Houston Journal of Mathematics

Volume 25, No. 4, 1999

LOCAL REGULARITY RESULTS FOR SOME PARABOLIC EQUATIONS

MARIA MICHAELA PORZIO
Communicated by Haim Brezis

Abstract. In this paper we prove the local L^s regularity (where s depends on the summability of the data) for local "unbounded" weak solutions of a class of nonlinear parabolic equations including the p-Laplacian equation.

1. Introduction and main results

The aim of this paper is to prove L^s_{loc}-regularity for unbounded weak solutions of nonlinear parabolic equations whose prototype is

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \quad \text{in} \quad \mathcal{D}'(\Omega_T), \tag{1.1}$$

where $p > 1$ and s is finite and depends on the local summability of the data f_i. Here Ω is an open bounded set in $\mathbb{R}^{\mathbb{N}}$ and for $0 < T < +\infty$ we have set $\Omega_T \equiv \Omega \times (0,T)$. By a local weak solution of (1.1) in Ω_T we mean a measurable function u satisfying

$$u \in C_{loc}(0,T; L^2_{loc}(\Omega)) \cap L^p_{loc}(0,T; W^{1,p}_{loc}(\Omega)), \tag{1.2}$$

1991 *Mathematics Subject Classification.* 35B45, 35B65, 35k55.

Key words and phrases. Local regularity, nonlinear parabolic equations, unbounded solutions.

.

and such that for every compact subset K of Ω and for every subinterval $[t_1, t_2]$ of $(0, T]$ it results

$$\begin{aligned} &\int_K u\psi \,|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \left\{-u\psi_t + |Du|^{p-2}DuD\psi\right\} dxd\tau = \\ &= \sum_{i=1}^{N}\int_{t_1}^{t_2}\int_K f_i \frac{\partial\psi}{\partial x_i} dxd\tau, \end{aligned} \tag{1.3}$$

for all testing functions $\psi \in W^{1,2}_{loc}(0,T;L^2(K)) \cap L^p_{loc}(0,T;W^{1,p}_0(K))$.
There is an extensive literature concerned with this kind of problems when the solution u of (1.1) is a global solution that takes sufficiently regular Dirichlet data on the parabolic boundary of Ω_T. First of all we recall that Aronson and Serrin in [2] have proved that if $p = 2$ and $f_i \in L^r(0,T;L^q(\Omega))$ where r and q satisfy

$$\frac{2}{r} + \frac{N}{q} < 1, \tag{1.4}$$

then every weak solution u of

$$\begin{cases} u_t - div(a(x,t,u,Du)) = -\sum_{i=1}^N \frac{\partial f_i}{\partial x_i} & \text{in } D'(\Omega_T) \\ u = g(x,t) & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

is bounded in Ω_T if u_0 and g are bounded and $-div(a(x,t,u,Du))$ is a coercive, uniformly elliptic operator acting from $L^p(0,T;W^{1,p}_0(\Omega))$ to its dual space $L^{p'}(0,T;W^{-1,p'}(\Omega))$. We notice that no linearity assumption is done in [2] on $a(x,t,u,Du)$ and thus Aronson and Serrin extend results for linear equations proved earlier by Ladyzenskaja and Ural'ceva (see [15]), Guglielmino (see [10]), Aronson (see [1]), Ivanov (see [11]) and Ivanov, Ladyzenskaja, Treskunov and Ural'ceva (see [12]). The same boundedness result holds in the general case $p \neq 2$ if

$$\frac{p}{r} + \frac{N}{q} < p - 1. \tag{1.6}$$

Besides, if the operator in divergence form is of the kind

$$-\text{div}(a(x,t,u,Du)) = -\text{div}(A(x,t)Du), \tag{1.7}$$

and suppose that r and q don't satisfy (1.6), that is

$$\frac{2}{r} + \frac{N}{q} > 1, \tag{1.8}$$

then every weak solution of (1.7) belongs to $L^s(\Omega_T)$, where

$$s = \frac{(N+2)qr}{Nr + 2q - qr}, \tag{1.9}$$

if u_0 and g are bounded (see [14], theorem 9.1).

Recently Boccardo, Dall'Aglio, Gallouët and Orsina have proved in [4], (see also [3]), the previous result if $p \neq 2$. More precisely if it results

$$p - 1 < \frac{N}{q} + \frac{p}{r} \leq \frac{N}{r} + p - 1, \quad r \geq p', \quad q \geq p', \tag{1.10}$$

and $g = u_0 = 0$ then $u \in L^s(\Omega_T)$, where

$$s = \frac{(N+2)(p-1)qr + N(r-q)(p-2)}{Nr - pq(r-1) + qr}. \tag{1.11}$$

In absence of information about the behaviour of a solution u on the parabolic boundary $\Omega \times [0, T)$, it is still possible to show that u is locally bounded when (1.6) holds. We refer to [11], [12], [13] and [17] for the linear case and to [2], [14] and [7] for the nonlinear case .

We notice that when $q = r$ condition (1.6) becomes

$$r > \frac{N+p}{p-1}, \tag{1.12}$$

and that (1.12) is sharp in order to have local bounded weak solutions (and Hölder continuity) in the sense that if the opposite relation holds, i.e.

$$r < \frac{N+p}{p-1}, \tag{1.13}$$

then for example the heat equation may have unbounded local weak solutions (see [7]).

To our knowledge the only information available on the local summability of the local solutions when (1.12) is violated regards operators in divergence form as in (1.7) (see again [14]). Here we prove that if (1.13) holds then every local weak solution of (1.1) belongs to L^s_{loc}, where s is given by formula (1.11). We notice especially that also this kind of regularity has a "purely local" thrust and so we don't need any assumption on the behaviour of the solution on the boundary of Ω_T and any information on the initial datum u_0.

More in details our results are the following.
Let f_i and $g(x,t)$ functions belonging to the space $L^r_{loc}(\Omega_T)$, $r > 1$ and let $a(x,t,s,\xi) : \Omega_T \times \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, a Carathéodory function satisfying

$$a(x,t,s,\xi)\xi \geq m_0|\xi|^p, \tag{1.14}$$

$$|a(x,t,s,\xi)| \leq m_1 \left[|\xi|^{p-1} + |s|^{p-1} + g(x,t)\right], \tag{1.15}$$

where $p > 1$, m_0 and m_1 are positive constants.
Consider the following parabolic equation

$$u_t - \operatorname{div}(a(x,t,u,Du)) = -\sum_{i=1}^{N} \frac{\partial f_i}{\partial x_i} \quad \text{in} \quad D'(\Omega_T). \tag{1.16}$$

Theorem 1.1. *Let (1.14) and (1.15) hold with $p > \frac{2N}{N+2}$. If the coefficient r satisfies*

$$p' < r < \frac{N+p}{p-1}, \tag{1.17}$$

then every local weak solution u of (1.16) belongs to $L^s_{loc}(\Omega_T)$, where

$$s = \frac{(N+2)r(p-1)}{N+p-r(p-1)}. \tag{1.18}$$

Remark 1.1. *For sake of simplicity we have considered only the case when f_i belong to $L^r_{loc}(0,T;L^q_{loc}(\Omega))$ with $q = r$, but theorem 1.1 can be extended to the case $q \neq r$. Besides the value of s in (1.18) is the same obtained in (1.11) when $q = r$.*

We notice that these regularity results, as in the global case (see [4]), can be proved also when in the right hand side of (1.16) a function f_0 (not in divergence form) appears. We haven't treated this case not to complicate further the proof.

The proof of theorem 1.1 is in section 4 and is completely different from those given in [4] and in [14]. The main tool is played by a double summation technique whose role is to "adjust" the powers in an integral estimate of energy type (derived in section 3) a suitable use of "parabolic cylinder" and an iterative argument that permits to conclude the process in a finite number of steps.

We recall that a double summation technique appeared in [6] and then in [9] in the framework of the elliptic equations. Here we adapted it to the evolution case. The main difficulty was how to handle the term involving the time derivative (see lemma 2.3 in section 2).

