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ATTRACTORS IN EUCLIDEAN SPACES AND SHIFT MAPS ON POLYHEDRA

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ABSTRACT. In this paper, we prove that if P is a compact polyhedron in the Euclidean n-dimensional space R^n and $f : P \to P$ is any map, then there is a homeomorphism $F : R^{2n} \to R^{2n}$ such that the inverse limit (P, f) of f is contained in R^{2n}, F is an extension of the shift map $\tilde{f} : (P, f) \to (P, f)$ of f, and (P, f) is an attractor of F. Moreover, if P is contractible, then F can be chosen so that (P, f) is a global attractor of F. For a special case, we prove that if S is the unit circle and $f : S \to S$ is any map of S, then there is a homeomorphism $F : R^3 \to R^3$ such that the inverse limit (S, f) of f is contained in R^3, F is an extension of the shift map $\tilde{f} : (S, f) \to (S, f)$ of f, and (S, f) is an attractor of F. As a corollary, we show that it is possible to characterize attractors (of cascades) on topological manifolds. This is an answer to a problem of Günther and Segal [4, Problems, p.328].

1. INTRODUCTION

All spaces considered in this paper are assumed to be separable metric spaces. *Maps* are continuous functions. By a *compactum* we mean a compact metric space. A *continuum* is a connected, nondegenerate compactum. By an ANR, we mean an absolute neighborhood retract. Let R be the real line and R^n the Euclidean n-dimensional space. For a topological manifold M, ∂M denotes the manifold boundary. Let $F : Y \to Y$ be a homeomorphism of a space Y (onto itself) with metric d and let Λ be a compact subset of Y. Then Λ is said to be an *attractor* of F provided that there exists an open neighborhood of U of Λ in Y such that

$$F(\mathrm{Cl}(U)) \subset U \text{ and } \Lambda = \cap_{n \geq 0} F^n(U).$$

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Note that $F(\Lambda) = \Lambda$. Moreover, if for each $y \in Y$ $\lim_{n\to\infty} d(F^n(y), \Lambda) = 0$, then we say that Λ is a *global attractor* of F, where $d(A, B) = \inf\{d(a, b) | \, a \in A, b \in B\}$ for sets A, B. Let $f : X \to X$ and $g : Y \to Y$ be maps. Then f is *topologically conjugate to* g if there is a homeomorphism $\phi : X \to Y$ such that $\phi \cdot f = g \cdot \phi$.

Let $\mathbf{X} = \{X_n, p_{i,i+1} | \, i = 1, 2, \dots\}$ be an inverse sequence of compacta X_i and maps $p_{i,i+1} : X_{i+1} \to X_i (i = 1, 2, \dots)$ and let

$$\operatorname{invlim} \mathbf{X} = \{(x_i)_{i=1}^{\infty} | \, x_i \in X_i, \, p_{i,i+1}(x_{i+1}) = x_i \text{ for each } i\} \subset \textstyle\prod_{i=1}^{\infty} X_i.$$

Then $\operatorname{invlim} \mathbf{X}$ is a topological space as a subspace of the product space $\prod_{i=1}^{\infty} X_i$. Then $\operatorname{invlim} \mathbf{X}$ is a compactum. Let $f : X \to X$ be a map of a compactum X. Consider the following special inverse limit space:

$$(X, f) = \{(x_i)_{i=1}^{\infty} | \, x_i \in X \text{ and } f(x_{i+1}) = x_i \text{ for each } i \geq 1\}.$$

Define a map $\tilde{f} : (X, f) \to (X, f)$ by $\tilde{f}(x_1, x_2, \dots,) = (f(x_1), x_1, \dots,)$. Then $\tilde{f}$ is a homeomorphism and it is called the *shift map* of f.

A map $f : X \to Y$ of compacta is a *near homeomorphism* if f can be approximated arbitrarily closely by homeomorphisms from X onto Y.

