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INVERSE LIMITS WHICH ARE THE PSEUDOARC

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Abstract. Let $C_s(I,I)$ denote the space of surjective continuous maps of the compact interval I to itself with the uniform topology. Given a map f in $C_s(I,I)$, let (I,f) denote the inverse limit space obtained from the inverse sequence all of whose maps are f and all of whose spaces are I. We show that the set of f in $C_s(I,I)$ such that (I,f) is homeomorphic to the pseudoarc is nowhere dense in $C_s(I,I)$. Also, we show that if f is any continuous map of I to itself such that f has a periodic point of period two or larger, but f has no periodic point of odd period larger than one, then (I,f) is not homeomorphic to the pseudoarc.

It follows that if f is any continuous map of I to itself with (I,f) the pseudoarc and with topological entropy positive, then the topological entropy of f is greater than $\frac{\log(2)}{2}$.

1. Introduction

Given a sequence $(f_1, f_2, \dots)$ of continuous maps of the compact interval $I = [0,1]$ to itself, we let $(I,(f_1,f_2,f_3,\dots))$ denote the inverse limit space associated to the following inverse system.

$$I \xleftarrow{f_1} I \xleftarrow{f_2} I \xleftarrow{f_3} I \xleftarrow{f_4} \cdots$$

This space consists of all sequences $(x_i \mid i = 0,1,2,\dots)$ such that $f_i(x_i) = x_{i-1}$ for i=1,2,3, ... In fact $(I,(f_1,f_2,f_3,\dots))$ is a metric space with the metric

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inherited as a subspace of the infinite product space I^∞. It is a well-known fact that $(I,(f_1,f_2,f_3,\dots))$ is a chainable continuum.

We let $C(I,I)$ denote the space of continuous maps of I to itself with metric $d(f,g)=\sup\{|f(x)-g(x)| \mid x\in I\}$. Let $C_s(I,I)$ denote the subspace of $C(I,I)$ consisting of those maps f which are surjective.

Our first result is the following proposition.

Proposition 3.1. *For any chainable continuum K, $\{(f_1,f_2,\dots)\in\prod_{i=1}^\infty C_s(I,I) \mid (I,(f_1,f_2,\dots))$ is homeomorphic to $K\}$ is dense in $\prod_{i=1}^\infty C_s(I,I)$.*

The dissertation of D. Kuykendall [9] also used function spaces to study inverse limits of continua in a different setting. He obtained some results about the hereditary indecomposability of the limit.

We let P denote the important continuum called the pseudoarc. For some historical discussion see [7]. Classical results about the pseudoarc were obtained in [1], [2], [3], [4], [8], and [10].

Let Q denote the Hilbert cube. Let $\mathcal{C}(Q)$ denote the space of continuum subsets of Q with the Hausdorff metric. It is well known that the set of all subcontinua of $\mathcal{C}(Q)$ which are homeomorphic to P is a dense G_δ in $\mathcal{C}(Q)$. Using this fact we obtain the following corollary to Proposition 3.1.

Corollary 3.2. *If $K=P$, then the subset of $\prod_{i=1}^\infty C_s(I,I)$ given in Proposition 3.1 is a dense G_δ.*

Next, let $f\in C(I,I)$, and let $(f_1,f_2,\dots)=(f,f,\dots)$. We denote $(I,(f_1,f_2,\dots))$ by (I,f) in this case. In light of the previous discussion and Corollary 3.2, it is reasonable to conjecture that (I,f) is homeomorphic to the pseudoarc for a dense G_δ of $C_s(I,I)$. A result along these lines is the following.

Proposition 3.3. *For any chainable continuum K, $\{f\in C_s(I,I) \mid (I,f)$ contains a subcontinuum homeomorphic to $K\}$ is dense in $C_s(I,I)$.*

However, we prove the following.

Theorem 3.5. *$\{f\in C_s(I,I) \mid (I,f)$ is homeomorphic to the pseudoarc$\}$ is nowhere dense in $C_s(I,I)$.*

To help put our next result in context, we recall the Theorem of Sharkovsky [12]. See [5] (Chapter 1) for a proof and discussion of this classical theorem.

Sharkovsky's Theorem. *Let $f\in C(I,I)$. Consider the following total ordering of the positive integers.*

$$3\lhd 5\lhd 7\lhd\cdots\lhd 2\cdot 3\lhd 2\cdot 5\lhd\cdots\lhd 2^2\cdot 3\lhd 2^2\cdot 5\lhd\cdots\lhd 2^3\lhd 2^2\lhd 2\lhd 1$$

If f has a periodic orbit of period n and if $n \lhd m$, then f also has a periodic orbit of period m.

