HOUSTON JOURNAL OF MATHEMATICS, Volume 1, No. 1, 1975.

A d-PSEUDOMANIFOLD WITH f_0 VERTICES HAS AT LEAST df_0-(d-1)(d+2) d-SIMPLICES

Victor Klee

Barnette was the first to prove that if f_k is the number of k-faces of a simple (d+1)-polytope P then (*) $f_0 \geqslant df_d - (d-1)(d+2)$. He later extended (*) to a graph-theoretic setting and was thereby enabled to prove the dual inequality for triangulated d-manifolds. Here his methods are used to provide a different graph-theoretic extension of (*) and thus extend the dual inequality to simplicial d-pseudomanifolds.

Introduction. A d-*polytope* is a d-dimensional set (in a real vector space) that is the convex hull of a finite set, and it is *simple* if each of its vertices is incident to precisely d edges. The inequality (*) becomes an equality when $d \in \{1,2\}$, as follows readily from Euler's theorem, and also when P is obtained from a d-simplex by successive truncations of vertices. Though (*) was first stated for d = 3 by Brückner [6] in 1909, his "proof" was incorrect and the problem remained open for sixty years. See Grünbaum [8] for historical details and an indication of the importance of the "lower bound conjecture."

The inequality (*) was first proved for $d \in \{3,4\}$ by Walkup [12] in 1970 and for arbitrary d by Barnette [3] in 1971. (see Barnette [4], Grünbaum [8], Klee [9], McMullen and Walkup [10], and Walkup [12] for related inequalities.) In the present note, Barnette's methods are used to establish (*) in a different graph-theoretic setting that makes it possible to extend the dual inequality to pseudomanifolds.

Preliminaries. The notion of an LB-system (LB for "lower bound") provides a suitable framework for our inductive arguments. For $d \geqslant 1$ an LB-*system of type* d, sometimes called a d-*system* for brevity, is a nonempty finite collection $\underset{\sim}{F}$ of undirected finite graphs (called *facets*) such that

(a) each facet is d-valent and d-connected,

The preparation of this paper was supported by the Office of Naval Research.

(b) in the graph $\underset{\sim}{F}^u = \cup \underset{\sim}{F}$, each vertex is d-valent (an *external vertex*) or (d+1)-valent (an *internal vertex*), and

(c) for each vertex v and set X of d vertices of $\underset{\sim}{F}^u$ adjacent to v, at most one facet contains $\{v\} \cup X$.

Note that every nonempty subcollection of a d-system is itself a d-system, and the above conditions imply

(d) any two intersecting facets have at least d common vertices, and

(e) each external vertex belongs to a single facet, each internal vertex to at least two and at most d + 1 facets.

The above axioms capture some important aspects of the boundary complex of a simple (d+1)-polytope but they are much less restrictive than that notion. Suppose, for example, that $\underset{\sim}{P}$ is a nonempty finite collection of simple d-polytopes in a real vector space and the intersection of any two members of $\underset{\sim}{P}$ is a face of both. Let $\underset{\sim}{F} = \{F_P : P \in \underset{\sim}{P}\}$, where F_P is the graph formed by the vertices and edges of P. Then condition (c) is obvious and (a) follows from Balinski's theorem [2] that the graph of a d-polytope is d-connected. Thus $\underset{\sim}{F}$ is a d-system if condition (b) is satisfied. In this setting (b) is equivalent to the conjunction of

(b′) each (d-1)-face of a member of $\underset{\sim}{P}$ is incident to at most one additional member of $\underset{\sim}{P}$, and

(b″) any two intersecting members of $\underset{\sim}{P}$ have a (d-1)-face in common.

A d-system $\underset{\sim}{F}$ is said to be *connected* if the graph $\underset{\sim}{F}^u$ is connected. Note that $\underset{\sim}{F}^u$ is actually (d+1)-connected when $\underset{\sim}{F}$ is obtained from the boundary complex of a simple (d+1)-polytope in the manner of the preceding paragraph.

The following two lemmas and their proofs are inspired by observations of Sallee [11, p. 470] and Barnette [3, p. 123] respectively.

1. LEMMA. *If $\underset{\sim}{F}$ is a connected LB-system of type d then the graph $\underset{\sim}{F}^u$ is d-connected.*

PROOF. By (d) and $\underset{\sim}{F}$'s connectedness, the facets can be arranged in a sequence such that each facet after the first has at least d vertices in common with one of its predecessors. Then use the fact that a graph is d-connected if it is the union of two d-connected subgraphs that have at least d vertices in common. This follows readily from a characterization of d-connectedness that was established by Dirac [7, p. 151] and is also derived easily from the max-flow min-cut theorem.

