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DISSIPATIVE MATRICES AND THE MATRIX $A^{-1}A^*$

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ABSTRACT. An $n \times n$ matrix A is dissipative if the imaginary component K in $A = H + iK$ with H, K Hermitian is positive definite. In this paper all relationships between the eigenvalues of $A^{-1}A^*$, $A_n^{-1}A_n^*$, $(A^{-1}A^*)_n$ are characterized when A is dissipative, and where, in general, A_n denotes the principal submatrix of A obtained by deleting the last row and column.

An n-square complex matrix A, written as $A = H + iK$ where H, K are Hermitian, is said to be *dissipative* if its imaginary component K is positive definite. When A is dissipative, A is nonsingular and Fan [1] proved that $A^{-1}A^*$ is similar to a unitary matrix. Thus the eigenvalues of $A^{-1}A^*$ lie on the unit circle. For any $n \times n$ matrix B, let B_n denote the principal submatrix of B obtained by deleting the last row and column. Then $A_n = H_n + iK_n$ is dissipative as well, and therefore $A_n^{-1}A_n^*$ also has eigenvalues on the unit circle. In [1] Fan established an interesting connection between these two sets of eigenvalues: the eigenvalues of $A^{-1}A^*$ are interlaced on the unit circle by the eigenvalues of $A_n^{-1}A_n^*$. (The order on the unit circle implied by this statement is obtained by deleting the point 1 and taking the increasing sense to be the counterclockwise direction.) This interlacing property resembles the interlacing property linking the eigenvalues of an n-square Hermitian matrix to the eigenvalues of any one of its principal (n-1)-square submatrices. The resemblence is not an exact analogy, however, because $A_n^{-1}A_n^*$ is not usually a principal submatrix of $A^{-1}A^*$. It is natural, therefore, to ask what connections exist between the eigenvalues of $A^{-1}A^*$, the eigenvalues of its principal submatrix $(A^{-1}A^*)_n$ and the eigenvalues of the matrix $A_n^{-1}A_n^*$ constructed from a principal submatrix of A, when A is dissipative. Two relationships must exist, an obvious one being the interlacing property just mentioned between the eigenvalues of $A^{-1}A^*$ and those of $A_n^{-1}A_n^*$. To see what the second is, let $d_{n-1}(\lambda)$ denote the greatest common divisor of the n-1 rowed minors of $\lambda I - A^{-1}A^*$,

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where λ is an indeterminate. Then $d_{n-1}(\lambda)$ divides $\det(\lambda I_{n-1} - (A^{-1}A^*)_n)$. By elementary divisor theory, the minimal polynomial of $A^{-1}A^*$ is

$$m(\lambda) = \det(\lambda I - A^{-1}A^*) / d_{n-1}(\lambda).$$

It follows that

$$\frac{\det(\lambda I - A^{-1}A^*)}{m(\lambda)} \Bigg| \det(\lambda I_{n-1} - (A^{-1}A^*)_n).$$

Because $A^{-1}A^*$ is similar to a unitary matrix, its minimal polynomial $m(\lambda)$ has simple roots. The second relationship is now evident from the divisibility formula displayed above: a multiple eigenvalue of $A^{-1}A^*$ is an eigenvalue of $(A^{-1}A^*)_n$ with multiplicity reduced by not more that one.

We shall demonstrate in this paper that a modest sharpening of these two relationships constitute the only connections between the eigenvalues of $A^{-1}A^*$, $(A^{-1}A^*)_n$, and $A_n^{-1}A_n^*$ when A is dissipative. This means, in particular, that when $A^{-1}A^*$ has simple eigenvalues (necessarily on the unit circle), the eigenvalues of its principal submatrix $(A^{-1}A^*)_n$ may be arbitrary numbers in the complex plane, even when the eigenvalues of $A_n^{-1}A_n^*$ are prescribed numbers on the unit circle interlacing (and distinct from) the eigenvalues of $A^{-1}A^*$. For the precise statement, see Corollary 1.

We begin with a lemma.