In [6, Theorem 1], Isbell proved that if $X = \operatorname{invlim}\{X_i, p_{i,i+1}\}$ where each X_i is a compactum which can be embedded into R^n (n fixed), then X can be embedded into R^{2n}. In [2], Barge and Martin proved that if $f : I \to I$ is any map of the unit interval $I = [0, 1]$, then there is a homeomorphism $F : R^2 \to R^2$ such that (I, f) is contained in R^2, F is an extension of the shift map $\tilde{f} : (I, f) \to (I, f)$, and (I, f) is a global attractor of F. In [7], we proved that if $f : S \to S$ is a map of the unit circle S with $|\deg(f)| \leq 1$, then the shift map $\tilde{f}$ can be extended to a homeomorphism of the plane R^2 whose attractor is (S, f).

In this paper, we show the following results: (i) If P is a compact polyhedron in R^n, then for any map $f : P \to P$ there is a homeomorphism $F : R^{2n} \to R^{2n}$ such that (P, f) is contained in R^{2n}, F is an extension of $\tilde{f}$, and (P, f) is an attractor of F. Moreover, if P is contractible, then F can be chosen so that (P, f) is a global attractor of F. (ii) If $f : S \to S$ of the unit circle S is any map, then there is a homeomorphism $F : R^3 \to R^3$ such that (S, f) is contained in R^3, F is an extension of $\tilde{f} : (S, f) \to (S, f)$, and (S, f) is an attractor of F.

As a corollary, we show that it is possible to characterize attractors (of cascades) on topological manifolds. This is an answer to a problem of Günther and Segal [4, Problems, p.328].

2. Shift maps of compact polyhedra in R^n

One of the main results of this paper is the following theorem which is a generalization of Barge-Martin's theorem [2], and which is related to Isbell's theorem [6, Theorem 1].

Theorem 2.1. *If P is a compact polyhedron in R^n and $f : P \to P$ is any map, then there is a homeomorphism $F : R^{2n} \to R^{2n}$ such that (P, f) is contained in R^{2n}, F is an extension of the shift map $\tilde{f} : (P, f) \to (P, f)$ of f, and (P, f) is an attractor of F. Moreover, if P is contractible, then F can be chosen so that (P, f) is a global attractor of F.*

To prove the above theorem, we need the following lemma which was proved by Brown [3].

Lemma 2.2. *Let $\mathbf{X} = \operatorname{invlim}\{X_i, p_{i,i+1}\}$ be an inverse sequence of compacta X_i. If each $p_{i,i+1} : X_{i+1} \to X_i$ is a near homeomorphism, then $\operatorname{invlim} \mathbf{X}$ is homeomorphic to X_i for each i.*

By using Lemma 2.2, we obtain the following.

Lemma 2.3. *Suppose that X is a compact subset of a compactum Y and $f : X \to X$ is a map of X. If there is an extension $h : Y \to Y$ of f such that h is a near homeomorphism and there is a neighborhood N of X in Y such that $h(N) \subset X$, then there is a homeomorphism $F : Y \to Y$ such that (X, f) is contained in Y, F is an extension of the shift map $\tilde{f} : (X, f) \to (X, f)$ of f, and (X, f) is an attractor of F.*

Proof of Lemma 2.3. Consider the inverse limit (Y, h) and the shift map $\tilde{h} : (Y, h) \to (Y, h)$ of h. By Lemma 2.2, we know that (Y, h) is homeomorphic to Y. Then $F = \tilde{h} : Y = (Y, h) \to Y = (Y, h)$ is a desired homeomorphism.