Remark 1.2. *The proof of theorem 1.1 shows that really every function satisfying a certain integral inequality belongs to L^s_{loc} also if such a function does not resolve any parabolic equation. More precisely what happens is the following.*

Fix $(x_0, t_0) \in \Omega_T$ and let $1 \geq R_1 > 0$ be so small that $\overline{B_{R_1}} \times [t_0, t_0 + R_1^p] \subset \Omega_T$, where B_{R_1} is the ball of radius R_1 centered at x_0. We denote with $Q_r(\tau, t)$ the cylinder $B_r \times (\tau, t)$ and with Q_r the "parabolic" cylinder $Q_r(t_0, t_0 + r^p)$. Take $0 < \rho < R \leq R_1$ and construct a piecewise smooth cut-off function $\eta : \Omega_T \to \mathbb{R}$ such that

$$\left\{ \begin{array}{l} \operatorname{supp}(\eta) \subset Q_R, \quad 0 \leq \eta \leq 1, \quad \eta = 1 \text{ in } Q_\rho, \\ |D\eta| \leq 2(R-\rho)^{-1}, \quad |\eta_t| \leq 2^p (R-\rho)^{-p}. \end{array} \right. \tag{1.19}$$

Moreover, for $k > 0$ let

$$A_k = \{(x, \tau) \in \Omega_T : |u(x, \tau)| > k\}, \tag{1.20}$$

$$A^{\tau,t}_{k,r} \equiv A_k \cap Q_r(\tau, t), \quad A_{k,r} = A_k \cap Q_r. \tag{1.21}$$

Theorem 1.1. *Let $p > \frac{2N}{N+2}$ and let F be a function in $L^r_{loc}(\Omega_T)$, where r is as in (1.17). Assume that u is a function belonging to $C_{loc}(0, T; L^2_{loc}(\Omega)) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(\Omega))$ satisfying, for every fixed $(x_0, t_0) \in \Omega_T$, $t' \in (t_0, t_0 + R^p)$*

$$\begin{aligned} &\int_{B_R} \eta^p \left[u - T_k(u)\right]^2 (t') dx + \int\int_{A^{t_0,t'}_{k,R}} \eta^p |Du|^p \leq \\ &c \int\int_{A^{t_0,t'}_{k,R}} \left(\frac{|u|^2}{(R-\rho)^\alpha} + \frac{|u|^p}{(R-\rho)^\beta} + |F|^{p'} \right), \end{aligned} \tag{1.22}$$

where $T_k(u)$ is the usual truncation of u at levels $\pm k$, that is

$$T_k(f) = max\{-k, min\{k, f\}\}, \tag{1.23}$$

and α, β, and c are non-negative constants. Then u is in $L^s_{loc}(\Omega_T)$, where s given by formula (1.18).

2. Preliminary results

We briefly recall some lemmas that will play an essential role in the proof of theorem 1.1.

The first is a very simple and useful lemma for real functions of one variable, the second is an immersion theorem of Gagliardo-Nirenberg type and the third is a rather technical lemma that, as said in the introduction, permits us to extend the "summation techniques" to the evolution case.

Lemma 2.1. *Let $f(\tau)$ be a non-negative bounded function defined for $0 \leq R_0 \leq \tau \leq R_1$. Suppose that for $R_0 \leq \tau < t \leq R_1$ we have*

$$f(\tau) \leq A(t-\tau)^{-\alpha} + B + \theta f(t) \tag{2.24}$$

where A, B, α, θ are non-negative constants, and $\theta < 1$. Then there exists a constant c, depending only on α and θ such that for every $\rho, R, R_0 \leq \rho < R \leq R_1$ we have

$$f(\rho) \leq c\left[A(R-\rho)^{-\alpha} + B\right]. \tag{2.25}$$

The proof of lemma 2.1 is very easy and can be found in [8], pp.161, Lemma 3.1.

Lemma 2.2. *Let $v \in L^\infty(0,T;L^h(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. Then $v \in L^q(\Omega_T)$ where*

$$q = p\frac{N+h}{N}\ , \tag{2.26}$$

and there exists a constant β that depends only upon N, p and h such that

$$\int\!\!\int_{\Omega_T} |v|^q \leq \beta\left(\sup_{0<t<T}\int_\Omega |v|^h(x,t)dx\right)^{\frac{p}{N}} \int\!\!\int_{\Omega_T} |Dv|^p. \tag{2.27}$$

For the proof see Proposition 3.1 of [7].

Lemma 2.3. *For every fixed positive α there exists a constant $n_0 = n_0(\alpha) \in \mathbb{N}$ such that for every function f in $L^2(A)$, for every positive bounded function g, for every $j \geq n_0$ and for every $n \in \mathbb{N}, \ltimes \geq \ltimes_{\nvdash}$, it results*

$$\begin{aligned}\int_{B(n)}\sum_{k=0}^{n}(2+k)^{\alpha-1}(f(x)-T_k(f))^2\delta_k g(x)dx \geq \\ \frac{1}{\alpha(\alpha+1)(\alpha+2)}\int_{B(n)}|T_{j+1}(f)|^{\alpha+2}g(x)dx,\end{aligned} \tag{2.28}$$

where

$$B(n) = \{x \in A : n \leq |f(x)| < n+1\}, \qquad \delta_k = \begin{cases} 1 & \text{if } k \leq j, \\ 0 & \text{if } k > j, \end{cases} \tag{2.29}$$

and $T_k(f)$ is the truncation of f at levels $\pm k$ defined in (1.23).

Proof. Let $n \in \mathbb{N}$ be arbitrary fixed. Denote with $\chi(B)$ the characteristic function of the set B. For every $k \in \mathbb{N}$ it results

$$(f - T_k(f))^2 = (|f|-k)^2\,\chi\,(|f|>k)\,, \tag{2.30}$$

and so it follows

(2.31)
$$\int_{B(n)} \sum_{k=0}^{n} (2+k)^{\alpha-1} \left(f - T_k(f)\right)^2 \delta_k g(x) = \int_{B(n)} \sum_{k=0}^{n} (2+k)^{\alpha-1} \left(|f| - k\right)^2 \delta_k g(x).$$

We prove now that there exists a constant $n_0(\alpha) \in \mathbb{N}$ such that the following inequality holds

$$\sum_{k=0}^{m} (2+k)^{\alpha-1} (m-k)^2 \geq \frac{(m+1)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}, \quad \forall m \geq n_0(\alpha). \tag{2.32}$$

Notice that (2.31) and (2.32) imply (2.28). As a matter of fact if $n \geq j+1$ then on $B(n)$ we have $|f| \geq n \geq j+1$ that with (2.32) implies

$$\sum_{k=0}^{n} (2+k)^{\alpha-1} (|f|-k)^2 \delta_k \geq \sum_{k=0}^{j} (2+k)^{\alpha-1} (j-k)^2 \geq \frac{(j+1)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} =$$

$$\frac{|T_{j+1}(f)|^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}, \quad \forall j \geq n_0(\alpha). \tag{2.33}$$

Otherwise, if $n < j+1$, then again using (2.32) and observing that on $B(n)$ we have $n+1 > |f|$, it results

$$\sum_{k=0}^{n} (2+k)^{\alpha-1} (|f|-k)^2 \delta_k \geq \sum_{k=0}^{n} (2+k)^{\alpha-1} (n-k)^2 \geq \frac{|f|^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} =$$

$$\frac{|T_{j+1}(f)|^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}, \quad \forall n \geq n_0(\alpha). \tag{2.34}$$

To prove (2.32) it is sufficient to notice that if $\alpha \geq 1$ then we have

$$\begin{aligned} &\sum_{k=0}^{m} (2+k)^{\alpha-1} (m-k)^2 \geq \sum_{k=0}^{m-1} \int_{k-1}^{k} (2+s)^{\alpha-1} [m-(s+1)]^2 ds = \\ &\int_{-1}^{m-1} (2+s)^{\alpha-1} [m-(s+1)]^2 ds = \\ &= \begin{cases} \frac{2(m+1)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} - \left[\frac{m^2}{\alpha} + \frac{2m}{\alpha(\alpha+1)} + \frac{2}{\alpha(\alpha+1)(\alpha+2)}\right], & \text{if} \quad \alpha > 1, \\ \frac{m^3}{3}, & \text{if} \quad \alpha = 1. \end{cases} \end{aligned} \tag{2.35}$$

Otherwise, if $0 < \alpha < 1$, it results

$$\sum_{k=0}^{m}(2+k)^{\alpha-1}(m-k)^2 \geq \sum_{k=0}^{m-1}\int_k^{k+1}\frac{(m-s)^2}{(2+s)^{1-\alpha}}ds = \frac{2(m+2)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} - \left[\frac{2^\alpha m^2}{\alpha} + \frac{2m2^{\alpha+1}}{\alpha(\alpha+1)} + \frac{2^{\alpha+3}}{\alpha(\alpha+1)(\alpha+2)}\right]. \tag{2.36}$$

□

3. An integral inequality

We prove now an integral inequality of the type (1.22) that will be essential in the proof of theorem 1.1. More precisely, using the notation introduced in section 1, we have the following result.