2. LEMMA. *Suppose that F is an LB-system of type d, Q is a connected proper subsystem of F, and C is a subsystem of $F \sim Q$ such that*

(i) *there is a vertex t of Q^u that is not in C^u, and*

(ii) *there is a vertex common to Q^u and C^u. Then at least d vertices of C^u are internal in $Q \cup C$ but external in C.*

PROOF. By (d) there are at least d vertices common to Q^u and C^u, and by (e) they are all internal in $Q \cup C$. Suppose that some vertex s of $Q^u \cap C^u$ is internal in C. By the preceding lemma there are d independent paths in Q^u that join s to t, and on each path the last vertex in C^u is internal in $Q \cup C$ but external in C.

Main result.

3. THEOREM. *The number of internal vertices of a connected LB-system F of type d is at least $d|F| - d$ when there is an external vertex and at least $n = d|F| - (d-1)(d+2)$ when all vertices are internal.*

PROOF. For an artibrary fixed $d \geqslant 1$, we first employ induction on $|F|$ to prove the theorem with n replaced by n - 1. The assertions are obvious when $|F| = 1$. For the inductive step, let v be a vertex of F^u, external if possible, let Q be the collection of all facets incident to v, and let $C_1, \cdots, C_m$ be the connected components of $F \sim Q$. By the preceding lemma, each subgraph C_i^u of F^u has at least d vertices that are internal in $Q \cup C_i$ but external in C_i, whence by the inductive hypothesis C_i^u also has $d|C_i| - d$ vertices that are internal in C_i. Thus at least $d|C_i|$ vertices of C_i^u are internal in F, and since the C_i's are pairwise disjoint the total number of these vertices is at least

$$d\Sigma_1^m |C_i| = d(|F|-|Q|).$$

That is the desired conclusion when v is external in F, for then $|Q| = 1$, and it is also the stated conclusion when all vertices are internal, because $|Q| \leqslant d + 1$ by (e) and hence

$$d(|F|-|Q|) + 1 \geqslant d|F| - d(d+1) + 1 = n - 1.$$

It remains only to show there is no connected d-system F with precisely

$$k = d|F| - d^2 - d + 1$$

vertices. Suppose there is such an F, let v and Q be as in the preceding paragraph, let F' be an isomorph of F disjoint from F, and let v′ and Q' be the correspondents in F' of v and Q. By the dual of a construction used by Barnette [4, p. 351] for a similar purpose, it is possible to eliminate v and v′ and meld each facet in $\underset{\sim}{Q}$ with its correspondent in $\underset{\sim}{Q}'$ (simply combine corresponding edges) so as to obtain from $\underset{\sim}{F} \cup \underset{\sim}{F}'$ a connected d-system with precisely 2k-2 vertices and $2|\underset{\sim}{F}| - |\underset{\sim}{Q}|$ facets. From the inequality of the preceding paragraph it follows that

$$2d|\underset{\sim}{F}| - 2d^2 - 2d = 2k - 2 \geqslant d(2|\underset{\sim}{F}|-|\underset{\sim}{Q}|) - d^2 - d + 1,$$

whence $d|\underset{\sim}{Q}| > d^2 + d$ and (e) is contradicted.

For the case in which there are external vertices, the theorem's inequality becomes an equality if the members of $\underset{\sim}{F}$ are the graphs of simple d-polytopes $P_1, \cdots, P_m$ which are pairwise disjoint except that for $1 < j \leqslant m$ the intersection $P_{j-1} \cap P_j$ is a (d-1)-simplex. When all vertices are internal, equality arises if the members of $\underset{\sim}{F}$ are the graphs of the various d-faces of one of the truncation (d+1)-polytopes mentioned earlier.

The dual inequality for pseudomanifolds. As the term is used here a (simplicial) d-*complex* is a nonempty finite collection $\underset{\sim}{C}$ of (d+1)-sets, the d-*simplices* of $\underset{\sim}{C}$, together with all subsets of these d-simplices. The *dual graph* $DG(\underset{\sim}{C})$ has as its vertices the d-simplices of $\underset{\sim}{C}$, two such vertices being joined by an edge of $DG(\underset{\sim}{C})$ if and only if the two d-simplices intersect in a (d-1)-simplex. For each k-simplex K of $\underset{\sim}{C}$, star $(K,\underset{\sim}{C})$ is the d-complex generated by all d-simplices of $\underset{\sim}{C}$ that contain K and link $(K,\underset{\sim}{C})$ is the (d-k-1)-complex consisting of all members of star $(K,\underset{\sim}{C})$ that are disjoint from K. Note that the dual graphs $DG(\text{star}(K,\underset{\sim}{C}))$ and $DG(\text{link}(K,\underset{\sim}{C}))$ are isomorphic.