LEMMA. *Let x be a row n-tuple, y a column n-tuple. Then a nonsingular matrix Y exists with y the last column of Y and x the last row of Y^{-1} if and only if $xy = 1$.*

PROOF. Computing the (n,n) entry of $Y^{-1}Y = I$ shows that $xy = 1$ is necessary. Suppose that $xy = 1$. Since $y \neq 0$, there is a nonsingular n-square matrix S such that $Sy = \tilde{y} = [0,0,0,\ldots,0,1]'$. Let $\tilde{x} = xS^{-1} = [x_1,\ldots,x_n]$. Then $\tilde{x}\tilde{y} = 1$, so that $x_n = 1$. Let $\tilde{X}$, $\tilde{Y}$ be identity matrices except in the last row, which in $\tilde{X}$ is $\tilde{x}$, and in $\tilde{Y}$ is $[-x_1,\ldots,-x_{n-1},1]$. Then $\tilde{X}\tilde{Y} = I$; hence $\tilde{Y}^{-1} = \tilde{X}$. Set $Y = S^{-1}\tilde{Y}$. Then $Y^{-1} = \tilde{Y}^{-1}S = \tilde{X}S$. The last column of Y is $S^{-1}\tilde{y} = y$, and the last row of $Y^{-1} = \tilde{X}S$ is $\tilde{x}S = x$, as required.

Let $\beta_1,\ldots,\beta_n$ be numbers on the unit circle, different from 1, and numbered such that $0 < \arg\beta_n \leqslant \cdots \leqslant \arg\beta_1 < 2\pi$. Let $\tilde{\beta}_1,\ldots,\tilde{\beta}_{n-1}$ be further numbers on the unit circle, with $0 \leqslant \arg\tilde{\beta}_{n-1} \leqslant \cdots \leqslant \arg\tilde{\beta}_1 < 2\pi$. We say that $\tilde{\beta}_1,\ldots,\tilde{\beta}_{n-1}$ interlace $\beta_1,\ldots,\beta_n$ if

$$\arg\beta_n \leqslant \arg\tilde{\beta}_{n-1} \leqslant \arg\beta_{n-1} \leqslant \cdots \leqslant \arg\beta_2 \leqslant \arg\tilde{\beta}_1 \leqslant \arg\beta_1.$$

Denote by f the Mobius function given by

$$f(z) = (z - i)(z + i)^{-1}$$

for all complex numbers z. This function f is an order preserving bijection of the real axis (∞ excluded) onto the ordered circumference of the unit circle (cut by excluding 1), counterclockwise being the increasing direction on the cut unit circle. This function f will be used in the proof of the main result of this paper, which we now state.

THEOREM. *Let $\beta_1,\ldots,\beta_n$, $\widetilde{\beta}_1,\ldots,\widetilde{\beta}_{n-1}$, $\gamma_1,\ldots,\gamma_{n-1}$ be complex numbers. Then an $n \times n$ dissipative matrix A exists such that*

(i) *$A^{-1}A^*$ has eigenvalues $\beta_1,\ldots,\beta_n$,*

(ii) $A_n^{-1}A_n^*$ has eigenvalues $\widetilde{\beta}_1,\ldots,\widetilde{\beta}_{n-1}$,

(iii) $(A^{-1}A^*)_n$ has eigenvalues $\gamma_1,\ldots,\gamma_{n-1}$,

if and only if

(a) *$\beta_1,\ldots,\beta_n$ are on the unit circle and not equal to 1,*

(b) *$\widetilde{\beta}_1,\ldots,\widetilde{\beta}_{n-1}$ are also on the unit circle and on this circle cut at 1 interlace $\beta_1,\ldots,\beta_n$,*

(c) *the multiplicity of β_j among $\gamma_1,\ldots,\gamma_{n-1}$ is not less than { the multiplicity of β_j among $\beta_1,\ldots,\beta_n$ } - 1, with strict inequality whenever β_j has multiplicity among $\widetilde{\beta}_1,\ldots,\widetilde{\beta}_{n-1}$ as great as among $\beta_1,\ldots,\beta_n$; $j = 1,\ldots,n$.*