Lemma 3.1. *Under the assumptions of theorem 1.1 every local weak solution u of (1.16) satisfies the following estimate*

$$\frac{1}{2}\int_{B_R}\eta^p\left[u-T_k(u)\right]^2(t')dx + \frac{m_0}{2}\int\int_{A_{k,R}^{t_0,t'}}\eta^p|Du|^p \leq \int\int_{A_{k,R}^{t_0,t'}}\left(2^p p\frac{|u|^2}{(R-\rho)^p} + \frac{c_2|u|^p}{(R-\rho)^p} + c_3|F|^{p'}\right), \tag{3.37}$$

for every fixed $(x_0,t_0) \in \Omega_T$ and $t' \in (t_0, t_0+R^p)$, where the constants $c_i,\ i = 1,2,3$, depend only upon the data and we have set $|F|^{p'} = |f|^{p'} + |g|^{p'}$.

Proof. Using $v = \eta^p(u-T_k(u))$ as a test function in (1.16) and integrating on $Q_R(t_0,t')$, where $t' \in (t_0, t_0+R^p)$ is arbitrary fixed, we have (the use of v as a test function can be made rigorous using the Steklov averaging process, see for example [7] pp. 18 and pp. 25 or [14] pp.85)

$$\frac{1}{2}\int_{B_R}\eta^p[u-T_k(u)]^2(t')dx - p\int\int_{Q_R(t_0,t')}\eta^{p-1}(u-T_k(u))^2\eta_t dxd\tau + + \int\int_{Q_R(t_0,t')}a(x,\tau,u,Du)\cdot D[\eta^p(u-T_k(u))]dxd\tau = = \int\int_{Q_R(t_0,t')}\sum_{i=1}^{N}f_i\frac{\partial[\eta^p(u-T_k(u))]}{\partial x_i}dxd\tau. \tag{3.38}$$

We estimate now the integrals in (3.38).

Using (1.19), it results

$$-p\int\int_{Q_R(t_0,t')}\eta^{p-1}(u-T_k(u))^2\eta_t dxd\tau \geq \frac{-2^p p}{(R-\rho)^p}\int\int_{Q_R(t_0,t')}(u-T_k(u))^2\eta^{p-1}.$$

Besides, using assumptions (1.14) and (1.15), we deduce

$$\begin{aligned}
&\int\int_{Q_R(t_0,t')} a(x,\tau,u,Du)\cdot D[\eta^p(u-T_k(u))]dxd\tau \geq m_0\int\int_{A_{k,R}^{t_0,t'}}\eta^p|Du|^p\\
&-c_1\int\int_{A_{k,R,\rho}}\frac{\eta^{p-1}}{R-\rho}[|Du|^{p-1}+|u|^{p-1}+g(x,t)]|u-T_k(u)|\geq\\
&\geq m_0\int\int_{A_{k,R}^{t_0,t'}}\eta^p|Du|^p-\epsilon c_1\int\int_{A_{k,R,\rho}}\eta^p|Du|^p+\\
&-c_1\int\int_{A_{k,R,\rho}}\left(|u|^p+|g(x,t)|^{p'}\right)\eta^p+ \qquad (3.39)\\
&-c_1[2+C(\epsilon)]\int\int_{A_{k,R,\rho}}\frac{|u-T_k(u)|^p}{(R-\rho)^p},
\end{aligned}$$

where $A_{k,R,\rho}=A_{k,R}^{t_0,t'}\setminus A_{k,\rho}$, ϵ is a positive constant to be determined, $C(\epsilon)=\epsilon^{1/(p-1)}=c(\epsilon,p)$ and $c_1=2pm_1=c(p,m_1)$. Finally, using Young inequality we can deal with the right-hand side of (3.38) as follows

$$\begin{aligned}
&\int\int_{Q_R(t_0,t')}\sum_{i=1}^N f_i\frac{\partial[\eta^p(u-T_k(u))]}{\partial x_i}\leq\int\int_{A_{k,R,\rho}}2p\frac{|f||u-T_k(u)|}{(R-\rho)}\\
&+\int\int_{A_{k,R}^{t_0,t'}}|f||Du|\eta^p\leq\int\int_{A_{k,R,\rho}}2p[|f|^{p'}+\frac{|u|^p}{(R-\rho)^p}]\\
&+\epsilon\int\int_{A_{k,R}^{t_0,t'}}\eta^p|Du|^p+C(\epsilon)\int\int_{A_{k,R}^{t_0,t'}}|f|^{p'}, \qquad (3.40)
\end{aligned}$$

where ϵ and $C(\epsilon)$ are as in (3.39) and $f\equiv(f_1,f_2,\cdots,f_N)$.
Using the previous estimates in (3.38) we have

$$\begin{aligned}
&\frac{1}{2}\int_{B_R}\eta^p[u-T_k(u)]^2(t')+m_0\int\int_{A_{k,R}^{t_0,t'}}\eta^p|Du|^p\leq\\
&\frac{2^p p}{(R-\rho)^p}\int\int_{Q_R(t_0,t')}(u-T_k(u))^2\eta^{p-1}+\epsilon(c_1+1)\int\int_{A_{k,R}^{t_0,t'}}\eta^p|Du|^p \qquad (3.41)\\
&+\frac{c_2}{(R-\rho)^p}\int\int_{A_{k,R,\rho}}|u|^p+c_3\int\int_{A_{k,R}^{t_0,t'}}|F|^{p'},
\end{aligned}$$

where $c_2 = [c_1(3 + C(\epsilon)) + 2p] = c(p, m_1, \epsilon)$, $c_3 = 2p + C(\epsilon) + c_1 = c(p, m_1, \epsilon)$ and $|F|^{p'} = |f|^{p'} + |g(x,t)|^{p'}$.
Thus choosing $\epsilon = \frac{m_0}{2(c_1+1)}$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_{B_R} \eta^p [u - T_k(u)]^2(t')dx + \frac{m_0}{2}\int\int_{A_{k,R}^{t_0,t'}} \eta^p |Du|^p \leq \\ &\frac{2^p p}{(R-\rho)^p}\int\int_{Q_R(t_0,t')} (u - T_k(u))^2 \eta^{p-1} + \frac{c_2}{(R-\rho)^p}\int\int_{A_{k,R}^{t_0,t'}} |u|^p + \\ &+ c_3 \int\int_{A_{k,R}^{t_0,t'}} |F|^{p'}, \end{aligned} \tag{3.42}$$

from which (3.37) follows. □

4. Proof of theorems 1.1 and 1.2

Proof of theorem 1.1 Let u be a local weak solution of (1.16). By lemma 3.1, u satisfies inequality (3.37). Let $m > 0$ to be chosen later (m=c(N, p, r)) and $j \geq n_0$ arbitrary fixed, where $n_0 = n_0(pm)$ is as in lemma 2.3. Analogously to [6] we multiply inequality (3.37) by $(2+k)^{pm-1}\delta_k$, where δ_k is as in (2.29) and we sum on k. We obtain

$$\begin{aligned} &\frac{1}{2}\sum_{k=0}^{+\infty}(2+k)^{pm-1}\delta_k \int_{B_R} \eta^p [u - T_k(u)]^2(t')dx + \\ &+\frac{m_0}{2}\sum_{k=0}^{+\infty}(2+k)^{pm-1}\delta_k \int\int_{A_{k,R}^{t_0,t'}} \eta^p |Du|^p \leq \\ &\sum_{k=0}^{+\infty}(2+k)^{pm-1}\delta_k \int\int_{A_{k,R}^{t_0,t'}} \left\{ \frac{2^p p}{(R-\rho)^p}|u|^2 + \frac{c_2}{(R-\rho)^p}|u|^p + c_3|F|^{p'} \right\}. \end{aligned} \tag{4.43}$$