Proof of Theorem 2.1. Put $R^{2n} = (R_1 \times \cdots R_n) \times (R_1' \times \cdots R_n')$, where $R_i = R_i' = R$. We may assume that $P \subset R_1 \times \cdots R_n \times \{\mathbf{0}\}(= R^n) \subset R^{2n}$, where $\mathbf{0} = (0, \ldots, 0) \in R_1' \times \cdots R_n'$. Choose a positive number $a > 0$ such that $P \subset B^n(a) - \partial B^n(a)$, where $B^n(a) = \{\mathbf{x} \in R^n \mid |\mathbf{x}| \leq a\}$. Choose a sufficiently large positive number $b > 0$ such that $B^n(a) \times B^n(a) \subset \operatorname{Int}(B^{2n}(b))$, where $B^{2n}(b) = \{\mathbf{x} \in R^{2n} \mid |\mathbf{x}| \leq b\}$.

Let N be a regular neighborhood of P in $\operatorname{Int}(B^{2n}(b))$ (e.g., see [9, Theorem 1.6.4] or [5]). By [9, Lemma 1.6.2], there is a retraction $r : N \to P$. Moreover, by the proof of [9, Lemma 1.6.2], we see that there is an extension $h_1 : B^{2n}(b) \to B^{2n}(b)$ of r such that h_1 is a near homeomorphism and $h_1|\partial(B^{2n}(b)) \cup P = \mathrm{id}$.

In fact, since regular neighborhoods of locally finite complexes in PL manifolds are mapping cylinder neighborhoods (see [12]), we can also obtain such a near homeomorphism h_1.

Let $T_i : R_i \times R_i' \to R_i \times R_i'$ be a homeomorphism of the plane $R_i \times R_i'$ such that $T_i(x, 0) = (0, x)$ for each $x \in [-c, c] \subset R_i, [-c, c]^2 \subset B_i^2(d)$, $T_i|\mathrm{Cl}(R_i \times R_i' - B_i^2(d)) = \mathrm{id}$, where $d > c > 0$ and $B_i^2(d) = \{\mathbf{x} \in R_i \times R_i' | |\mathbf{x}| \leq d\}$. We may assume that

$$P \subset E = [-c, c]^n \times \{\mathbf{0}\} \subset \textstyle\prod_{i=1}^n B_i^2(d) \subset B^n(a) \times B^n(a).$$

By considering the product map of $T_i (i = 1, 2, .., n)$, we obtain the homeomorphism $h_2 : R^{2n} \to R^{2n}$ such that if $\mathbf{x} \in P$, then

$$h_2(x_1, \ldots, x_n, 0, \ldots, 0) = (0, \ldots, 0, x_1, \ldots, x_n),$$

and $h_2|\mathrm{Cl}(R^{2n} - (B^n(a) \times B^n(a))) = \mathrm{id}$.

Since $B^n(a) - \partial B^n(a)$ is an absolute retract, we can choose an extension $f^* : R^n \to (B^n(a) - \partial B^n(a))$ of $f : P \to P \subset B^n(a) - \partial B^n(a)$ such that $f^*(\mathbf{x}) = \mathbf{0}$ for each $\mathbf{x} \in \mathrm{Cl}(R^n - B^n(a))$. Define a map $h_3 : R^{2n} \to R^{2n}$ by $h_3(\mathbf{x}, \mathbf{y}) = (\psi_{\mathbf{y}}(\mathbf{x}), \mathbf{y})$, where $\psi_{\mathbf{y}} : R^n \to R^n (\mathbf{y} \in R^n)$ is the map defined as follows:

If $\mathbf{x} \in B^n(a)$ and $\mathbf{x} = t \cdot \mathbf{x}'$ for some $\mathbf{x}' \in \partial B^n(a)$ and $0 \leq t \leq 1$,
$\psi_{\mathbf{y}}(\mathbf{x}) = (1 - t) \cdot f^*(\mathbf{y}) + t \cdot \mathbf{x}'$, and if $\mathbf{x} \in R^n - B^n(a)$, $\psi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x}$.