PROOF. Let A be a dissipative matrix such that (i), (ii), (iii) all hold. We wish to prove that (a), (b), (c) all hold. Take $\alpha_1 \geqslant \cdots \geqslant \alpha_n$ to be the eigenvalues of the Hermitian matrix $K^{-1/2}HK^{-1/2}$, where $A = H + iK$. Then a nonsingular matrix X exists such that

$$XKX^* = I, \; XHX^* = \mathrm{diag}(\alpha_1,\ldots,\alpha_n),$$

so that

$$A = X^{-1}\mathrm{diag}(\alpha_1+i,\ldots,\alpha_n+i)X^{*-1}.$$

Hence

(1) $A^{-1}A^* = X^*\mathrm{diag}(\beta_1,\ldots,\beta_n)X^{*-1}$,

where

(2) $\beta_t = f(\alpha_t), \quad t = 1,\ldots,n.$

Since f maps the real axis to the unit circle cut at 1, this proves (a).

With λ an indeterminate, from (1) we get

$$\lambda I - A^{-1}A^* = X^* \,\mathrm{diag}(\lambda-\beta_1,\ldots,\lambda-\beta_n)X^{*-1},$$

and thus

$$(\lambda I - A^{-1}A^*)^{-1} = X^* \operatorname{diag}((\lambda-\beta_1)^{-1},\ldots,(\lambda-\beta_n)^{-1})X^{*-1}.$$

Multiplying by

$$g(\lambda) = \det(\lambda I - A^{-1}A^*) = (\lambda-\beta_1) \cdots (\lambda-\beta_n),$$

we get

(3) $\operatorname{adj}(\lambda I - A^{-1}A^*) = X^* \operatorname{diag}(\ldots, g(\lambda)/(\lambda-\beta_t), \ldots)X^{*-1}$,

where adj indicates adjugate. The (n,n) element of $\operatorname{adj}(\lambda I - A^{-1}A^*)$ is the characteristic polynomial of $(A^{-1}A^*)_n$. Let the last row of X^* be $(x_1,\ldots,x_n)$ and let the last column of X^{*-1} be $(y_1,\ldots,y_n)'$. Equating the (n,n) elements of each side of (3), we get

(4) $\det(\lambda I - (A^{-1}A^*)_n) = \sum_{t=1}^{n} x_t y_t g(\lambda)/(\lambda-\beta_t).$

From (1) we see that the eigenvalues $\beta_1,\ldots,\beta_n$ of $A^{-1}A^*$ are linked to the roots $\alpha_1,\ldots,\alpha_n$ of

$$\det(H - \lambda K) = \det K \cdot (\alpha_1-\lambda) \cdots (\alpha_n-\lambda)$$

by (2). Applying this fact to the (n-1)-square dissipative matrix $A_n = H_n + iK_n$, we see that eigenvalues $\tilde{\beta}_1,\ldots,\tilde{\beta}_{n-1}$ of $A_n^{-1}A_n^*$ are linked to the roots (call them $\tilde{\alpha}_1,\ldots,\tilde{\alpha}_{n-1}$) of $\det(H_n - \lambda K_n)$ by

(5) $\tilde{\beta}_t = f(\tilde{\alpha}_t), \qquad t = 1,\ldots,n-1.$

In particular, $\tilde{\beta}_1,\ldots,\tilde{\beta}_{n-1}$ are also on the cut unit circle, establishing the first part of (b).

From

$$H - \lambda K = X^{-1}\operatorname{diag}(\alpha_1-\lambda,\ldots,\alpha_n-\lambda)X^{*-1}$$

we get

$$(H - \lambda K)^{-1} = X^*\operatorname{diag}((\alpha_1-\lambda)^{-1},\ldots,(\alpha_n-\lambda)^{-1})X.$$

Multiplying by

$$h(\lambda) = \det(H - \lambda K) = \det K \cdot (\alpha_1-\lambda) \cdots (\alpha_n-\lambda),$$

yields

(6) $\operatorname{adj}(H - \lambda K) = X^*\operatorname{diag}(\ldots, h(\lambda)/(\alpha_t-\lambda),\ldots)X.$

The (n,n) entry of $\operatorname{adj}(H - \lambda K)$ is $\det(H_n-\lambda K_n)$. Equating the (n,n) entry of each side of (6) thus produces

(7) $\det(H_n - \lambda K_n) = \sum_{t=1}^{n} |x_t|^2 h(\lambda)/(\alpha_t-\lambda).$