Using the equalities

$$\sum_{k=0}^{+\infty} a_k\delta_k \sum_{n=k}^{+\infty}\int_{B(n)} |\psi| = \sum_{n=0}^{+\infty}\int_{B(n)} |\psi| \sum_{k=0}^{n} a_k\delta_k, \tag{4.44}$$

$$\sum_{k=0}^{+\infty} a_k\delta_k \sum_{n=k}^{+\infty}\int_{B(n)} |\psi_k| = \sum_{n=0}^{+\infty}\int_{B(n)} \sum_{k=0}^{n} a_k\delta_k|\psi_k|, \tag{4.45}$$

the previous inequality becomes

$$\frac{1}{2}\sum_{n=0}^{+\infty}\int_{B_R\times\{t'\}\cap B(n)}\sum_{k=0}^{n}(2+k)^{pm-1}\delta_k\eta^p[u-T_k(u)]^2(t')+$$

$$\frac{m_0}{2}\sum_{n=0}^{+\infty}\int\int_{Q_R(t_0,t')\cap B(n)}\eta^p|Du|^p\sum_{k=0}^{n}(2+k)^{pm-1}\delta_k\leq \tag{4.46}$$

$$\sum_{n=0}^{+\infty}\int\int_{Q_R(t_0,t')\cap B(n)}\left\{\frac{2^p p|u|^2}{(R-\rho)^p}+\frac{c_2|u|^p}{(R-\rho)^p}+c_3|F|^{p'}\right\}\sum_{k=0}^{n}(2+k)^{pm-1}\delta_k,$$

where here $B(n)=\{(x,\tau)\in Q_R: n\leq|u(x,\tau)|<n+1\}$. We estimate now the terms in (4.46). Using lemma 2.3 (with $f=u,\ g=\eta^p$ and $\alpha=pm$) in the first term in the left-hand side we have

$$\frac{1}{2}\sum_{n=0}^{+\infty}\int_{B_R\times\{t'\}\cap B(n)}\sum_{k=0}^{n}(2+k)^{pm-1}\delta_k\eta^p[u-T_k(u)]^2(t')\geq \tag{4.47}$$

$$\frac{1}{2}\sum_{n=n_0}^{+\infty}\frac{1}{pm(pm+1)(pm+2)}\int_{B_R\times\{t'\}\cap B(n)}|T_{j+1}(u)|^{pm+2}\eta^p dx\geq$$

$$\frac{1}{2pm(pm+1)(pm+2)}\int_{B_R\times\{t'\}}|T_{j+1}(u)|^{pm+2}\eta^p dx-c_0|B_R|,$$

where $c_0=c(pm)\equiv\frac{(n_0+1)^{pm+2}}{2pm(pm+1)(pm+2)}$. Let us define $\gamma=\max\left\{\frac{p(m+1)}{pm+2},1\right\}$. Then the second term in the left-hand side of (4.46) can be estimate as follows

$$\frac{m_0}{2}\sum_{n=0}^{+\infty}\int\int_{Q_R(t_0,t')\cap B(n)}\eta^p|Du|^p\sum_{k=0}^{n}(2+k)^{pm-1}\delta_k\geq$$

$$c_4\sum_{n=0}^{j}\int\int_{Q_R(t_0,t')\cap B(n)}\eta^p|Du|^p(1+n)^{pm}\geq$$

$$c_4\sum_{n=0}^{j}\int\int_{Q_R(t_0,t')\cap B(n)}\eta^{p\gamma}|Du|^p|u|^{pm}=$$

$$=c_5\int\int_{Q_R(t_0,t')\cap\{|u|<j+i\}}|\eta^\gamma D\left[|T_{j+1}(u)|^m\cdot T_{j+1}(u)\right]|^p$$

$$\geq\frac{c_5}{2^p}\int\int_{Q_R(t_0,t')}|D\left[|T_{j+1}(u)|^m\cdot T_{j+1}(u)\eta^\gamma\right]|^p+$$

$$-\frac{(2\gamma)^p c_5}{(R-\rho)^p}\int\int_{Q_R(t_0,t')}|T_{j+1}(u)|^{(m+1)p},$$

where $c_4 = \frac{m_0}{2pm}(1 - 2^{-pm}) = c(m_0, pm)$ and $c_5 = c_4(m+1)^{-p} = c(m_0, p, m)$. In the same way we can rewrite the right-hand side of (4.46) as

$$\sum_{n=0}^{+\infty} \int\int_{Q_R(t_0,t')\cap B(n)} \left\{ \frac{2^p p|u|^2}{(R-\rho)^p} + \frac{c_2|u|^p}{(R-\rho)^p} + c_3|F|^{p'} \right\} \sum_{k=0}^{n} (2+k)^{pm-1}\delta_k$$

$$\leq \sum_{n=0}^{j} \int\int_{Q_R(t_0,t')\cap B(n)} \left\{ \frac{2^p p|u|^2}{(R-\rho)^p} + \frac{c_2|u|^p}{(R-\rho)^p} + c_3|F|^{p'} \right\} c_6(1+n)^{pm} +$$

$$\sum_{n=j+1}^{+\infty} \int\int_{Q_R(t_0,t')\cap B(n)} \left\{ \frac{2^p p|u|^2}{(R-\rho)^p} + \frac{c_2|u|^p}{(R-\rho)^p} + c_3|F|^{p'} \right\} c_6(1+j)^{pm} \leq$$

$$c_6 \int\int_{Q_R(t_0,t')} \left\{ \frac{2^p p|u|^2}{(R-\rho)^p} + \frac{c_2|u|^p}{(R-\rho)^p} + c_3|F|^{p'} \right\} (1+|T_{j+1}(u)|)^{pm},$$

where $c_6 = \frac{2^{pm}}{\min\{pm,1\}} = c(pm)$. Using the previous estimates in (4.46) we obtain

$$\begin{aligned} & c_7 \int_{B_R\times\{t'\}} |T_{j+1}(u)|^{pm+2}\eta^p dx + \\ & \frac{c_5}{2^p} \int\int_{Q_R(t_0,t')} |D\left[|T_{j+1}(u)|^m \cdot T_{j+1}(u)\eta^\gamma\right]|^p \\ & \leq c_0|B_R| + \frac{c_8}{(R-\rho)^p} \int\int_{Q_R(t_0,t')} |T_{j+1}(u)|^{(m+1)p} + \\ & c_6 \int\int_{Q_R(t_0,t')} \left\{ \frac{2^p p|u|^2}{(R-\rho)^p} + \frac{c_2|u|^p}{(R-\rho)^p} + c_3|F|^{p'} \right\} (1+|T_{j+1}(u)|)^{pm}, \end{aligned} \tag{4.48}$$

where $c_7 = [2pm(pm+1)(pm+2)]^{-1} = c(pm)$ and $c_8 = (2\gamma)^p c_5 = c(p, m_0, m)$. Recalling that $t' \in (t_0, t_0 + R^p)$ is arbitrary fixed and that by the definition of γ it follows that $\eta^{p/\nu} \geq \eta^\gamma$, where we have set $\nu = \frac{pm+2}{m+1}$, we derive

$$\begin{aligned} & \min\{c_7, \frac{c_5}{2^p}\} \sup_{\tau\in(t_0,t_0+R^p)} \int_{B_R} ||T_{j+1}(u)|^m \cdot T_{j+1}(u)\eta^\gamma|^\nu(\tau)dx + \\ & \min\{c_7, \frac{c_5}{2^p}\} \int\int_{Q_R} |D\left[|T_{j+1}(u)|^m \cdot T_{j+1}(u)\eta^\gamma\right]|^p \leq \\ & \leq c_0|B_R| + \frac{c_8}{(R-\rho)^p} \int\int_{Q_R} |T_{j+1}(u)|^{(m+1)p} + \\ & c_6 \int\int_{Q_R} \left\{ \frac{2^p p|u|^2 + c_2|u|^p}{(R-\rho)^p} + c_3|F|^{p'} \right\} (1+|T_{j+1}(u)|)^{pm}. \end{aligned} \tag{4.49}$$