Then $h_3 : R^{2n} \to R^{2n}$ is a homeomorphism such that $h_3|[R^{2n} - (B^n(a) \times B^n(a))] = \mathrm{id}$. Note that $proj \cdot h_3 \cdot h_2|P = f$, where $proj : R^{2n} \to R^n$ is the projection defined by $proj(x_1, \ldots, x_{2n}) = (x_1, \ldots, x_n)$.

Choose a $2n$-ball H in $\mathrm{Int}(B^{2n}(b))$ such that $B^n(a) \times B^n(a) \subset \mathrm{Int}\, H$. Then there is a near homeomorphism $h_4 : R^{2n} \to R^{2n}$ such that $h_4|\mathrm{Cl}(R^{2n} - H) = \mathrm{id}, h_4(\mathbf{x}) = (x_1, \ldots, x_n, 0, \ldots, 0)$ for each $\mathbf{x} = (x_1, \ldots, x_{2n}) \in B^n(a) \times B^n(a)$.

Put $h = h_4 \cdot h_3 \cdot h_2 \cdot h_1|B^{2n}(b) : B^{2n}(b) \to B^{2n}(b)$. Then we see that h is a near homeomorphism, $h|P = f$ and $h(N) \subset P$, and $h|\partial B^{2n}(b) = \mathrm{id}$. By Lemma 2.3, we see that $(B^{2n}(b), h)$ is homeomorphic to a $2n$-ball B, $F' = \tilde{h} : B = (B^{2n}(b), h) \to B = (B^{2n}(b), h)$ is an extension of $\tilde{f}$, (P, f) is an attractor of F', and $F'|\partial B = \mathrm{id}$. Put $F = F'|(B - \partial B)(= R^{2n})$.

Next, suppose that P is contractible. Take a n-simplex Δ in R^n containing P. Since P is an AR, there is a retraction $r : \Delta \to P$. Put $f' = f \cdot r : \Delta \to P \subset \Delta$. Then $(P, f) = (\Delta, f')$ and $\tilde{f} = \tilde{f}'$. Hence we may assume that P is collapsible. Note that every regular neighborhood of P is a $2n$-ball (see [9, Corollary 1.6.4]). Hence we may assume that $N \supset H$. Hence, moreover we can choose a near homeomorphism $h_1 : B^{2n}(b) \to B^{2n}(b)$ such that if $\mathbf{x} \in B^{2n}(b) - \partial B^{2n}(b)$, then there is some natural number $i > 0$ such that $h_1^i(\mathbf{x}) \in N$, and hence $h_1^{i+1}(\mathbf{x}) \in P$.

Note that $h_4 \cdot h_3 \cdot h_2|\mathrm{Cl}(B^{2n}(b) - H) = \mathrm{id}$. This implies that if $\mathbf{x} \in B^{2n}(b) - \partial B^{2n}(b)$, then there is some natural number $j > 0$ such that $h^j(\mathbf{x}) \in P$. This implies that (P, f) is a global attractor of F. This completes the proof.

Remark. Concerning the above theorem, we have the following question: For what f is it the case that F can be chosen as a diffeomorphism ? Especially, in [1, p.177, Problem (1.5)] Barge asked: For what $f : I \to I$ is it the case that (I, f) can be embedded in the plane so that $\tilde{f}$ extends to a diffeomorphism ? In [11], Szczechla has shown that if $f : I \to I$ is a piecewise monotone map, then the plane homeomorphism F can be taken to be C^1. Naturally, we have the following question: Is it true that if $f : P \to P$ is a piecewise linear map, then the homeomorphism $F : R^{2n} \to R^{2n}$ can be taken to be C^1 ?

3. Shift maps of the unit circle S

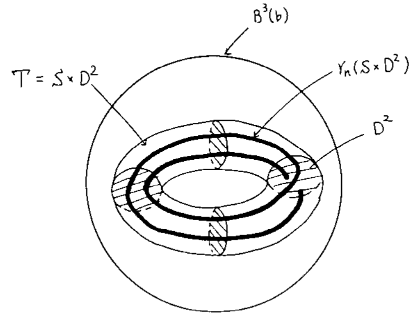
In this section, as a special case we show the following theorem.