Let $\mu_1 > \cdots > \mu_s$ be the distinct numbers among $\alpha_1 \geqslant \cdots \geqslant \alpha_n$, with μ_t having multiplicity e_t, for t = 1,...,s. Set

$$\nu_t = f(\mu_t), \qquad t = 1,\ldots,s,$$

so that by (2) $\nu_1,\ldots,\nu_s$ are the distinct numbers among $\beta_1,\ldots,\beta_n$, with ν_t having multiplicity e_t, t = 1,...,s. Using

$$g(\lambda) = \prod_{t=1}^{s} (\lambda-\nu_t)^{e_t},$$

we may rewrite (4) as

$$(8)\quad \det(\lambda I_n - (A^{-1}A^*)_n) = \left\{ \prod_{t=1}^{s} (\lambda-\nu_t)^{e_t-1} \right\} \left\{ \sum_{t=1}^{s} \left[\sum_{\substack{j \\ \alpha_j=\mu_t}} x_j y_j \right] \prod_{\substack{k=1 \\ k\neq t}}^{s} (\lambda-\nu_k) \right\}.$$

The sum in square parentheses here (and in (9) below) is over all e_t values of j for which $\alpha_j = \mu_t$. Using

$$h(\lambda) = \det K \cdot \prod_{t=1}^{s} (\mu_t-\lambda)^{e_t},$$

we may rewrite (7) as

$$(9)\quad \det(H_n - \lambda K_n) = \det K \left\{ \prod_{t=1}^{s} (\mu_t-\lambda)^{e_t-1} \right\} \left\{ \sum_{t=1}^{s} \left[\sum_{\substack{j \\ \alpha_j=\mu_t}} |x_j|^2 \right] \prod_{\substack{k=1 \\ k\neq t}}^{s} (\mu_k-\lambda) \right\}.$$

In the next paragraph we derive consequences of the basic formulas (8) and (9).

By hypothesis (iii)

$$(10)\quad \det(\lambda I - (A^{-1}A^*)_n) = (\lambda-\gamma_1) \cdots (\lambda-\gamma_{n-1}).$$

Comparing (8) and (10), we see that the numbers $\gamma_1,\ldots,\gamma_{n-1}$ consist of

(11) ν_1 (e_1-1 times),...,ν_s (e_s-1 times),

together with s-1 further numbers which we denote by

$$\eta_1,\ldots,\eta_{s-1}.$$

Cancelling the common factors

$$(\lambda-\nu_t)^{e_t-1}, \qquad t = 1,\ldots,s,$$

from the equality produced by equating the right-hand sides of (8) and (10), we are led to

$$(12)\quad (\lambda-\eta_1) \cdots (\lambda-\eta_{s-1}) = \sum_{t=1}^{s} \theta_t \prod_{\substack{k=1 \\ k\neq t}}^{s} (\lambda-\nu_k),$$

where

$$(13)\quad \theta_t = \sum_{\substack{j \\ \alpha_j=\mu_t}} x_j y_j, \qquad t = 1,\ldots,s.$$

By hypothesis (ii), (5), and the remarks above (5),

$$(14)\quad \det(H_n - \lambda K_n) = \det K_n(\widetilde{\alpha}_1 - \lambda) \cdots (\widetilde{\alpha}_{n-1}-\lambda).$$

Comparing (9) and (14), we see that the numbers $\widetilde{\alpha}_1,\ldots,\widetilde{\alpha}_{n-1}$ consist of

$$(15)\quad \mu_1 \ (e_1\text{-}1 \text{ times}),\ldots,\mu_s \ (e_s\text{-}1 \text{ times})$$

together with s-1 further real numbers which we denote by

$$\xi_1,\ldots,\xi_{s-1}.$$

Applying f then shows that the numbers $\widetilde{\beta}_1,\ldots,\widetilde{\beta}_{n-1}$ consist of the numbers (11) together with s-1 additional numbers $f(\xi_1),\ldots,f(\xi_{s-1})$. Denote these latter numbers by $\zeta_1,\ldots,\zeta_{s-1}$, so that $\zeta_t = f(\xi_t)$, t = 1,...,s-1. Cancelling the common factors

$$(\mu_t-\lambda)^{e_t-1}$$

from the equality produced by equating the right-hand sides of (9) and (14), we are led to

$$(16)\quad \det K_n \det K^{-1}(\xi_1-\lambda) \cdots (\xi_{s-1}-\lambda) = \sum_{t=1}^{s} \varphi_t \prod_{\substack{k=1 \\ k\neq t}}^{s} (\mu_k-\lambda)$$

where

$$(17)\quad \varphi_t = \sum_{\substack{j \\ \alpha_j=\mu_t}} |x_j|^2, \qquad t = 1,\ldots,s.$$