Applying lemma 2.2 and the previous estimate we obtain

$$
\begin{aligned}
(4.50)\qquad & \iint_{Q_R} ||T_{j+1}(u)|^m \cdot T_{j+1}(u)\eta^\gamma|^q dx d\tau \le \\
& \beta \left(\sup_{\tau\in(t_0,t_0+R^p)} \int_{B_R} |v|^\nu(\tau)dx \right)^{p/N} \iint_{Q_R} |Dv|^p \le \\
& c_9 \Bigg[c_0|B_R| + \frac{c_8}{(R-\rho)^p} \iint_{Q_R} |T_{j+1}(u)|^{(m+1)p} + \\
& c_6 \iint_{Q_R} \left\{ \frac{2^p p|u|^2 + c_2|u|^p}{(R-\rho)^p} + c_3|F|^{p'} \right\} (1+|T_{j+1}(u)|)^{pm} \Bigg]^{1+p/N},
\end{aligned}
$$

where $q = p\frac{N+\nu}{N}$, $c_9 = \beta^{-1} min\{c_7, c_5 2^{-p}\}^{1+p/N} = c(p,N,m,m_0)$, β is as in lemma 2.2 and $v = |T_{j+1}(u)|^m \cdot T_{j+1}(u)\eta^\gamma$. Since it results

$$
(4.51)\qquad \frac{c_8}{(R-\rho)^p} \iint_{Q_R} |T_{j+1}(u)|^{(m+1)p} \le \iint_{Q_R} \frac{c_8|u|^p}{(R-\rho)^p} |T_{j+1}(u)|^{pm},
$$

using (4.51) in (4.50) we deduce

$$
\begin{aligned}
(4.52)\qquad & \iint_{Q_R} |T_{j+1}(u)|^{q(m+1)} \eta^{\gamma q} dx d\tau \le \\
& c_{10} \left(c_0|B_R|\right)^{1+p/N} + c_{11} \left(\iint_{Q_R} \frac{2^p p|u|^2}{(R-\rho)^p} (1+|T_{j+1}(u)|)^{pm} \right)^{1+p/N} + \\
& c_{12} \left(\iint_{Q_R} \frac{|u|^p}{(R-\rho)^p} (1+|T_{j+1}(u)|)^{pm} \right)^{1+p/N} + \\
& c_{13} \left(\iint_{Q_R} |F|^{p'} (1+|T_{j+1}(u)|)^{pm} \right)^{1+p/N},
\end{aligned}
$$

where $c_{10} = c_9 \cdot 4^{1+p/N} = c(p,N,m,m_0)$, $c_{11} = c_{10} \cdot (c_6)^{1+p/N} = c(p,N,m,m_0)$, $c_{12} = c_{10}[c_6c_2 + c_8]^{1+p/N} = c(p,N,m,m_0,m_1)$ and $c_{13} = c_{10}(c_6c_3)^{1+p/N} = c(p,N,m,m_0,m_1)$. We estimate now the terms that appear in the right hand side of (4.52). As $r > p'$ we obtain

$$\left(\int\int_{Q_R} |F|^{p'}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} \leq \tag{4.53}$$
$$c_{14}\left(\int\int_{Q_R} |F|^r\right)^{(p'/r)(1+p/N)} \Big[|Q_R|^{(1-p'/r)(1+p/N)}+$$
$$\left(\int\int_{Q_R} |T_{j+1}(u)|^{pm(r/p')'}\right)^{(1-p'/r)(1+p/N)}\Big],$$

where $c_{14} = 2^{pm(1+p/N)} = c(p, N, m)$. We notice that by assumption (1.17) it results $(1 - \frac{p'}{r})(1 + \frac{p}{N}) < 1$. Thus, choosing $m = \frac{(N+2)[r(p-1)-p]}{p(N+p)-r(p-1)p}$, i.e. choosing m such that it results $pm(\frac{r}{p'})' = q(m + 1) = s$, where s is as in (1.18), we deduce

$$\left(\int\int_{Q_R} |F|^{p'}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} \leq \tag{4.54}$$
$$c_{14}\left(\int\int_{Q_R} |F|^r\right)^{(p'/r)(1+p/N)} \Big[|Q_R|^{(1-p'/r)(1+p/N)}+$$
$$\epsilon\int\int_{Q_R} |T_{j+1}(u)|^s + C(\epsilon)\Big],$$

where ϵ is a positive constant that will be chosen later and $C(\epsilon) = c(\epsilon, p, N, r)$. To estimate the other terms in (4.52) we distinguish two cases: $p \geq 2$, and $\frac{2N}{N+2} < p < 2$.

First case: if $p \geq 2$, it results

$$\left(\int\int_{Q_R} \frac{2^p p|u|^2}{(R-\rho)^p}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} \leq \tag{4.55}$$
$$\left(\int\int_{Q_R} \frac{2^p p(|u|^p+1)}{(R-\rho)^p}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} \leq$$
$$c_{15}\Bigg\{\left(\int\int_{Q_R} \frac{|u|^p}{(R-\rho)^p}\right)^{1+p/N} + \left(\int\int_{Q_R} \frac{|u|^p}{(R-\rho)^p}|T_{j+1}(u)|^{pm}\right)^{1+p/N}$$
$$+\frac{1}{(R-\rho)^{p(1+p/N)}}\left[\left(\int\int_{Q_R} |T_{j+1}(u)|^{pm}\right)^{1+p/N} + |Q_R|^{1+p/N}\right]\Bigg\},$$

where $c_{15} = (2^p p)^{1+p/N} c_{14} = c(p, N, m)$ and c_{14} is as in (4.53). We observe that using assumption (1.17) and the choice done for m it follows

$$(4.56)\ \frac{1}{(R-\rho)^{p(1+p/N)}} \left(\int\!\!\int_{Q_R} |T_{j+1}(u)|^{pm} \right)^{1+p/N} \leq$$
$$\left[\frac{1}{(R-\rho)^p} \left(\int\!\!\int_{Q_R} |T_{j+1}(u)|^{q(m+1)} \right)^{pm/[q(m+1)]} |Q_R|^{1-pm/[q(m+1)]} \right]^{1+p/N}$$
$$\leq \epsilon \int\!\!\int_{Q_R} |T_{j+1}(u)|^s + c_{16} \frac{1}{(R-\rho)^{r(p-1)\lambda}} |Q_R|^{\lambda},$$

where we have set $\lambda = (N+p)/[N+p-r(p-1)]$, ϵ is as in (4.54) and $c_{16} = c(\epsilon, p, N, r)$. Moreover, we have

$$(4.57)\ c_{12} \left(\int\!\!\int_{Q_R} \frac{|u|^p}{(R-\rho)^p} (1 + |T_{j+1}(u)|)^{pm} \right)^{1+p/N} +$$
$$c_{11} c_{15} \left(\int\!\!\int_{Q_R} \frac{|u|^p}{(R-\rho)^p} |T_{j+1}(u)|^{pm} \right)^{1+p/N} \leq c_{16} \left(\int\!\!\int_{Q_R} \frac{|u|^p}{(R-\rho)^p} \right)^{1+p/N}$$
$$+ \frac{c_{17}}{(R-\rho)^{p(1+p/N)}} \cdot \left(\int\!\!\int_{Q_R} |u|^p |T_{j+1}(u)|^{pm} \right)^{1+p/N},$$

where $c_{16} = c_{12} c_{14} = c(p, N, m, m_0, m_1)$, c_{14} is as in (4.53) and $c_{17} = c_{16} + c_{11} c_{15} = c(p, N, m, m_0, m_1)$. Using (4.53)-(4.57) in (4.52) we obtain

$$(4.58) \qquad \int\!\!\int_{Q_\rho} |T_{j+1}(u)|^s dx d\tau \leq$$
$$c_{10} \left(c_0 |B_R| \right)^{1+p/N} + c_{16} \left(\int\!\!\int_{Q_R} \frac{|u|^p}{(R-\rho)^p} \right)^{1+p/N} + \frac{c_{18} |Q_R|^{1+p/N}}{(R-\rho)^{p(1+p/N)}}$$
$$+ c_{16} c_{18} \frac{1}{(R-\rho)^{r(p-1)\lambda}} |Q_R|^{\lambda} +$$
$$+ c_{19} \left(\int\!\!\int_{Q_R} |F|^r \right)^{\frac{p'}{r}(1+p/N)} \left[|Q_R|^{(1-p'/r)(1+p/N)} + C(\epsilon) \right] +$$
$$\epsilon \left[c_{18} + c_{19} \left(\int\!\!\int_{Q_{R_1}} |F|^r \right)^{(p'/r)(1+p/N)} \right] \int\!\!\int_{Q_R} |T_{j+1}(u)|^s +$$
$$\frac{c_{17}}{(R-\rho)^{p(1+p/N)}} \cdot \left(\int\!\!\int_{Q_R} |u|^p |T_{j+1}(u)|^{pm} \right)^{1+p/N},$$

where $c_{18} = c_{15}c_{11} = c(p, N, m, m_0)$ and $c_{19} = c_{13}c_{14} = c(p, N, m, m_0, m_1)$. Thus it remains to estimate "only" the last integral in the right hand side of (4.58). We proceed by steps.