As the term is used here, a d-*pseudomanifold* is a d-complex $\underset{\sim}{M}$ such that (i) each (d-1)-simplex of $\underset{\sim}{M}$ is contained in precisely two d-simplices and (ii) the dual graph $DG(\underset{\sim}{M})$ is connected. Note that if a d-complex $\underset{\sim}{C}$ satisfies (i) then each subcomplex of $\underset{\sim}{C}$ that is generated by a connected component of $DG(\underset{\sim}{C})$ is a d-pseudomanifold; such a complex is called a *strong component* of $\underset{\sim}{C}$. Note also that if (i) holds for $\underset{\sim}{C}$ and K is a k-simplex of $\underset{\sim}{C}$ then (i) holds for $\text{link}(K,\underset{\sim}{C})$--that is, each (d-k-2)-simplex of link $(K,\underset{\sim}{C})$ is contained in precisely two (d-k-1)-simplices of the link.

The following lemma and proof were presented by Barnette [5, p. 65] for his graph manifolds and by Adler, Dantzig and Murty [1] for a special class of pseudomanifolds called abstract polytopes.

4. LEMMA. *If $\underset{\sim}{M}$ is a d-pseudomanifold, x is a vertex (0-simplex) of $\underset{\sim}{M}$, and S and T are d-simplices of $\underset{\sim}{M}$ not containing x, then S and T are joined in $DG(\underset{\sim}{M})$ by a*

path that does not use any d-simplex containing x.

PROOF. The assertion is obvious when $d = 1$, for the 1-pseudomanifolds and their dual graphs are simple circuits. Now suppose $d > 1$ and let $\underset{\sim}{M}$, x, S and T be as described. By the connectedness of $DG(\underset{\sim}{M})$ there is in $\underset{\sim}{M}$ a sequence of successively adjacent d-simplices $S = S_o, S_1, \cdots, S_\ell = T$. Let $S_i' = S_i$ when $x \notin S_i$. When $x \in S_i$ let $R_i = S_i \sim \{x\}$ and let S_i' be the unique d-simplex of $\underset{\sim}{M}$ that contains R_i but not x. Since $S_o' = S$ and $S_\ell' = T$ it suffices to show for $1 \leqslant i \leqslant \ell$ that $S_{i-1}' = S_i'$ or $DG(\underset{\sim}{M})$ admits a path from S_{i-1}' to S_i' that does not use any d-simplex incident to x.

The simplices S_{i-1}' and S_i' are adjacent ⟨resp. identical⟩ when neither ⟨resp. precisely⟩ one of S_{i-1} and S_i contains x. For the remaining case, let $K_i = R_{i-1} \cap R_i$ and note that the sequence S_{i-1}', S_{i-1}, S_i, S_i' describes a path from S_{i-1}' to S_i' in $DG(star(K_i,\underset{\sim}{M}))$, whence $S_{i-1}' \sim K_i$ and $S_i' \sim K_i$ belong to the same strong component $\underset{\sim}{Q}_i$ of link $(K_i,\underset{\sim}{M})$. The desired conclusion then follows from the fact that $\underset{\sim}{Q}_i$ is a 2-pseudomanifold.

5. THEOREM. *The dual graph of a d-pseudomanifold is (d+1)-connected.*

PROOF. The assertion is obvious when $d = 1$ and we proceed by induction to show that if S and T are distinct d-simplices of $\underset{\sim}{M}$ and $\underset{\sim}{X}$ is a set of d d-simplices in $\underset{\sim}{M} \sim \{S,T\}$ then $DG(\underset{\sim}{M})$ admits a path from S to T that uses no member of $\underset{\sim}{X}$. Supposing the contrary, let $S = S_o, S_1, \cdots, S_\ell = T$ be a path from S to T that uses the minimum number of members of $\underset{\sim}{X}$, let j be the smallest index for which $S_{j+1} \in \underset{\sim}{X}$, and with $m = \min\{j+d, \ell\}$ let $x \in \cup_{i=j}^{m} S_i$.

If all members of $\underset{\sim}{X}$ contain x let S' and T' be d-simplices of $\underset{\sim}{M}$ that do not contain x and are identical with or adjacent to S and T respectively. By applying the preceding lemma to S' and T' we see that S and T are joined in $DG(\underset{\sim}{M})$ by a path not using any member of $\underset{\sim}{X}$.

If, on the other hand, some member of $\underset{\sim}{X}$ fails to contain x, then $\underset{\sim}{X}$ includes at most $d - 1$ d-simplices of star $(x,\underset{\sim}{M})$ and there exists k such that $j + 2 \leqslant k \leqslant m$ and $S_k \notin \underset{\sim}{X}$. By applying the inductive hypothesis to the appropriate strong component of link $(x,\underset{\sim}{M})$ we see that $DG(star(x,\underset{\sim}{M}))$ admits a path $S_j = T_1, \cdots T_r = S_k$ that uses no member of $\underset{\sim}{X}$, whence the path

$$S_o, \cdots, S_j, T_2, \cdots, T_{r-1}, S_k, \cdots, S_\ell$$

contradicts the minimizing property for which the S_i's were chosen.