Theorem 3.1. *Let $f : S \to S$ be any map of the unit circle S, then there is a homeomorphism $F : R^3 \to R^3$ such that (S, f) is contained in R^3, F is an extension of the shift map $\tilde{f} : (S, f) \to (S, f)$, and (S, f) is an attractor of F.*

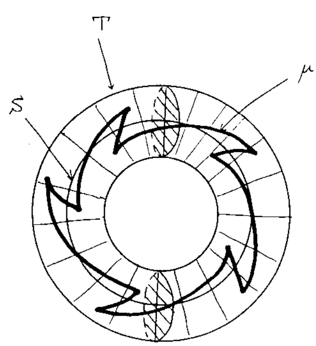
Proof of Theorem 3.1. Let $e : R \to S$ be the natural covering projection, i.e., $e(x) = exp(2\pi i x)$. Let $L(f) : R \to R$ be a lifting of f, i.e., $L(f) : R \to R$ is a map such that $e \cdot L(f) = f \cdot e$. By [7, (3.2)], we may assume that $|\deg(f)| = |L(f)(1) - L(f)(0)| \geq 2$. Also we may assume that $\deg(f) = n \geq 2$. Define a map $g : R \to R$ by $g(x) = L(f)(x)/n$. Clearly there is a map $f_1 : S \to S$ such that $e \cdot g = f_1 \cdot e$. As in the proof of [8, p.329], we can define a map $\mu : S \to S \times R$ by $\mu((e(t)) = (f_1(e(t)), g(t) - t)$ for each $t \in R$. Note that the map $g(t) - t$ has period 1, and hence μ is well defined. Then μ is an embedding (see [8]). In fact, if $0 \leq t < t' < 1$ and $g(t) - t = g(t') - t'$, then $0 < t' - t = g(t') - g(t) < 1$. Hence $f_1(e(t)) = e \cdot g(t) \neq e \cdot g(t') = f_1(e(t'))$ which implies that $\mu(e(t)) \neq \mu(e(t'))$. Therefore μ is an embedding.

We consider $S \subset R^2 \times \{0\} \subset R^3$, Let $T = S \times D^2$ be the solid torus which is naturally embedded in R^3 such that $S = S \times \{*\}$, where D^2 is a 2-ball and $* \in D^2 - \partial(D^2)$. Take a 3-ball $B^3(b)$ in R^3 such that $T \subset \mathrm{Int}(B^3(b))$. Since R is homeomorphic to a sufficiently small open interval, we may assume that $\mu(S) \subset T \cap (R^2 \times \{0\})$. Also, we may assume that $\mu : S \to T \cap (R^2 \times \{0\})$ satisfies $q \cdot \mu = f_1$, where $q : R^3 \to R^3$ is a near homeomorphism such that $q(S \times D^2) \subset S$, $q(s \times D^2) = s$ for $s \in S$ and $q|\mathrm{Cl}(R^3 - B^3(b)) = \mathrm{id}$. Then there is a homeomorphism $r_n : R^3 \to R^3$ such that $r_n(T) \subset \mathrm{Int}\, T$, $q \cdot r_n(e(t) \times D^2) =$

$e(n \cdot t)$ for $e(t) \in S(t \in R)$ and $r_n|\partial B^3(b) = \mathrm{id}$ (see Figure 1). By using the Schönflies theorem, we can choose a homeomorphism $k : B^3(b) \to B^3(b)$ such that $k|S = \mu, k|\partial B^3(b) = \mathrm{id}$ (see Figure 2). Put $h = q \cdot r_n \cdot k \cdot q : B^3(b) \to B^3(b)$. Note that $h|S = f$, $h|\partial B^3(b) = \mathrm{id}$, and h is a near homeomorphism. Then $(B^3(b), h) = B^3$ is a 3-ball. Put $F = \tilde{h}|B^3 - \partial B^3$. Then F is a desired homeomorphism. This completes the proof.