Step 1. If $r \le r_1 = p' \frac{N+2}{N}$ we deduce

$$
\begin{aligned}
C\left(\iint_{Q_R} |u|^p |T_{j+1}(u)|^{pm}\right)^{1+p/N} &\le \\
C\left(\iint_{Q_R} |u|^{p\frac{N+2}{N}}\right)^{\frac{N+p}{N+2}} \left(\iint_{Q_R} |T_{j+1}(u)|^{pm\frac{N+2}{2}}\right)^{\frac{2(N+p)}{(N+2)N}} &\le \\
C\left(\iint_{Q_R} |u|^{p\frac{N+2}{N}}\right)^{\frac{N+p}{N+2}} \left(\iint_{Q_R} |T_{j+1}(u)|^{s}\right)^{\frac{(N+p)pm}{Ns}} |Q_R|^{\theta} &\le \\
\epsilon \iint_{Q_R} |T_{j+1}(u)|^s + c_{20}\left[C\left(\iint_{Q_R} |u|^{p\frac{N+2}{N}}\right)^{\frac{N+p}{N+2}}\right]^{\Lambda}&,
\end{aligned}
\tag{4.59}
$$

where we have set $\Lambda = \frac{Nr(p-1)}{p[N+p-r(p-1)]}$, ϵ is as in (4.54), $c_{20} = c(\epsilon, p, N, r)$ being $|Q_R| \le |Q_{R_1}| \le c(N)$, $\theta = \frac{2(N+p)}{(N+2)N}\left(1 - \frac{pm(N+2)}{2s}\right)$ and

$$
C = \frac{c_{17}}{(R-\rho)^{p(1+p/N)}}. \tag{4.60}
$$

Let $0 < \delta < \sigma \le R_1$ arbitrary fixed. We recall that also ρ and R are arbitrary fixed verifying $0 < \rho < R \le R_1$. Let us choose ρ and R satisfying $\delta \le \rho < R \le \sigma$ and $\epsilon = \frac{1}{2}\left[1 + c_{18} + c_{19}\left(\iint_{Q_{R_1}} |F|^r\right)^{\frac{p'}{r}(1+\frac{p}{N})}\right]^{-1}$. From (4.58) and (4.59) we obtain that for every ρ and R, satisfying $\delta \le \rho < R \le \sigma$ it results

$$\begin{aligned}(4.61)\qquad &\iint_{Q_\rho} |T_{j+1}(u)|^s dx d\tau \le \frac{1}{2}\iint_{Q_R} |T_{j+1}(u)|^s + \\ &+c_{10}\left(c_0|B_{R_1}|\right)^{1+p/N} + c_{16}\left(\iint_{Q_\sigma}\frac{|u|^p}{(R-\rho)^p}\right)^{1+p/N} + \\ &\frac{c_{18}|Q_{R_1}|^{1+p/N}}{(R-\rho)^{p(1+p/N)}} + \frac{c_{16}c_{18}}{(R-\rho)^{r(p-1)\lambda}}|Q_\sigma|^\lambda + \\ &c_{19}\left(\iint_{Q_{R_1}}|F|^r\right)^{(p'/r)(1+p/N)}\left[|Q_{R_1}|^{(1-p'/r)(1+p/N)} + C(\epsilon)\right] + \\ &c_{20}\left\{\frac{c_{17}}{(R-\rho)^{p(1+p/N)}}\left(\iint_{Q_\sigma}|u|^{p\frac{N+2}{N}}\right)^{\frac{N+p}{N+2}}\right\}^{\Lambda} \\ &\le \frac{1}{2}\iint_{Q_R}|T_{j+1}(u)|^s + \frac{A}{(R-\rho)^{r(p-1)\lambda}} + B,\end{aligned}$$

where

$$A = c_{16}\left(\iint_{Q_\sigma}|u|^p\right)^{1+p/N} + c_{21} + c_{20}\left\{c_{17}\left(\iint_{Q_\sigma}|u|^{p\frac{N+2}{N}}\right)^{\frac{N+p}{N+2}}\right\}^{Lambda},$$

and

$$B = c_{22}\left[\left(\iint_{Q_{R_1}}|F|^r\right)^{(p'/r)(1+p/N)} + 1\right],$$

whith $c_{21} = c(p, N, m, m_0, m_1, r)$, $c_{22} = c(p, N, m, m_0, m_1, r, \|F\|_{L^r(Q_{R_1})})$ as it results $|B_{R_1}| + |Q_{R_1}| \le c(N)$. Applying lemma 2.1 to the previous inequality we deduce that there exists a constant $c_{23} = c(r, p, N)$ such that

$$(4.62)\qquad \iint_{Q_\rho}|T_{j+1}(u)|^s dx d\tau \le c_{23}\left(\frac{A}{(R-\rho)^{r(p-1)\lambda}} + B\right),$$

for every $0 < \delta \le \rho < R \le \sigma \le R_1$. Passing to the limit as j tends to infinity in (4.62) we obtain that $u \in L^s(Q_\rho)$, for every $0 < \delta \le \rho < R \le \sigma \le R_1$, that is for every $0 < \rho < R_1$.

We notice that if $r_1 = p'\frac{N+2}{N} \ge \frac{N+p}{p-1}$, that is if $p \ge \frac{N^2}{2}$, then the proof of theorem 1.1 is complete, otherwise we can proceed as follows.

Step 2. If $\frac{N+p}{p-1} > r > r_1 = p'\frac{N+2}{N}$ then by step 1 it follows that $u \in L^{s_1}(Q_\sigma)$, for every $\sigma < R_1$, where $s_1 = s(r_1) = \frac{p(N+2)^2}{N^2-2p}$. Thus if $0 < \delta \le \rho < R \le \sigma < R_1$ are arbitrary fixed, we can estimate the left hand side of (4.59) as follows

$$(4.63)\quad C\left(\iint_{Q_R}|u|^p|T_{j+1}(u)|^{pm}\right)^{1+p/N}\le C\left(\iint_{Q_\sigma}|u|^{s_1}\right)^{\frac{p(N+p)}{s_1N}}\left(\iint_{Q_R}|T_{j+1}(u)|^{pm\frac{s_1}{s_1-p}}\right)^{(s_1-p)(N+p)/(s_1N)}.$$

Thus, if we assume that $r_1 < r \le r_2 = \frac{s_1}{p-1} = \frac{p(N+2)^2}{(N^2-2p)(p-1)}$, that is $pm\frac{s_1}{s_1-p} \le s$, we can increase the right hand side of (4.63) with the following integrals

$$(4.64)\quad C\left(\iint_{Q_\sigma}|u|^{s_1}\right)^{\frac{p(N+p)}{s_1N}}\left(\iint_{Q_R}|T_{j+1}(u)|^{s}\right)^{\frac{pm(N+p)}{sN}}|Q_{R_1}|^{h(s_1)}\le C(\epsilon)\left[C\left(\iint_{Q_\sigma}|u|^{s_1}\right)^{p(N+p)/(s_1N)}\right]^{\Lambda}+\epsilon\iint_{Q_R}|T_{j+1}(u)|^{s},$$

where again ϵ is as in (4.54), $C(\epsilon) = c(\epsilon, N, p, r)$ and $h(s_1) = \frac{(s_1p'-rp)(N+p)}{rs_1N} = c(p, N, r)$. Proceeding as in step1 we can conclude that $u \in L^s(Q_\rho)$, for every $0 < \rho < R_1$.

As in the previous step the proof of theorem 1.1 is complete if $r_2 \ge \frac{N+p}{p-1}$, otherwise we can proceed exactly as before, i.e..