Finally, we consider the following question. If (I, f) is homeomorphic to the pseudoarc, what is the possible dynamic behavior of f? Examples have been given to show the dynamic behavior of f may be very simple [6] or very complex [11]. Surprisingly, we show that certain behavior in between is ruled out.

Theorem 3.7. *Let $f : I \to I$ be continuous. Suppose that f has a periodic orbit of period two or larger, but f has no periodic orbit of odd period larger than one. Then (I, f) is not homeomorphic to the pseudoarc.*

The following problem remains open.

Problem 1. *Is there a map $f \in C(I, I)$ such that f has a periodic orbit of odd period larger than one, f has no periodic orbit of period three, and (I, f) is homeomorphic to the pseudoarc?*

Let $h(f)$ denote the topological entropy of the map f. See [5], (Chapter 8) for the definition and basic properties of this concept. As a corollary to Theorem 3.7 we obtain the following.

Corollary 3.8. *Let $f \in C(I, I)$, and suppose (I, f) is the pseudoarc. Suppose that $h(f) > 0$. Then $h(f) > \frac{\log(2)}{2}$.*

This leads to the following open problem.

Problem 2. *Let λ denote the greatest lower bound of the set of positive real numbers r with $r = h(f)$ for some $f \in C(I, I)$ such that (I, f) is homeomorphic to the pseudoarc. Determine the value of λ.*

Does there exist a map $f \in C(I, I)$ such that (I, f) is homeomorphic to the pseudoarc and $h(f) = \lambda$?

2. Preliminaries

Given a sequence of maps $f_1, f_2, \ldots$, one may think of the inverse limit space $(I, (f_1, f_2, \ldots))$ as an element of $\mathcal{C}(Q)$. The same holds for the inverse limit space (I, f). The metric we will use on the product space $\prod_{i=1}^{\infty} C_s(I, I)$ will be given by

$$d((f_1, f_2, \ldots), (g_1, g_2, \ldots)) = \sum_{i=0}^{\infty} \frac{\sup\{|f_i(x) - g_i(x)| \mid x \in I\}}{2^i}$$

We will use the Hausdorff metric on $\mathcal{C}(Q)$ given by

$$D(A,B) = \inf\{\epsilon > 0 \mid N_\epsilon(A) \supset B \text{ and } N_\epsilon(B) \supset A\}$$

We leave for the reader the straightforward proofs of the following two propositions. We remark, however, that surjectivity is needed.

Proposition 2.1. *Define* $F : C_s(I,I) \to \mathcal{C}(Q)$ *by* $F(f) = (I,f)$. *Then* F *is an embedding (a homeomorphism onto its image).*

Proposition 2.2. *Define*

$$G : \prod_{i=1}^{\infty} C_s(I,I) \to \mathcal{C}(Q)$$

by $G(f_1, f_2, \dots) = (I, (f_1, f_2, \dots))$. *Then* G *is an embedding.*

We now recall some basic definitions. A *continuum* is a compact, connected, metric space containing more than one point. A continuum is *indecomposable* if it cannot be written as the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if every subcontinuum is indecomposable. A space Y is *chainable* if given any open cover $\mathcal{V}$ there is a refinement $\mathcal{W}$ of $\mathcal{V}$ whose elements form a finite chain, *i.e.*, $\mathcal{W} = \{W_1, W_2, \dots, W_n\}$ where $W_i \cap W_j \neq \emptyset$ if and only if $|i-j| = 1$.

The following facts are well-known. The proof of Proposition 2.3 is in [2], Theorem 1 and the proof of Proposition 2.4 can be obtained by an adaptation of the proof in [2] of Theorem 2.

Proposition 2.3. *The pseudoarc* P *is the unique continuum which is chainable and hereditarily indecomposable.*

Proposition 2.4. *The set* $\{K \in \mathcal{C}(Q) \mid K \text{ is homeomorphic to } P\}$ *is a dense* G_δ, *i.e., a countable intersection of open dense subsets of* $\mathcal{C}(Q)$.

Proposition 2.5. *A continuum* Y *is chainable if and only if* Y *is the inverse limit space associated with a system*

$$I_0 \xleftarrow{f_1} I_1 \xleftarrow{f_2} I_2 \xleftarrow{f_3} I_3 \xleftarrow{f_4} \cdots$$

where each I_j *is a compact interval.*

Proposition 2.6. *A continuum* Y *is chainable if and only if there exists* $(g_1, g_2, \dots) \in \prod_{i=1}^{\infty} C_s(I,I)$ *such that* Y *is homeomorphic to* $(I, (g_1, g_2, \dots))$.