Evaluating (16) at μ_t yields

$$(18)\quad \det K_n \det K^{-1}(\xi_1-\mu_t) \cdots (\xi_{s-1}-\mu_t) = \varphi_t \prod_{\substack{k=1 \\ k\neq t}}^{s} (\mu_k-\mu_t).$$

Since $\varphi_t \geqslant 0$ and $\mu_1 > \cdots > \mu_s$, the right-hand side of (18) has sign $(-1)^{s-t}$ whenever

$\varphi_t \neq 0$. If all of $\varphi_1,...,\varphi_s$ are nonzero the polynomial on the right-hand side of (16) therefore takes alternate signs at $\mu_1,...,\mu_s$, so that its roots interlace $\mu_1,...,\mu_s$. By a continuity argument this still holds if some of $\varphi_1,...,\varphi_s$ are zero. Thus $\xi_1,...,\xi_{s-1}$ interlace $\mu_1,...,\mu_s$, and hence $\xi_1,...,\xi_{s-1}$ augmented with the numbers (15) interlace $\mu_1,...,\mu_s$ augmented with (15); that is, $\widetilde{\alpha}_1,...,\widetilde{\alpha}_{n-1}$ interlace $\alpha_1,...,\alpha_n$. Therefore $\widetilde{\beta}_1 = f(\widetilde{\alpha}_1),...,\widetilde{\beta}_{n-1} = f(\widetilde{\alpha}_{n-1})$ interlace $\beta_1 = f(\alpha_1),...,\beta_n = f(\alpha_n)$ on the cut unit circle. This finishes the proof of (b). We also see from (18) that at least one of $\xi_1,...,\xi_{s-1}$ equals μ_t if and only if $\varphi_t = 0$; thus (by (17)),

(19) μ_t lies among $\xi_1,...,\xi_{s-1}$ iff $x_j = 0$ for all j with $\alpha_j = \mu_t$.

Let j be fixed and suppose t is such that $\beta_j = \nu_t$, so that the multiplicity of β_j among $\beta_1,...,\beta_n$ is e_t. From (11) we observed that the multiplicity of ν_t (and therefore β_j) among $\gamma_1,...,\gamma_{n-1}$ is at least e_t-1. This proves the first part of (c). If β_j has multiplicity at least e_t among $\widetilde{\beta}_1,...,\widetilde{\beta}_{n-1}$ then $\alpha_j = f^{-1}(\beta_j) = \mu_t$ has multiplicity at least e_t among $\widetilde{\alpha}_1 = f^{-1}(\widetilde{\beta}_1),...,\widetilde{\alpha}_{n-1} = f^{-1}(\widetilde{\beta}_{n-1})$, implying (see (15)) that at least one of $\xi_1,...,\xi_{s-1}$ equals μ_t and therefore by (19) that $x_j = 0$ for *every* j with $\alpha_j = \mu_t$. By (13) this implies $\theta_t = 0$. Evaluating (12) at ν_t then produces

$$(\nu_t-\eta_1) \cdots (\nu_t-\eta_{s-1}) = \theta_t \prod_{\substack{k=1 \\ k\neq t}}^{s} (\nu_t-\nu_k) = 0,$$

and yields the conclusion that at least one of $\eta_1,...,\eta_{s-1}$ equals ν_t. But this means (see the discussion above and below (11)) that the multiplicity of $\nu_t = f(\mu_t) = f(\alpha_j) = \beta_j$ among $\gamma_1,...,\gamma_{n-1}$ is at least e_t. This proves the second part of assertion (c) and completes the proof that (i), (ii), (iii) together imply (a), (b), (c).