Step 3. If $\frac{N+p}{p-1} > r > r_2$, then by step 2 it follows that $u \in L^{s_2}(Q_\sigma)$, for every $\sigma < R_1$, where $s_2 = s(r_2) = \frac{p(N+2)^3}{N(N^2-6p)-2p(p+2)}$. Thus if again $0 < \delta \le \rho < R \le \sigma < R_1$ are arbitrary fixed, we can estimate the left hand side of (4.59) as follows

$$(4.65)\quad C\left(\iint_{Q_R}|u|^p|T_{j+1}(u)|^{pm}\right)^{1+p/N}\le C\left(\iint_{Q_\sigma}|u|^{s_2}\right)^{p(N+p)/(s_2N)}\left(\iint_{Q_R}|T_{j+1}(u)|^{pm\frac{s_2}{s_2-p}}\right)^{(s_2-p)(N+p)/(s_2N)}.$$

Thus if we suppose that $r_2 < r \le r_3 = \frac{s_2}{p-1}$, that is $pm\frac{s_2}{s_2-p} \le s$ we can increase the right hand side of (4.65) with the following integrals

$$(4.66)\quad C\left(\iint_{Q_\sigma}|u|^{s_2}\right)^{\frac{p(N+p)}{s_2N}}\left(\iint_{Q_R}|T_{j+1}(u)|^{s}\right)^{\frac{pm(N+p)}{sN}}|Q_{R_1}|^{h(s_2)}\le C(\epsilon)\left[C\left(\iint_{Q_\sigma}|u|^{s_2}\right)^{\frac{p(N+p)}{s_2N}}\right]^{\Lambda}+\epsilon\iint_{Q_R}|T_{j+1}(u)|^{s},$$

where $C(\epsilon) = c(\epsilon, N, r, p)$ and $h(\cdot)$ is as in (4.64). Proceeding as in the previous steps we can conclude that $u \in L^s(Q_\rho)$, for every $0 < \rho < R_1$.

Thus the proof of theorem 1.1 is complete if $r_3 \geq \frac{N+p}{p-1}$, otherwise we can proceed exactly as before.

Step 4. To conclude the proof we show that this process finishes after a finite number of steps, that is that at a certain step i, where i depends only on N and p, it results $r_i \geq \frac{N+p}{p-1}$. To do this we observe that for every $n \in \mathbb{N}$ it results

$$r_{n+3} = \frac{(N+2)^n s_2}{(p-1)\left[(N+p)^n - s_2 \sum_{k=0}^{n-1}(N+p)^k (N+2)^{n-1-k}\right]}. \tag{4.67}$$

(The proof of the previous formula can be obtained easily by induction on n and so we omit it). From (4.67) it follows (for every $n \in \mathbb{N}$)

$$r_{n+3} > \frac{N+p}{p-1} \quad \Leftrightarrow \quad s_2 \sum_{k=0}^{n} \left(\frac{N+2}{N+p}\right)^k > N+p. \tag{4.68}$$

Notice that if $p = 2$ we have

$$s_2 \sum_{k=0}^{n} \left(\frac{N+2}{N+p}\right)^k = s_2(n+1) \to +\infty, \quad \text{as} \quad n \to +\infty, \tag{4.69}$$

while if $p > 2$ it results

$$s_2 \sum_{k=0}^{n} \left(\frac{N+2}{N+p}\right)^k = s_2 \frac{1 - \left(\frac{N+2}{N+p}\right)^{n+1}}{1 - \frac{N+2}{N+p}} \to s_2 \frac{N+p}{p-2}, \quad \text{as} \quad n \to +\infty, \tag{4.70}$$

where the limit satisfies

$$s_2 \frac{N+p}{p-2} > N+p. \tag{4.71}$$

Thus from (4.68)-(4.71) we deduce the existence of the desired r_i which concludes the proof in the case $p \geq 2$.

Second case: If $\frac{2N}{N+2} < p < 2$ it results

$$c_{12}\left(\iint_{Q_R} \frac{|u|^p}{(R-\rho)^p}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} \leq \tag{4.72}$$
$$c_{12}\left(\iint_{Q_R} \frac{|u|^2+1}{(R-\rho)^p}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} \leq$$
$$c_{24}\left(\iint_{Q_R} \frac{|u|^2}{(R-\rho)^p}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} +$$
$$c_{24}\left(\iint_{Q_R} (1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N},$$

where $c_{24} = c(p, N, m, m_0, m_1)$. Besides as $p > \frac{2N}{N+2}$, we can estimate the left hand side of (4.55) as follows

$$\left(\iint_{Q_R} \frac{2^p p|u|^2}{(R-\rho)^p}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} \leq \tag{4.73}$$
$$c_{25}\left[\left(\iint_{Q_R} \frac{|u|^{\frac{p(N+2)}{N}}}{(R-\rho)^{\frac{p^2(N+2)}{2N}}}\right)^{\frac{2N}{p(N+2)}}\right]^{1+p/N} \times$$
$$\left[\left(\iint_{Q_R} |T_{j+1}(u)|^{pm\left(\frac{p(N+2)}{2N}\right)'}\right)^{1-\frac{2N}{p(N+2)}}\right]^{1+p/N} +$$
$$c_{25}\left[\iint_{Q_R} \frac{|u|^2}{(R-\rho)^p}\right]^{1+p/N},$$

where $c_{25} = c(p, m, N)$. As in the previous case the proof proceeds by steps.
Step 1. If $r \leq \tilde{r}_1 = \frac{p^2(N+2)}{(p-1)2N}$, i.e. if $pm\left(\frac{p(N+2)}{2N}\right)' \leq s = q(m+1)$, we can increase the first term in the right hand side of (4.73) with the following integrals

$$c_{26}\left[\left(\iint_{Q_R} \frac{|u|^{\frac{p(N+2)}{N}}}{(R-\rho)^{\frac{p^2(N+2)}{2N}}}\right)^{\frac{2N}{p(N+2)}} \left(\iint_{Q_R} |T_{j+1}(u)|^s\right)^{1-p'/r}\right]^{1+p/N}$$
$$\leq c_{27}\left(\iint_{Q_R} \frac{|u|^{\frac{p(N+2)}{N}}}{(R-\rho)^{\frac{p^2(N+2)}{2N}}}\right)^{\kappa} + \epsilon \iint_{Q_R} |T_{j+1}(u)|^s, \tag{4.74}$$

where $c_{26} = c(p, m, N)$, $\kappa = c(N, p, r) = \frac{2(N+p)r(p-1)N}{p^2(N+2)[N+p-r(p-1)]}$ and $c_{27} = c(p, r, N, \epsilon)$. Thus proceeding as in the first step of case 1 we deduce that $u \in L^s(Q_\rho)$, for every $0 < \rho < R_1$.

We notice that if $\tilde{r}_1 \geq \frac{N+p}{p-1}$, that is if $p^2(N+2) \geq 2N(N+p)$, then the proof of theorem 1.1 is complete, otherwise we can proceed as follows.

Step 2. If $\frac{N+p}{p-1} > r > \tilde{r}_1$ then by step 1 we have that $u \in L^{\tilde{s}_1}(Q_\sigma)$, for every $\sigma < R_1$, where $\tilde{s}_1 = s(\tilde{r}_1) = \frac{(N+2)^2p^2}{(N+p)2N-p^2(N+2)}$. Thus assuming $0 < \delta \leq \rho < R \leq \sigma < R_1$ arbitrary fixed, we can estimate the left hand side of (4.73) as follows

$$\left(\iint_{Q_R} \frac{|u|^2}{(R-\rho)^p}(1+|T_{j+1}(u)|)^{pm}\right)^{1+p/N} \leq c_{14}\left[\iint_{Q_\sigma} \frac{|u|^2}{(R-\rho)^p}\right]^{1+p/N} +$$

$$c_{14}\left[\left(\iint_{Q_\sigma} \frac{|u|^{\tilde{s}_1}}{(R-\rho)^{\frac{p\tilde{s}_1}{2}}}\right)^{\frac{2}{\tilde{s}_1}} \left(\iint_{Q_R} |T_{j+1}(u)|^{pm\left(\frac{\tilde{s}_1}{2}\right)'}\right)^{1-\frac{2}{\tilde{s}_1}}\right]^{1+p/N}. \tag{4.75}$$