Proposition 2.7. *Let X be the inverse limit space associated with the inverse system*

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} \cdots .$$

Let Y be the inverse limit space associated with the system

$$Y_0 \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xleftarrow{g_3} Y_3 \xleftarrow{g_4} \cdots .$$

Suppose that for each $i = 0, 1, 2, \dots$ there is a homeomorphism $h_i : X_i \to Y_i$ such that for each $i = 1, 2, \dots$ and for each $x \in X_i$, $h_{i-1}(f_i(x)) = g_i(h_i(x))$. Then X and Y are homeomorphic.

Finally, let $f \in C(I, I)$. We define f^n for each positive integer n inductively by $f^1 = f$ and $f^{n+1} = f \circ f^n$. A point $x \in I$ is said to be periodic of period k if $f^k(x) = x$ while $f^j(x) \neq x$ for each positive integer $j < k$. In this case, we also say that the orbit of x is a periodic orbit of period k.

We say that f is turbulent if there exist compact subintervals J, K of I with at most one common point such that $J \cup K \subset f(J) \cap f(K)$. If x is a fixed point of f, we define the unstable manifold of x to be the set

$$W(x, f) = \bigcap_{\epsilon > 0} \bigcup_{m \geq 0} f^m(x - \epsilon, x + \epsilon).$$

These concepts will be used in the proof of Theorem 3.7.

3. Proof of Main Results

Now we proceed to the proofs of our main results.

Proposition 3.1. *For any chainable continuum K, $\{(f_1, f_2, \dots) \in \prod_{i=1}^{\infty} C_s(I, I) \mid (I, (f_1, f_2, \dots))$ is homeomorphic to $K\}$ is dense in $\prod_{i=1}^{\infty} C_s(I, I)$.*

Proof. Let K be a chainable continuum, let $(f_1, f_2, \dots) \in \prod_{i=1}^{\infty} C_s(I, I)$, and let $\epsilon > 0$. By Proposition 2.6, there is a $(g_1, g_2, \dots) \in \prod_{i=1}^{\infty} C_s(I, I)$ such that K is homeomorphic to $(I, (g_1, g_2, \dots))$. Choose a positive integer N such that

$$\frac{1}{2^N} + \frac{1}{2^{N+1}} + \cdots < \epsilon.$$

Define $(h_1, h_2, \dots) \in \prod_{i=1}^{\infty} C_s(I, I)$ by $h_i = f_i$ for $i = 1, 2, \dots, N$ and $h_{N+i} = g_i$ for $i = 1, 2, 3, \dots$. Then $d((h_1, h_2, \dots), (f_1, f_2, \dots)) < \epsilon$. It is easy to verify that the map

$$\varphi : (I, (h_1, h_2, \dots)) \to (I, (g_1, g_2, \dots))$$

defined by $\varphi((x_0, x_1, \dots)) = (x_N, x_{N+1}, \dots)$ is a homeomorphism. □

Corollary 3.2. *If $K = P$, then the subset of $\prod_{i=1}^{\infty} C_s(I,I)$ given in Proposition 3.1 is a dense G_δ.*

PROOF. Let W denote $\{(f_1, f_2, \dots) \in \prod_{i=1}^{\infty} C_s(I,I) \mid (I,(f_1,f_2,\dots))$ is homeomorphic to $P\}$. By Proposition 3.1, W is a dense subset of $\prod_{i=1}^{\infty} C_s(I,I)$. Let G be the map defined in Proposition 2.2. By Proposition 2.4, $\{K \in (C)(Q) \mid K$ is homeomorphic to $P\}$ is a G_δ in $(C)(Q)$. Now, $\{K \in (C)(Q) \mid K$ is homeomorphic to $P\} \cap G(\prod_{i=1}^{\infty} C_s(I,I) = G(W)$. Thus, $G(W)$ is a G_δ in $G(\prod_{i=1}^{\infty} C_s(I,I))$. By Proposition 2.2, W is a G_δ in $\prod_{i=1}^{\infty} C_s(I,I)$. □

Proposition 3.3. *For any chainable continuum K, $\{f \in C_s(I,I) \mid (I,f)$ contains a subcontinuum homeomorphic to $K\}$ is dense in $C_s(I,I)$.*

PROOF. Let K be a chainable continuum, let $f \in C_s(I,I)$, and let $\epsilon > 0$. We must show that there is a map $g \in C_s(I,I)$ with $d(f,g) < \epsilon$ such that (I,g) contains a subcontinuum homeomorphic to K.

