cup P_{12} \cup P_{21} \cup P_{22})$ such that $x \in \mathrm{cl}C_2$ and $C_2 \subset W$. In this way, by induction, we obtain countable many countable subsets $C_n (n \in N)$ such that $x \in \mathrm{cl}C_n \subset W$, and no $P \in \mathcal{U}$ meets infinitely many C_n. Let $A_n = \cup\{C_i : i \geq n\}$. Then $\{A_n : n \in N\}$ is decreasing, and $x \in \mathrm{cl}(A_n - \{x\})$ for all $n \in N$. Since X is an inner-closed A-space, there exists $B_n \subset A_n$ which are closed in X, but, $B = \cup\{B_n : n \in N\}$ is not closed in X. Since X is sequential by Lemma 2.2(i), there exists a sequence in B converging to a point not in B. Then, since the sets B_n are closed in X, there exists a convergent sequence $\{x_i : ii \in N\}$ in B such that $x_i \in B_{n(i)}$, hence $x_i \in A_{n(i)}$, where $i(n) < i(n+1)$ for each $n \in N$. Then, the sequence $\{x_i : i \in N\}$ converges to a point in $\mathrm{cl}W$, hence in V. Since $\mathcal{P}$ is a k-network, there exists $Q \in \mathcal{U}$ such that Q contains a subsequence L of $\{x_i : i \in N\}$. But, L meets infinitely many C_n. Then $Q \in \mathcal{U}$ meets infinitely many C_n. This is a contradiction. Hence, property (*) holds. □

As applications of Theorem 2.15, the following hold. We will also apply 2.15 to "k-space property of products" in next Section. Corollary 2.16 is due to [4]. For Corollary 2.17 due to [15], use Fact (d), and for Corollary 2.18, use Lemma 2.4(ii).

Corollary 2.16. *Let X be a strongly Fréchet space (or countably bi-k-space in the sense of [15]) with a point-countable k-network. Then X has a point-countable base.*

Corollary 2.17. *Let X be a quotient s-image of a space with a point-countable base. If Y is an inner-closed A-space, then X has a point-countable base.*

Corollary 2.18. *Let X be a k-space with a σ-compact-finite k-network. If X is an inner-closed A-space, then X is metric.*

3. k-SPACE PROPERTY OF PRODUCT SPACES.

We shall consider necessary and sufficient conditions for $X \times Y$ to be a k-space when X and Y have compact-countable or point-countable k-networks.

Lemma 3.1. *[21]. Let X have countable tightness. If $X \times Y$ is a k-space, then X is an inner-one A-space, or every first countable, closed subset of Y is locally countably compact.*

Lemma 3.2. *Let X and Y have compact-countable k-networks. If $X \times Y$ is a k-space, then the following (a), (b), or (c) holds.*

(a) X and Y have point-countable bases.

(b) X or Y is a locally compact metric space.

(c) X and Y have star-countable k-networks.

PROOF. Since X and Y are k-spaces, they have countable tightness by Lemma 2.2. Now, suppose that neither (a) nor (b) holds. We will show that every first countable, closed subsets of X and Y are locally compact. Since $X \times Y$ is a k-space, by Lemmas 2.2 and 3.1, X is an inner-one A-space, otherwise, every first countable, closed subset of Y is locally compact. Let X be an inner-one A-space. Then, by Theorem 2.15, X has a point-countable base. On the other hand, $Y \times X$ is a k-space. Thus, we have the following cases by Lemma 3.1.

Case 1: Y is an inner-one A-space.

Y has a point-countable base by Theorem 2.15. Then (a) holds, a contradiction.

Case 2: Every first countable, closed subset of X is locally compact.

Since X is first countable, X is locally compact. Then X is locally compact metric by Lemma 2.2. Thus (b) holds, a contradiction.