Thus if $r \leq \tilde{r}_2 = \frac{p'}{2}\tilde{s}_1 = \frac{p'}{2}\frac{(N+2)^2p^2}{(N+p)2N-p^2(N+2)}$, that is if $pm\left(\frac{\tilde{s}_1}{2}\right)' \leq q(m+1) = s$, we can estimate from above the second term in the right hand side of (4.75) with the following integrals

$$\left[\left(\iint_{Q_\sigma} \frac{|u|^{\tilde{s}_1}}{(R-\rho)^{\frac{p\tilde{s}_1}{2}}}\right)^{2/\tilde{s}_1} \left(\iint_{Q_R} |T_{j+1}(u)|^s\right)^{1-p'/r} |Q_\sigma|^l\right]^{1+p/N} \leq$$

$$c_{28}\left(\iint_{Q_\sigma} \frac{|u|^{\tilde{s}_1}}{(R-\rho)^{\frac{p\tilde{s}_1}{2}}}\right)^{\frac{2r(p-1)}{p\tilde{s}_1}} + \epsilon\iint_{Q_R} |T_{j+1}(u)|^s, \tag{4.76}$$

where $c_{28} = c(\epsilon, p, r, N)$ and $l = \left(1 - \frac{2}{\tilde{s}_1}\right)\left(1 - \frac{pm\left(\frac{\tilde{s}_1}{2}\right)'}{s}\right) = c(p, N, r)$. Then proceeding as in the step 1 we conclude that that $u \in L^s(Q_\rho)$, for every $0 < \rho < R_1$.

We notice that if $\tilde{r}_2 \geq \frac{N+p}{p-1}$ then the proof of theorem 1.1 is complete, otherwise we can proceed exactly as before.

Step 3. As in the first case, to conclude the proof we show that this process finishes after a finite number of steps, that is that there exists a certain step h, where h depends only on N and p, such that $\tilde{r}_h \geq \frac{N+p}{p-1}$. To do this we observe that as for every $n \in \mathbb{N}$ it results $\tilde{r}_n = \frac{p'}{2}\tilde{s}_{n-1}$, it is very easy to derive by induction that for every $n \geq 3$ we have

$$\tilde{r}_n = \frac{p'}{2} \frac{p^{n-2}(N+2)^{n-2}\tilde{s}_1}{2^{n-2}(N+p)^{n-2} - \tilde{s}_1 \sum_{j=0}^{n-3} \left[2(N+p)\right]^j p^{n-2-j}(N+2)^{n-3-j}}. \tag{4.77}$$

From (4.77) it follows that

$$\tilde{r}_n > \frac{N+p}{p-1} \quad \Leftrightarrow \quad \frac{p}{2}\tilde{s}_1 \sum_{j=0}^{n-2} \left[\frac{p(N+2)}{2(N+p)}\right]^k > N+p, \tag{4.78}$$

that is

$$\frac{p}{2}\tilde{s}_1 \frac{1 - \left[\frac{p(N+2)}{2(N+p)}\right]^{n-1}}{1 - \frac{p(N+2)}{2(N+p)}} > N+p. \tag{4.79}$$

Being $p < 2$, the left hand side of (4.79) for $n \to +\infty$ tends to

$$\frac{p}{2}\tilde{s}_1 \frac{1}{1 - \frac{p(N+2)}{2(N+p)}},$$

and thus we have proved the result if

$$\frac{p}{2}\tilde{s}_1 \frac{1}{1 - \frac{p(N+2)}{2(N+p)}} > N+p,$$

that is if

$$G(p) = p^3(N+2) + 2p^2N(N+1) + p(N^3 - 2N^2) - 2N^3 > 0. \tag{4.80}$$

We observe that $G(\frac{2N}{N+2}) = 0$ and that $G'(p) > 0$ for every $N \geq 2$ which means that $G(p)$ is strictly increasing in $(\frac{2N}{N+2}, 2)$ and so inequality (4.80) is satisfied.

Proof of theorem 1.2 The proof of theorem 1.1, as said before, is analogous at all to that of theorem 1.1 and so we omit it.

Acknowledgements I would like to thank the referee for his useful remarks. For example if we substitute the constant $\frac{1}{\alpha(\alpha+1)(\alpha+2)}$ in (2.32) with $c(\alpha) = \frac{1}{2^{2\alpha+5}\alpha}$, then such a formula can be also proved as follows. Set $m = 2p + \theta$, with $\theta = 0$ or

1. Then it results

$$\sum_{k=0}^{m}(2+k)^{\alpha-1}(m-k)^2 \geq \sum_{k=0}^{p}(2+k)^{\alpha-1}(m-k)^2 \geq$$
$$p^2\sum_{k=0}^{p}(2+k)^{\alpha-1} \geq p^2\int_0^p(2+s)^{\alpha-1}ds = p^2\,\frac{(2+p)^\alpha-2^\alpha}{\alpha}$$
$$\geq p^2\,\frac{(2+p)^\alpha}{2\alpha} \geq \frac{(m+1)^{\alpha+2}}{2^{2\alpha+5}\alpha},$$

since $p \geq \frac{m+1}{4}$ as soon as $m \geq 2$.

References

[1] D.G. Aronson, *On the Green's function for second order parabolic differential equations with discontinuous coefficients* , Bulletin of the American Mathematical Society **69**, 841-847 (1963)

[2] D.G. Aronson, J. Serrin, *Local behaviour of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal., 25 (1967), 81-122.

[3] L. Boccardo, *Existence and regularity results for quasilinear parabolic equations*, Proceedings Trieste, 1995.

[4] L. Boccardo, A. Dall'Aglio, T. Gallouët, L. Orsina, *Regularity results for nonlinear parabolic equations*, to appear on Adv. Math. Sci. Appl.

[5] L. Boccardo, D. Giachetti, *Alcune osservazioni sulla regolarità delle soluzioni di problemi nonlineari e applicazioni*, Ricerche di Matematica, XXXIV, (1985).

[6] L. Boccardo, D. Giachetti, *L^s-regularity of solutions of some nonlinear elliptic problems*, preprint.

[7] E. Di Benedetto, *Degenerate parabolic equations*, Springer-Verlag, New York, (1993).

[8] M. Giaquinta, *Multiple integrals in the Calculus of Variations and nonlinear elliptic systems*, Princeton University Press, (1983).

[9] D. Giachetti, M.M. Porzio, *Local regularity results for minima of functionals of the Calculus of Variation*, to appear on Nonlinear Analysis T.M.A.

[10] F. Guglielmino, *Sulla regolarizzazione delle soluzioni deboli dei problemi al contorno per operatori parabolici* , Ricerche di Matematica **12**, 44-66, 140-150 (1963).

[11] A.V. Ivanov, *A priori estimates for solutions of linear equations of second order of elliptic and parabolic type* , Doklady Akademii Nauk S.S.S.R. **161**, 1270-1273 (1965).

[12] A.V. Ivanov, O. A. Ladyzenskaja, A.L. Treskunov, N.N. Uralcéva, *Certain properties of generalized solutions of parabolic equations of second order* , Doklady Akademii Nauk S.S.S.R. **168**, 17-20 (1966).

[13] S.N. Kruzkov, *A priori estimates and certain properties of solutions of elliptic and parabolic equations*, Matematiceskii Sbornik **65**, 522-570 (1964).

[14] O. A. Ladyzenskaja, N.A. Solonnikov, N.N. Uralcéva, *Linear and quasilinear equations of parabolic type*, Trans Math. Monogr., Am. Math. Soc. **23**, Providence, RI (1968).

[15] O. A. Ladyzenskaja, N.N. Uralcéva, *A boundary value problem for linear and quasilinear parabolic equations I, II, III.* Izvestija akademii Nauk S.S.S.R. Serija Matematiceskaja **26**, 5-52 (1962); **26**, 753-780 (1962); **27**, 161-240 (1963).

[16] J.L. Lions, *Quelques méthodes de resolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, (1969).

[17] J. Moser, *A Harnack inequality for parabolic differential equations*, Communications on Pure and Applied Mathematics **17**, 101-134 (1964); Correction ibid. **20**, 231-236 (1967).

Received January 12, 1998

Revised version received August 4, 1998

Dipartimento di Matematica "G. Castelnuovo", Università degli Studi di Roma "La Sapienza", P.le A. Moro, 2 – 00185 Roma, Italy

E-mail address: `e-mail:porzio@mat.uniroma1.it`