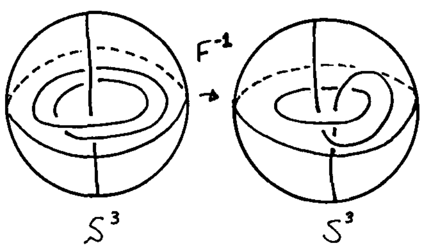
(Figure 1)



(Figure 2)

Also, we have the following corollary whose proof is essentially by Figure 3.

Corollary 3.2. *Let $f : S \to S$ be a map of the unit circle S with $|\deg(f)| \geq 1$, then there is a homeomorphism $F : S^3 \to S^3$ of the 3-sphere S^3 such that $(S, f) \subset S^3$, F is an extension of $\tilde{f}$, (S, f) is an attractor of F and if Λ' is the attractor of F^{-1}, then $F^{-1}|\Lambda' : \Lambda' \to \Lambda'$ is topologically conjugate to the shift map $\tilde{g} : (S, g) \to (S, g)$, where $g : S \to S$ is the natural covering map with* $\deg(g) = \deg(f) = n$. *In fact, Λ' is the n-adic solenoid.*



(Figure 3)

Note that there is a finite graph G which is naturally embedded into R^3 and a homeomorphism $f : G \to G$ such that there is no near homeomorphism $F : R^3 \to R^3$ which in an extension of f. Naturally, we have the following question: If $f : G \to G$ is a map of any finite graph G, does there exist a homeomorphism $F : R^3 \to R^3$ such that $(G, f) \subset R^3$, F is an extension of the shift map $\tilde{f} : (G, f) \to (G, f)$, and (G, f) is an attractor of F ?

4. A characterization of attractors

In [4], Günther and Segal proved that the class of compacta which can occur as attractors of flows on topological manifolds coincides with the class of finite dimensional compacta having the shape of a finite polyhedron. In [4], they asked the following problem: Is it possible to characterize attractors (of cascades) ? In this section, we answer to this problem.

Theorem 4.1. *Let $f : \Lambda \to \Lambda$ be a homeomorphism of a compactum Λ. Then the following are equivalent.*

1. *There is an Euclidean space E containing Λ and a homeomorphism $F : E \to E$ such that Λ is an attractor of F and F is an extension of f.*
2. *There is a topological manifold M containing Λ and a homeomorphism $F : M \to M$ such that Λ is an attractor of F and F is an extension of f.*
3. *There is a finite dimensional, locally compact ANR Y containing Λ and a map $F : Y \to Y$ such that F is an extension of f and there is an open neighborhood U of Λ in Y so that $F(Cl(U)) \subset U$ and $\cap_{n=1}^{\infty} F^n(U) = \Lambda$.*
4. *There is a compact polyhedron P and a map $g : P \to P$ such that f is topologically conjugate to the shift map $\tilde{g} : (P, g) \to (P, g)$.*
5. *There is a finite dimensional compact ANR X and a map $g' : X \to X$ such that f is topologically conjugate to the shift map $\tilde{g}' : (X, g') \to (X, g')$.*

PROOF. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. We show (3) $\Rightarrow$ (4). We may assume that Y is a closed subset of an Euclidean space E. Since Y is an ANR, there is a retraction $r : W \to Y$, where W is a neighborhood of Y in E. Take a compact polyhedron P in W such that $F(Cl(U)) \subset P$ and $r(P) \subset U$. Put $g = F \cdot r|P : P \to P$. Note that $\cap_{n=1}^{\infty} g^n(P) = \Lambda$. Define a map $K : (P, g) \to \Lambda$ by $K(x_1, x_2, ..,) = x_1$ (see [10, Theorem 37]). Then K is a homeomorphism and $f \cdot K = K \cdot \tilde{g}$. Hence f is topologically conjugate to $\tilde{g}$. (4) $\Rightarrow$ (5) is trivial. Finally, we show (5) $\Rightarrow$ (1). We may assume that X is in R^n. Then we can choose a compact polyhedron P containing X in R^n and a retraction $r : P \to X$. Put $g = g' \cdot r : P \to X \subset P$. Then $(P, g) = (X, g')$ and $\tilde{g} = \tilde{g}'$. Theorem 2.1 shows (5) $\Rightarrow$ (1). □