Therefore, it is impossible that X is an inner-one A-space. Thus, every first countable, closed subset of Y is locally compact. Similarly, we show that every first countable, closed subset of X is locally compact. Thus, Y has a star-countable k-network $\mathcal{P}$. Similarly, X has also a star-countable k-network. Then (c) holds. □

We recall a set-theoretic axiom $BF(\omega_2)$. Let ${}^{\omega}\omega$ be the set of all functions from ω to ω. For f and $g \in {}^{\omega}\omega$, we define $f \leq g$ if and only if the set $\{n \in \omega : f(n) > g(n)\}$ is finite. $BF(\omega_2)$ is the following set-theoretic axiom:

If $F \subset {}^\omega\omega$ has cardinality less than ω_2, then there exists $g \in {}^\omega\omega$ such that $f \leq g$ for all $f \in F$.

In other words, $BF(\omega_2)$ means "$\mathbf{b} \geq \omega_2$", where the cardinal $\mathbf{b}$ is defined by $\mathbf{b} = \min\{\gamma :$ there exists an unbounded family $A \subset {}^\omega\omega$ with $|A| = \gamma\}$. It is well-known that (MA) implies that "$\mathbf{b} = 2^\omega$".

In [11], the authors obtained the following result on space with a star-countable k-network. In [9], the first author obtained a similar result on spaces with a σ-hereditarily closure preserving k-network. We will extend these results to a more general case where spaces have compact-countable k-networks.

Lemma 3.3. *The following assertions (i) and (ii) are equivalent. When $X = Y$, (ii) holds without (i).*

(i) $BF(\omega_2)$ is false (equivalently, $\mathbf{b} = \omega_1$).

(ii) Suppose that X and Y are k-spaces having a star-countable k-network. Then $X \times Y$ is a k-space if and only if the following (a), (b), or (c) holds.

(a) X and Y have point-countable bases.

(b) X or Y is a locally compact metric space.

(c) X and Y are locally k_ω-spaces.

The following fairly general theorem holds by Lemmas 3.2 and 3.3. This gives an affirmative answer to the parenthetic parts of (a) and (b) of Question 8 in [12].

Theorem 3.4. *In the previous lemma, it is possible to replace "X and Y have star-countable k-networks" by "X and Y have compact-countable k-networks".*

In the following, the result for case where X and Y are dominated by Lašnev spaces (resp. X and Y are closed images of CW-complexes) is due to [25] (resp. [30].)

Corollary 3.5. *In Lemma 3.3, it is possible to replace "X and Y are star-countable k-networks" by "X and Y have property (*) below", and to replace "X and Y have point-countable bases" by "X and Y are metric spaces".*

(*) Space dominated by a cover of Lašnev spaces, or closed images of CW-complexes.

PROOF. Let X be dominated by $\{X_\alpha : \alpha \in A\}$, where each X_α is a Lašnev space, or a closed image of a CW-complex. Each X_α is a k-space and X is determined by $\{X_\alpha : \alpha \in A\}$, so X is a k-space. Each X_α has a compact-countable k-network by Facts (b) and (c), and a fact that every closed image of a CW-complex has a star-countable k-network [5]. But, every space dominated by spaces with a compact-countable k-network has a compact-countable k-network [12]. Hence, X is a k-spaces with a compact-countable k-network. Similarly, Y has the same property. Thus the corollary holds by Theorem 3.4. Next, let (a) of Lemma 3.3 hold. Then X and Y are first countable. Thus each X_α is metric. Indeed, if X_α is Lašnev, X_α is metric since it is first countable. If X_α is a closed image of a CW-complex, then X_α is dominated by compact metric subsets by [30; Lemma 2.5]. Then X_α is also metric, here note that every first countable space dominated by metric subsets is metric [20]. Hence, every X_α is metric. Since X is a first countable space dominated by metric subsets $X_\alpha (\alpha \in A), X$ is metric. Similarly Y is also metric. Thus X and Y are metric when (a) of Lemma 3.3 holds. □

Let us consider k-space property of $X \times X_\alpha$, where $\alpha = \omega, \omega_1$, or 2^ω. First, we recall the following basic lemma due to [3].