By Proposition 2.5, K is the inverse limit space associated with a system

$$I_0 \xleftarrow{f_1} I_1 \xleftarrow{f_2} I_2 \xleftarrow{f_3} I_3 \xleftarrow{f_4} \cdots$$

where each I_j is a compact interval. Also, f has a fixed point $w \in I$. We may choose a collection $L_0, L_1, L_2, \dots$ of pairwise disjoint closed subintervals of I all of which are suitably close to w on the same side of w, such that for each $i = 0,1,2\dots$, L_{i+1} is closer to w than L_i.

For each $i = 0,1,2,\dots$, let $h_i : I_i \to L_i$ be a homeomorphism. For each $i = 1,2,\dots$, define a map $g_i : L_i \to L_{i-1}$ by $g_i(x) = h_{i-1}(f_i(h_i^{-1}(x)))$. Then for each $i = 1,2,\dots$, and for each $x \in I_i$, $h_{i-1}(f_i(x)) = g_i(h_i(x))$.

Since the intervals L_i are all suitably close to w, there is a map $g \in C_s(I,I)$ with $d(f,g) < \epsilon$ such that for $i = 1,2,\dots$, $g|L_i = g_i$. By Proposition 2.7, (I,g) has a subcontinuum homeomorphic to K, namely the inverse limit of the system

$$L_0 \xleftarrow{g_1} L_1 \xleftarrow{g_2} L_2 \xleftarrow{g_3} L_3 \xleftarrow{g_4} \cdots .$$

□

We will use the following lemma in the proofs of Theorems 3.5 and 3.7.

Lemma 3.4. *Suppose $f : I \to I$ is continuous. Suppose K, L, and M are compact subintervals of I such that $K = L \cup M$, $f(L) \subset L$ and $f(M) \subset M$. Suppose there are elements $(x_0, x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$ of (I,f) such that $x_i \in K$ for each i, $y_i \in K$ for each i, $x_0 \in K \backslash M$, $y_0 \in K \backslash L$. Then (I,f) is not the pseudoarc.*

Proof. By hypothesis, $f(K) \subset K$, and $(K, f|K)$ is a nondegenerate subcontinuum of (I, f). Also, $(L, f|L)$ and $(M, f|M)$ are nonempty proper subcontinua of $(K, f|K)$.

We claim that $(K, f|K) = (L, f|L) \cup (M, f|M)$. Clearly, $(L, f|L) \cup (M, f|M) \subset (K, f|K)$. To show the reverse inclusion let $(z_0, z_1, \dots) \in (K, f|K)$. We have two cases.

Case 1. $z_i \in L \cap M$ for each i.

In this case we have $(z_0, z_1, \dots) \in (L, f|L) \cap (M, f|M)$ which puts $(z_0, z_1, \dots) \in (L, f|L) \cup (M, f|M)$.

Case 2. $z_k \notin L \cap M$ for some k.

Without loss of generality, we may assume that $z_k \notin L$. Then $z_k \in M$. Hence $z_0, z_1, z_2, \dots, z_{k-1}$ are also elements of M. Now $z_{k+1} \in K = L \cup M$. But $z_{k+1} \notin L$, as $z_{k+1} \in L$ would imply that $z_k \in L$. Thus, $z_{k+1} \in M$. By induction, $(z_0, z_1, z_2, \dots) \in (M, f|M)$.

This establishes our claim that $(K, f|K) = (L, f|L) \cup (M, f|M)$. Thus, $(K, f|K)$ is a decomposable subcontinuum of (I, f). It follows that (I, f) is not hereditarily indecomposable. By Proposition 2.3, (I, f) is not the pseudoarc. □

Theorem 3.5. *$\{f \in C_s(I, I) \mid (I, f)$ is the pseudoarc$\}$ is nowhere dense in $C_s(I, I)$.*

Proof. Fix $f \in C_s(I, I)$. Let $\epsilon > 0$. We must show that there is a map $g \in C_s(I, I)$ with $d(f, g) < \epsilon$ and a neighborhood $N(g)$ such that for all $h \in N(g)$, (I, h) is not homeomorphic to the pseudoarc.