Suppose now that numbers $\beta_1,...,\beta_n$, $\widetilde{\beta}_1,...,\widetilde{\beta}_{n-1}$, $\gamma_1,...,\gamma_{n-1}$ are given, satisfying conditions (a), (b), (c). We wish to construct a dissipative matrix A for which (i), (ii), (iii) are satisfied. We let

$$\nu_1 \text{ (}e_1\text{ times)},...,\nu_s \text{ (}e_s\text{ times)}$$

be the distinct numbers among $\beta_1,...,\beta_n$; then (by (b)) $\widetilde{\beta}_1,...,\widetilde{\beta}_{n-1}$ consist of the numbers displayed in (11) together with s-1 additional numbers interlacing $\nu_1,...,\nu_{s-1}$ on the cut unit circle and which we choose to denote by $\zeta_1,...,\zeta_{s-1}$; and by (c) $\gamma_1,...,\gamma_{n-1}$ consist of the numbers (11) together with s-1 additional numbers which we elect to denote by $\eta_1,...,\eta_{s-1}$. Furthermore, by the last part of (c), if at least one of

$\zeta_1,\ldots,\zeta_{s-1}$ equals ν_t then at least one of $\eta_1,\ldots,\eta_{s-1}$ equals ν_t, for each fixed $t = 1,2,\ldots,s$. Let

$$\alpha_t = f^{-1}(\beta_t), \quad t = 1,\ldots,n, \quad \mu_t = f^{-1}(\nu_t), \quad t = 1,\ldots,s,$$

so that the distinct numbers among $\alpha_1,\ldots,\alpha_n$ are

$$\mu_1 \ (e_1 \text{ times}),\ldots,\mu_s \ (e_s \text{ times}).$$

Then the numbers $f^{-1}(\widetilde{\beta}_1),\ldots,f^{-1}(\widetilde{\beta}_{n-1})$ consist of the numbers (15) together with s-1 additional numbers which we denote by $\xi_1 = f^{-1}(\zeta_1),\ldots,\xi_{s-1} = f^{-1}(\zeta_{s-1})$. Since $\zeta_1,\ldots,\zeta_{s-1}$ interlace $\nu_1,\ldots,\nu_s$ on the cut unit circle, $\xi_1,\ldots,\xi_{s-1}$ interlace $\mu_1,\ldots,\mu_s$ on the real axis.

We begin by choosing nonnegative numbers $\varphi_1,\ldots,\varphi_s$, not all zero, such that the polynomial

$$(20) \quad \sum_{t=1}^{s} \varphi_t \prod_{\substack{k=1 \\ k\neq t}}^{s} (\mu_k - \lambda)$$

has $\xi_1,\ldots,\xi_{s-1}$ as its roots, i.e., shall equal

$$(21) \quad (\xi_1-\lambda) \cdots (\xi_{s-1}-\lambda).$$

We determine φ_t by evaluating both (20) and (21) at μ_t; equating the results yields

$$(22) \quad \varphi_t = \prod_{k=1}^{s-1} (\xi_k - \mu_t) \Big/ \prod_{\substack{k=1 \\ k\neq t}}^{s} (\mu_k - \mu_t).$$

Because $\xi_1,\ldots,\xi_{s-1}$ interlace $\mu_1,\ldots,\mu_s$, the numerator of the righthand side of (22) has sign $(-1)^{s-t}$, or zero, and the denominator has sign $(-1)^{s-t}$. Thus the number φ_t given by (22) is nonnegative. For this choice of φ_t, $t = 1,\ldots,s$, the polynomials (20), (21), of degree at most s-1, are equal at s distinct values of λ and thus are equal polynomials. Observe that $\varphi_t = 0$ if and only if at least one of $\xi_1,\ldots,\xi_{s-1}$ is μ_t. Having found values for $\varphi_1,\ldots,\varphi_s$, we use (17) to obtain values for $x_1,\ldots,x_n$. These numbers $x_1,\ldots,x_n$ are not unique: any choice such that (17) holds for $t = 1,\ldots,s$ will do. From (17) and the fact that $\varphi_t = 0$ if and only if at least one of $\xi_1,\ldots,\xi_{s-1}$ is μ_t, we see that (19) holds. In due course $x_1,\ldots,x_n$ will be placed in the last row of a certain matrix.