Lemma 3.6. *(i) $S_\omega \times S_{\omega_1}$ is s k-space if and only if $BF(\omega_2)$ holds.*
(ii) $S_{\omega_1} \times S_{\omega_1}$ is not a k-space.

Theorem 3.7. *The following (1), (2), and (3) hold.*

(1) The following assertions are equivalent.

(i) $BF(\omega_2)$ is false.

(ii) Let X have a compact-countable k-network. Then $X \times S_\omega$ is a k-space if and only if X is a locally k_ω-space.

(iii) Let X have a point-countable k-network. Then $X \times S_{\omega_1}$ is a k-space if and only if X is locally compact metric.

(2) Let X have a compact-countable (resp. point-countable) k-network. If $X \times S_{\omega_1}$ is a k-space, then X is locally k_ω (resp. locally σ-compact).

(3) Let X have a point-countable k-network. Then $X \times S_c$, where $c = 2^\omega$, is a k-space if and only if X is locally compact metric.

PROOF. For (1), the implication (i) $\to$ (ii) holds by Theorem 3.4, and the implication (ii) or (iii) $\to$ (i) holds by means of Lemma 3.6(i). For the implication (i) $\to$ (iii), we show that the "only if" part of (iii) holds under $BF(\omega_2)$ being false. Since $X \times S_{\omega_1}$ is a k-space, by Lemmas 2.6 and 3.1, every first countable, closed subset of X is locally compact. Then, by Lemma 2.12, X has a point-countable k-network $\mathcal{P}$ such that the closure of each element of $\mathcal{P}$ is compact, thus, compact metric by Lemma 2.5(ii). Suppose that, for some $x \in X$, any *nbd* of x is not contained in any countable union of elements of $\mathcal{P}$. Since $\mathcal{P}$ is a point-countable k-network, in view of the proof of Lemma 2.2 in [11], there exists a disjoint family $\{N_\alpha : \alpha < \omega_1\}$ of countable of X such that $x \in \mathrm{cl}N_\alpha - N_\alpha$, and any compact subset of X meets only finitely many N_α. Thus, as in the proof of Lemma 5 in [3] (or Theorem 2.6 in [11]), there exists a subset H of $S \times S_{\omega_1}$, where $S = \cup\{N_\alpha : \alpha < \omega_1\} \cup \{x\}$, such that H is not closed in $X \times S_{\omega_1}$ but $H \cap C$ is closed in C for every compact subset C of $X \times S_{\omega_1}$. This is a contradiction. Thus, X has a point-countable k-network $\mathcal{P}$ such that each point of X had a *nbd* which is contained in a countable union of elements of $\mathcal{P}$, and each closure of elements of $\mathcal{P}$ is compact metric. Then, each point of X is a G_δ-set in X. While, under $BF(\omega_2)$ being false, $S_\omega \times S_{\omega_1}$ is not a k-space by Lemma 3.6(i). Then X contains no closed copy of S_ω, thus, no closed copy of S_2, because $S_\omega \times S_{\omega_1}$ is the perfect image of $S_2 \times S_{\omega_1}$, and every perfect pre-image of a k-space is a k-space. Then, a sequential space X is strongly Fréchet by Lemma 2.6. Then, since X has a point-countable k-network, X has a point-countable base by Corollary 2.16. Thus X is first countable, then, X is locally compact. Then, as is well-known, X is locally compact metric. For (2), suppose that X is not a locally k_ω-space. Then X has a star-countable k-network by Lemma 3.2. Thus, in view of the proof of Theorem 3.4(1) in [11], X contains a closed subset which is the perfect pre-image of S_{ω_1}. Hence, $S_{\omega_1} \times S_{\omega_1}$ is a k-space. This is a contradiction to Lemma 3.6(ii). Thus, X is a locally k_ω-space. The parenthetic part holds in view of the proof of (1). For (3), since $X \times S_c$ is a k-space, each separable closed subset of X is locally countable compact [22], and thus, locally compact metric. While, since $X \times S_{\omega_1}$ is closed in $X \times S_c$, $X \times S_{\omega_1}$ is a k-space, thus, X is locally separable by the parenthetic part of (2). Hence X is locally compact metric. □