6. THEOREM. *Suppose that $\underset{\sim}{M}$ is a d-pseudomanifold and for each vertex x of $\underset{\sim}{M}$ let Q_x denote the set of all strong components of star $(x,\underset{\sim}{M})$. Let $Q(\underset{\sim}{M}) = \cup_x Q_x$ and $\underset{\sim}{F} = \{DG(\underset{\sim}{Q}) : \underset{\sim}{Q} \in Q(\underset{\sim}{M})\}$. Then $\underset{\sim}{F}$ is a connected LB-system of type d with $\underset{\sim}{F}^u = DG(\underset{\sim}{M})$. The numbers of vertices and facets of $\underset{\sim}{F}$ are respectively $f_d(\underset{\sim}{M})$ and $\Sigma_x|Q_x|$.*

PROOF. It follows from the preceding theorem that $\underset{\sim}{F}$ satisfies axiom(a), and (b) is obvious. To verify (c), note that if S_0 is a vertex of $DG(\underset{\sim}{M})$ and $S_1,\cdots,S_d$ are vertices of $DG(\underset{\sim}{M})$ adjacent to S_0 then the intersection $\cap_{i=0}^{d} S_i$ consists of a unique vertex x of $\underset{\sim}{M}$. There is a strong component $\underset{\sim}{Q}$ of star $(x,\underset{\sim}{M})$ that includes all the S_i's, and $DG(\underset{\sim}{Q})$ is the unique member of $\underset{\sim}{F}$ that contains all the vertices $S_0,S_1,\cdots,S_d$.

Plainly $f_0(\underset{\sim}{F}^u) = f_d(\underset{\sim}{M})$. To see that $|\underset{\sim}{F}| = \Sigma_x|Q_x|$, note that if x_1 and x_2 are distinct vertices of $\underset{\sim}{M}$ and $\underset{\sim}{Q}_i$ is a strong component of star $(x_i,\underset{\sim}{M})$, then $\underset{\sim}{Q}_1 \neq \underset{\sim}{Q}_2$. Indeed, let $\underset{\sim}{S}$ be a d-simplex in $\underset{\sim}{Q}_1$, let $T = S$ if $y \notin S$, and if $y \in S$ let T be the d-simplex of $\underset{\sim}{M}$ that contains $S \sim \{y\}$ but not y. Then $T \in \underset{\sim}{Q}_1 \sim \underset{\sim}{Q}_2$.

7. COROLLARY. *A d-pseudomanifold with f_0 vertices has at least $df_0 - (d-1)(d+2)$ d-simplices.*

PROOF. Use Theorems 3 and 7.

REFERENCES

1. I. Adler, G. B. Dantzig and K. G. Murty, *Existence of A-avoiding paths in abstract polytopes,* Math. Programming Studies 1(1974), 41-42.
2. M. Balinski, *On the graph structure of convex polyhedra in n-space* , Pacific J. Math. 11(1961), 431-434.
3. D. Barnette, *The minimum number of vertices of a simple polytope,* Israel J. Math. 10(1971), 121-125.
4. D. Barnette, *A proof of the lower bound conjecture for convex polytopes,* Pacific J. Math. 46(1973), 349-354.
5. D. Barnette, *Graph theorems for manifolds,* Israel J. Math. 16(1973), 62-72.
6. M. Brückner, *Über die Ableitung der allgemeinen Polytope und die nach Isomorphisms verschiedenen Typen derallgemeinenachtzelle(Oktatope),* Verh. Nedrl. Akad. Wetensch. (Sect. I) 10(1909), 3-27.
7. G. A. Dirac, *Extensions of Menger's theorem,* J. London Math. Soc. 38(1963), 148-161.
8. B. Grünbaum, *Convex Polytopes,* Wiley-Interscience (London-New York-Sidney), 1967.
9. V. Klee, *Polyotope pairs and their relationship to linear programming,* Acta Math. 133(1974), 1-25.
10. P. McMullen and D. W. Walkup, *A generalized lower-bound conjecture for simplicial polytopes,* Mathematika 18(1971), 264-273.
11. G. T. Sallee, *Incidence graphs of polytopes,* J. Combinatorial Theory 2(1967), 466-506.
12. D. Walkup, *The lower bound conjecture for 3- and 4-manifolds,* Acta Math. 125(1970), 75-107.

University of Washington
Seattle, Washington

Received July 10, 1975