We next choose numbers $\theta_1,\ldots,\theta_s$ such that the polynomial identity (12) holds. Evaluating each side of (12) at ν_t leads to the choice

$$(23)\quad \theta_t = \prod_{k=1}^{s-1} (\nu_t-\eta_k) \Big/ \prod_{\substack{k=1\\k\neq t}}^{s} (\nu_t-\nu_k).$$

With θ_t defined by (23) for $t = 1,\dots,s$, the two sides of (12) are polynomials of degree at most s-1 equal for s distinct values of λ, and therefore are equal polynomials. Thus (12) will hold if $\theta_1,\dots,\theta_s$ are given by (23). Having specified by (23) the value of the θ_t, we construct numbers $y_1,\dots,y_n$ such that (13) holds for $t = 1,\dots,s$. (The quantities $x_1,\dots,x_n$ in equation (13) were selected in the last paragraph.) There always will be values of $y_1,\dots,y_n$ such that (13) holds, provided that $\theta_t = 0$ whenever $x_j = 0$ for each j such that $\alpha_j = \mu_t$. To see that this condition is satisfied, note that when $x_j = 0$ for all j with $\alpha_j = \mu_t$, we obtain from (19) that at least one of $\xi_1,\dots,\xi_{s-1}$ equals μ_t; therefore at least one of $\zeta_1 = f(\xi_1),\dots,\zeta_{s-1} = f(\xi_{s-1})$ equals $\nu_t = f(\mu_t)$, and hence (by a remark in the middle of the paragraph above (20)), at least one of $\eta_1,\dots,\eta_{s-1}$ equals ν_t. This means (by (23)) that θ_t is indeed zero. The numbers $y_1,\dots,y_n$ just constructed will in due course be placed in the last column of a certain matrix.

The numbers $x_1,\dots,x_n$, $y_1,\dots,y_n$ constructed in the last two paragraphs are such that if θ_t for $t = 1,\dots,s$ are defined by (13), then (12) holds and if φ_t for $t = 1,\dots,s$ are defined by (17) the right-hand side of (16) equals (21). Comparing the leading coefficients in these polynomial equations, we see that

$$(24)\quad \sum_{j=1}^{n} x_j y_j = 1, \qquad \sum_{j=1}^{n} |x_j|^2 = 1.$$

Now let X be a nonsingular matrix with $x_1,\dots,x_n$ in the last row of X^* and $y_1,\dots,y_n$ in the last column of X^{*-1}. This matrix exists by the first part of (24) and the lemma. For this X, let

$$H = X^{-1} \operatorname{diag}(\alpha_1,\dots,\alpha_n)X^{*-1}, \qquad K = X^{-1}X^{*-1},$$

and put $A = H + iK$. Then K is positive definite, so that A is dissipative. For this K we find that $K^{-1} = X^*X$ and a comparison of the (n,n) elements shows that $\det K_n/\det K = \Sigma_{j=1}^n |x_j|^2 = 1$; therefore $\det K_n = \det K$. We now apply to this A the calculations in the first part of the proof. By construction the eigenvalues of $A^{-1}A^*$ are $f(\alpha_1) = \beta_1,\dots,f(\alpha_n) = \beta_n$. The eigenvalues $(A^{-1}A^*)_n$ are the roots of the polynomial

(8). This polynomial shows roots given by (11), and s-1 further roots obtained from the polynomial on the right-hand side of (12). By construction the two sides of (12) are equal, meaning (by the definitions of $\nu_1,\dots,\nu_s$, $\eta_1,\dots,\eta_{s-1}$) that (8) equals $(\lambda-\gamma_1)\cdots(\lambda-\gamma_{n-1})$. Thus $(A^{-1}A^*)_n$ has the required eigenvalues. The eigenvalues of $A_n^{-1}A_n^*$ will be the function f applied to the roots of $\det(H_n - \lambda K_n)$, that is, applied to the roots of the right-hand side of (9). Polynomial (9) shows roots given by (15), and s-1 further roots given by the polynomial on the right-hand side of (16). By construction the right-hand side of (16) equals (21), and since $\det K = \det K_n$, we see that the right-hand side of (16) equals the left-hand side of (16). Thus the right-hand side of (9) has roots (15) and $\xi_1,\dots,\xi_{s-1}$. Applying f to these roots produces the numbers (11) and $\zeta_1,\dots,\zeta_{s-1}$, that is, $\widetilde{\beta}_1,\dots,\widetilde{\beta}_{n-1}$. The proof is complete.