Remark. In view of Lemma 3.6(i), it is impossible to replace "locally k_ω" by "locally compact" in (2) of the previous theorem, and thus, it is also impossible to replace "S_c" by "S_{ω_1}" in (3). For the parenthetic part of (2), if X is Fréchet, then it is possible to replace "locally σ-compact" by "locally k_ω" by means of (2) and Remark A. However, the authors don't know whether this replacement is possible without the Fréchetness of X.

As for the k-space property of product X^ω, as far as the authors know, the following theorem seems to be the most general type in the theory of k-networks. This theorem gives an affirmative answer to (C) of Question 7 in [12].

Theorem 3.8. *Let X have a point-countable k-network. Then X^ω is a k-space if and only if X has a point-countable base.*

PROOF. Since $X \times X^\omega$ is a copy of X^ω, $X \times X^\omega$ is a k-space. Then, by Lemmas 2.6(i) and 3.1, X is an inner-one A-space, or every first countable, closed subset of

X^ω is countably compact. For the latter case, X is countably compact. Indeed, suppose not. Then, X contains a countable closed discrete subset D. Thus X^ω contains a closed subset D^ω. But, D^ω is a first countable space which is not countably compact. This is a contradiction. Hence, X is countably compact, and thus, X is an inner-one A-space. Then, for any case, X is an inner-one A-space. Thus, X has a point-countable bases by Theorem 2.15. $\square$

The following holds by Fact (d) and Theorems 3.7 and 3.8. In (1) (resp. (2)), the result for case where X is dominated by metric subsets (resp. X is a quotient s-image of a metric space) is due to [29] (resp. [12]).

Corollary 3.9. *Let X have property (*) in Corollary 3.5, or let X be a quotient s-image of a metric space (or space with a point-countable base). Then the following (1) and (2) hold.*

(1) If $X \times S_c$ is a k-space, then X is locally compact metric.

(2) If X^ω is a k-space, then X has a point-countable base.

4. Questions

Question 4.1. Is every separable k-space X with a σ-compact-finite k-network $\mathcal{P}$ an $\aleph_0$-space (equivalently, ω_1-compact space)?

Question 4.2. Let X be a k-space with a point-countable (σ-compact-finite k-network; or compact-countable) k-network. If X contains no closed copy of S_ω, and no S_2, then does X have a point-countable base?

Remark. (1) Question 4.1 is affirmative if X is meta-Lindelöf; or $\mathcal{P}$ is σ-hereditarily closure preserving under (CH); see [9]. (In [9], the first author posed the question whether every separable k-space with a σ-HCP k-network is an $\aleph_0$-space). Question 4.2 is affirmative if each point of X is a G_δ-set by Lemma 2.5 and Corollary 2.17.

(2) In [13], under (CH), it is showed that Question 4.1 is affirmative, and so is Question 4.2 for the first case in the parenthetic part of it.

Every Fréchet space with a point-countable k-network of separable subsets has a star-countable k-network [18], hence, it has a compact-countable k-network. Also, every locally separable Fréchet space with a point-countable k-network is the topological sum of $\aleph_0$-spaces in view of [4], hence it has a compact-countable k-network in view of [4]. However, not every separable, Lindelöf, k-space with a point-countable k-network of compact metric subsets has a compact-countable

k-network; see [4, Example 9.3]. Thus, we have the following question, which is also posed in [12].

Question 4.3. Does every Fréchet space with a point-countable k-network have a compact-countable k-network?

Addendum: Quite recently, Question 4.2 was answered affirmatively by S. Lin, "A note on the Arens' space and sequential fan," (to appear in Topology and Appl.)

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