Now, f has a fixed point p. Choose $a < b < c < d$ suitably close to p all on one side of p. There is a map $g \in C_s(I, I)$ with $d(f, g) < \epsilon$ such that

(1): $a < g(x) < c$ for $x \in [a, c]$
(2): $b < g(x) < d$ for $x \in [b, d]$
(3): $g(w) < w$ for some $w \in [a, b]$
(4): $g(z) > z$ for some $z \in [c, d]$

See Figure 1.

There is a neighborhood $N(g)$ such that if $h \in N(g)$, then (1), (2), (3), and (4) hold with g replaced by h.

Let $h \in N(g)$. Let $L = [a, c]$ and $M = [b, d]$. By (1) and (2) we see that $h(L) \subset L$ and $h(M) \subset M$. Also, by (3), $h(w) < w$ for some $w \in (a, b)$. Since $h(a) > a$, $h(x) = x$ for some $x \in (a, b)$. Set $x_i = x$ for $i = 0, 1, \dots$. Then $(x_0, x_1, x_2, \dots)$ is an element of (I, h) such that $x_i \in K$ for each i, and $x_0 \in K \setminus M$.

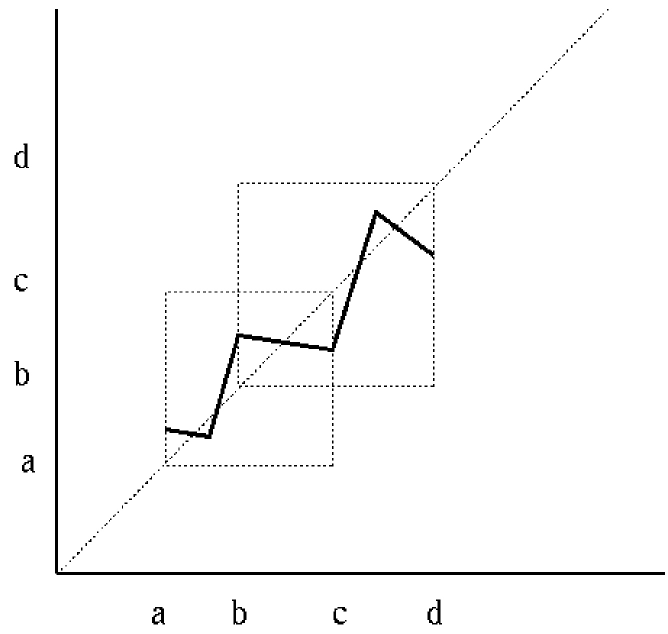

FIGURE 1. Graph of g

Similarly, there is an element $(y_0, y_1, y_2, \dots)$ of (I, h) such that $y_i \in K$ for each i, and $y_0 \in K \setminus L$. By Lemma 3.4, (I, h) is not the pseudoarc. $\square$

Lemma 3.6. *Let $f : I \to I$ be continuous. Suppose that f has a periodic orbit of period four, but f has no periodic orbit of odd period larger than one. Then (I, f) is not homeomorphic to the pseudoarc.*

PROOF. Suppose f satisfies the hypothesis. By [5], Theorem VII.18, page 184, f has a periodic orbit $\{z_1, z_2, z_3, z_4\}$ of period 4 with $z_1 < z_2 < z_3 < z_4$ such that $f(\{z_1, z_2\}) = \{z_3, z_4\}$ and $f(\{z_3, z_4\}) = \{z_1, z_2\}$. It follows that $f^2(z_1) = z_2$, $f^2(z_2) = z_1$, $f^2(z_3) = z_4$, and $f^2(z_4) = z_3$. Since $f(z_2) > z_2$ and $f(z_3) < z_3$, f has a fixed point $p \in (z_2, z_3)$.

We claim that $z_3 \notin \cup_{n=0}^{\infty} f^{2n}([z_2, p])$. To prove this, suppose that $z_3 \in f^{2n}([z_2, p])$ for some positive integer n. Then $f^{2n}(w) = z_3$ for some $w \in (z_2, p)$. Since $f^{2n}(z_2)$ is either z_1 or z_2, $f^{2n}([z_2, w]) \supset [p, z_3]$. There is a closed subinterval J of $[z_2, w]$ with $f^{2n}(J) = [p, z_3]$. Since $f(p) = p$ and $f(z_3)$ is either z_1 or z_2, it follows that $f^{2n+1}(J) \supset J$. Hence, there is a point $v \in J$ with $f^{2n+1}(v) = v$. Since $f^{2n}(J) \cap J = \emptyset$, $f^{2n}(v) \neq v$. Thus, v is a periodic point of f of odd period larger than one. This contradicts our hypothesis and establishes the claim.