COROLLARY 1. *Given distinct numbers* $\beta_1,\dots,\beta_n$, $\widetilde{\beta}_1,\dots,\widetilde{\beta}_{n-1}$ *on the unit circle cut at 1, with* $\widetilde{\beta}_1,\dots,\widetilde{\beta}_{n-1}$ *interlacing* $\beta_1,\dots,\beta_n$, there exists a dissipative matrix A such *that* $A^{-1}A^*$ *has eigenvalues* $\beta_1,\dots,\beta_n$, $A_n^{-1}A_n^*$ *has eigenvalues* $\widetilde{\beta}_1,\dots,\widetilde{\beta}_{n-1}$, *and* $(A^{-1}A^*)_n$ *has arbitrarily given complex numbers* $\gamma_1,\dots,\gamma_{n-1}$ *as eigenvalues.*

COROLLARY 2. *If* $A^{-1}A^*$ *has precisely s distinct eigenvalues, then* $(A^{-1}A^*)_n$ *has at most s-1 eigenvalues not on the unit circle. If these eigenvalues are denoted by* $\eta_1,\dots,\eta_{s-1}$, *then*

$$(25)\quad s\prod_{k=1}^{s-1}\big|\,|\eta_k|-1\,\big| / 2^{s-1} \leqslant (\lambda_{max}(K)/\lambda_{min}(K))^{1/2},$$

where $\lambda_{max}(K)$, $\lambda_{min}(K)$ *denote the greatest and least eigenvalues of the imaginary component K of A.*

PROOF. The fact that $(A^{-1}A^*)_n$ can have at most s-1 eigenvalues not on the unit circle follows from the first part of the proof of the theorem; in the notation of that proof these eigenvalues are $\eta_1,\dots,\eta_{s-1}$. The following argument will work even if some of $\eta_1,\dots,\eta_{s-1}$ are on the unit circle, but then yields a trivial result. Evaluating both sides of (12) at ν_t yields (23), with θ_t given by (13). Hence

$$(26)\quad \prod_{k=1}^{s-1}|\nu_t-\eta_k| \Big/ \prod_{\substack{k=1\\k\neq t}}^{s}|\nu_t-\nu_k| \leqslant \sum_{\substack{j\\ \alpha_j=\mu_t}} |x_j|\,|y_j|.$$

Since the ν_t are on the unit circle, $|\nu_t - \nu_k| \leqslant 2$. Since the least distance from η_k to the unit circle is $||\eta_k| - 1|$, from (26) we get

$$\prod_{k=1}^{s-1} \big| |\eta_k| - 1 \big| / 2^{s-1} \leqslant \sum_{\substack{j \\ \alpha_j = \mu_t}} |x_j|\, |y_j|.$$

Summing over t yields

$$s \prod_{k=1}^{s-1} \big| |\eta_k| - 1 \big| / 2^{s-1} \leqslant \sum_{j=1}^{n} |x_j|\, |y_j|$$

$$\leqslant \left\{ \sum_{j=1}^{n} |x_j|^2 \right\}^{1/2} \left\{ \sum_{j=1}^{n} |y_j|^2 \right\}^{1/2}.$$

Now $\Sigma_{j=1}^{n} |x_j|^2$ is the (n,n) element of $X^*X = K^{-1}$ and hence is bounded above by $\lambda_{max}(K^{-1}) = \lambda_{min}(K)^{-1}$. Also $\Sigma_{j=1}^{n} |y_j|^2$ is the (n,n) element of $X^{-1}X^{*-1} = K$, and thus is bounded above by $\lambda_{max}(K)$. Inserting these estimates yields the result.

Inequality (25) shows that assigning large eigenvalues to $(A^{-1}A^*)_n$ forces substantial structure on the imaginary component K of A.

Further inequalities linking the eigenvalues of $A^{-1}A^*$ and $A_n^{-1}A_n^*$ will be found in [1] and [2]. The author is indebted to Professor Fan for providing him with a preprint of [1].

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