By using the proof (5) $\Rightarrow$ (1) of the above theorem, we can generalize Theorem 2.1 as follows.

Corollary 4.2. *If X is a compact ANR in R^n and $f : X \to X$ is any map, then there is a homeomorphism $F : R^{2n} \to R^{2n}$ such that (X, f) is contained in R^{2n}, F is an extension of the shift map $\tilde{f} : (X, f) \to (X, f)$ of f, and (X, f) is an attractor of F. Moreover, if X is an AR, then F can be chosen so that (X, f) is a global attractor of F.*

Corollary 4.3. *A compactum Λ is an attractor on a topological manifold if and only if there is a map $g : X \to X$ of a finite dimensional compact ANR X such that Λ is homeomorphic to (X, g).*

Remark. The result of Günther and Segal [4] implies that a compactum Λ is an attractor of a flow on a manifold if and only if there is a map $g : X \to X$ of a finite dimensional compact ANR X such that g is homotopic to the identity 1_X and Λ is homeomorphic to (X, g).

Proposition 4.4. *Let Λ be a compactum. If Λ admits a homeomorphism $F : M \to M$ of a topological manifold M such that Λ is an attractor of F, then one has that rank($H^k(\Lambda)$) is finite, where $H^k(\Lambda)$ denotes the k-dimensional Čech cohomology group.*

PROOF. By Theorem 4.1, there is a compact polyhedron P and a map $g : P \to P$ such that (P, g) is homeomorphic to Λ. Note that $H^k(\Lambda) = \text{dirlim}\{H^k(P) \to H^k(P) \to \cdots\}$. Since $H^k(P)$ is finitely generated, rank$(H^k(\Lambda))$ is finite. □

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REFERENCES

[1] M. Barge and M. Brown, Problems in Dynamics on Continua, *Contemporary Math.* 117 (1991), 177-182.

[2] M. Barge and J. Martin, The construction of global attractors, *Proc. Amer. Math. Soc.*, 110 (1990), 523-525.

[3] M. Brown, Some applications of an approximation theorem for inverse limits, *Proc. Amer. Math. Soc.*, 11 (1960), 476-483.

[4] B. Günther and J. Segal, Every attractor of a flow on a manifold has the shape of finite polyhedron, *Proc. Amer. Math. Soc.* 119 (1993), 321-329.

[5] L. F. P. Hudson, Piecewise Linear Topology, *Menlo Park, Benjamin*, 1969.

[6] J. R. Isbell, Embeddings of inverse limits, *Ann. of Math.*, 70 (1959), 73-84.

[7] H. Kato, Everywhere chaotic homeomorphisms on manifolds and k-dimensional Menger manifolds," *Topology Appl.*, 72 (1996), 1-17.

[8] M. C. McCord, Embedding $\mathcal{P}$-like compacta in manifolds, *Canad. J. Math.*, 19 (1967), 321-332.

[9] T. B. Rushing, Topological embeddings, *Academic Press*, New York and London 52, 1973.

[10] R. M. Schori, Chaos: An introduction to some topological aspects, *Contemporary Math.* 117 (1991), 149-161.

[11] W. Szczechla, Inverse limits of certain interval mappings as attractors in two dimensions, *Fund. Math.*, 133 (1989), 1-23.

[12] J. H. C. Whitehead, Simplicial spaces, nuclei, and m-groups, *Proc. London Math. Soc.* 45 (1939), 243-327.

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