By a similar argument, $z_2 \notin \cup_{n=0}^{\infty} f^{2n}([p, z_3])$. Now, set $L = \overline{\cup_{n=0}^{\infty} f^{2n}([z_2, p])}$, $M = \overline{\cup_{n=0}^{\infty} f^{2n}([p, z_3])}$, and $K = L \cup M$. Then K, L, and M are compact subintervals of I, $f^2(L) \subset L$, and $f^2(M) \subset M$. Also, since $f^2(z_1) > z_1$ and $f^2(z_2) < z_2$, there is a point $q \in (z_1, z_2)$ with $f^2(q) = q$. Set $x_i = q$ for $i = 0, 1, \ldots$. Then $(x_0, x_1, x_2, \ldots) \in (I, f^2)$, $x_i \in K$ for each i, and $x_0 \in K \backslash M$. Similarly, we obtain a point $(y_0, y_1, y_2, \ldots) \in (I, f^2)$ with $y_i \in K$ for each i, and $y_0 \in K \backslash L$. By Lemma 3.4, (I, f^2) is not the pseudoarc. Since (I, f) is homeomorphic to (I, f^2), (I, f) is not the pseudoarc. □

Theorem 3.7. *Let $f : I \to I$ be continuous. Suppose that f has a periodic orbit of period two or larger, but f has no periodic orbit of odd period larger than one. Then (I, f) is not homeomorphic to the pseudoarc.*

Proof. By Lemma 3.6 we may assume that f has no periodic orbit of period four. Hence, by [5], Lemma II.3, page 26, f^2 is not turbulent. By hypothesis and Sharkovsky's Theorem, there are points z_1 and z_2 in I with $z_1 < z_2$, $f(z_1) = z_2$, and $f(z_2) = z_1$. Since f^2 is not turbulent, it follows from [5], Proposition III.24, page 66, that $z_2 \notin \overline{W(z_1, f^2)}$. Hence, there is an open interval A with $z_1 \in A$ and $z_2 \notin \overline{\bigcap_{n=0}^{\infty} f^{2n}(A)}$.

There is an open interval V with $z_2 \in V$ and $f(V) \subset A$. Let $E = V \cup f(A)$, $B = \overline{\bigcup_{n=0}^{\infty} f^{2n}(A)}$ and $D = \overline{\bigcup_{n=0}^{\infty} f^{2n}(E)}$. Then $f(B) \subset D$, $f(D) \subset B$, $f^2(B) \subset B$, and $f^2(D) \subset D$.

Let $B = [a, b]$ and $D = [c, d]$. Then $a < z_1 < z_2 < d$, and both points b and c lie in the open interval (z_1, z_2).

First, suppose that $b \geq c$. Then, if we set $L = B$ and $M = D$ and consider the function f^2, it is easy to verify that the hypothesis of Lemma 3.4 holds. Hence (I, f^2) is not homeomorphic to the pseudoarc. Since (I, f) is homeomorphic to (I, f^2), (I, f) is not homeomorphic to the pseudoarc.

Second, suppose that $b < c$. Since $f(b) \geq c$ and $f(c) \leq b$, there is a fixed point p of f with $b < p < c$. Since f has no periodic orbit of odd period larger than one, it follows as in the proof of Lemma 3.6 that $c \notin \bigcup_{n=0}^{\infty} f^{2n}([a, p])$ and $b \notin \bigcup_{n=0}^{\infty} f^{2n}([p, d])$. Hence if we set $L = \overline{\bigcup_{n=0}^{\infty} f^{2n}([a, p])}$ and $M = \overline{\bigcup_{n=0}^{\infty} f^{2n}([p, d])}$ and consider the function f^2, it is easy to verify that the hypothesis of Lemma 8 holds. As in the previous case, (I, f) is not homeomorphic to the pseudoarc. □

Corollary 3.8. *Let $f \in C(I, I)$, and suppose (I, f) is the pseudoarc. Suppose that $h(f) > 0$. Then $h(f) > \frac{\log(2)}{2}$.*

Proof. Since $h(f) > 0$, it follows from [5], Proposition VIII.34, page 218, that f has a periodic point of period larger than two. By Theorem 3.7, f has a periodic orbit of odd period larger than one. Finally, it follows from [5], Proposition VIII.21, page 206, that $h(f) > \frac{\log 2}{2}$. $\square$

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