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TRANSFERRED BOUNDS FOR SQUARE FUNCTIONS

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Abstract. **Let G be a locally compact abelian group, and let $u \to R_u$ be a uniformly bounded, strongly continuous representation of G in a closed subspace X of $L^p(\mu)$, where μ is an arbitrary measure and $1 \le p < \infty$. We show that under appropriate circumstances the representation R will transfer to X the bounds for square functions defined by sequences of translation-invariant operators on $L^p(G)$. In the case when G is the additive group of real numbers and $1 < p < \infty$, the circle of ideas from the abstract setting is refined so as to provide counterparts for X of classical square function inequalities such as Littlewood-Paley.**

1. Introduction. Let G be a locally compact abelian group with dual group $\hat{G}$, and suppose that $u \to R_u$ is a uniformly bounded, strongly continuous representation of G in a closed subspace X of $L^p(\mu)$, where μ is an arbitrary measure and $1 \le p < \infty$. Denote by $M_p(\hat{G})$ the Banach algebra of $L^p(G)$-Fourier multipliers. In this abstract framework we shall study circumstances under which the representation R will transfer to X estimates for square functions defined by sequences $\{\varphi_n\}_{n=1}^{\infty} \subseteq M_p(\hat{G})$. In the case when G is the additive group $\mathbb{Z}$ of integers, $1 < p < \infty$, and the sequence $\{\varphi_n\}_{n=1}^{\infty}$ consists of functions having bounded variation on the unit circle $\mathbf{T}$, the transference of square function estimates was initiated in [**2**, §§2.3] as a tool for treating the almost everywhere convergence properties of discrete averages defined by a power-bounded operator on X. The methods and results in [2] rely on features special to the concrete setting of $\mathbb{Z}$ and

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its dual group $\mathbb{T}$. Our aim in what follows below will be to formulate and develop the transference of square function estimates in the abstract setting described above, and thereby to incorporate square functions into the general transference theory initiated by R. R. Coifman and G. Weiss in [**7**], [**8**].

In §2 we take up the transference to X of bounds for square functions on G defined by sequences of convolution operators-that is, we consider the particular case when each $\varphi_n \in M_p(\hat{G})$ is the Fourier transform of a function belonging to $L^1(G)$. In §3 we pass to the general case when each $\varphi_n \in M_p(\hat{G})$ is only assumed to be continuous as a mapping of $\hat{G}$ into the complex numbers $\mathbb{C}$. In order to transfer such multipliers in this broad a framework, it is necessary to assume additionally in §3 that X coincides with $L^p(\mu)$ and that the representation R in $L^p(\mu)$ also has a uniformly bounded version acting in $L^2(\mu)$. In broad terms, the role of these two additional assumptions consists of providing enough spectral decomposability of R in $L^2(\mu)$ (*via* Stone's Theorem) to implement transference to $L^p(\mu)$ of individual $L^p(G)$-Fourier multipliers ([**6**, Theorem (2.1) and Remarks, p. 57]). Such auxiliary assumptions on R can largely by circumvented in the case when $1 < p < \infty$ and $G = \mathbb{Z}$, or G is the additive group $\mathbb{R}$ of real numbers, since under these circumstances the requisite spectral decomposition of the representation R automatically exists in $X \subseteq L^p(\mu)$ ([**3**, Theorems (4.8)-(ii) and (4.21)]), and can be used to transfer functions of bounded variation on $\mathbb{T}$ or $\mathbb{R}$, respectively. This theme is taken up in §4 below, where the general methods of §§2,3 are refined so as to provide directly the analogues for $G = \mathbb{R}$ of the square function transference results previously obtained for $G = \mathbb{Z}$ in [**2**, §§2,3].

Henceforth we shall denote the Fourier transform of a function f by $\hat{f}$. The set of positive integers will be denoted by $\mathbb{N}$. The set-theoretic (respectively, group-theoretic) difference of two sets A and B will be written $A \backslash B$ (respectively, $A - B$). The Banach algebra of all bounded linear mappings of a Banach space Y into itself will be denoted by $\mathfrak{B}(Y)$, and the identity operator on Y will be symbolized by I.

2. Transferred bounds for square functions defined by convolution operators. We return to the abstract setting described at the outset of §1. To fix notation for this section, let G be a locally compact abelian group with Haar measure λ, and let X be a closed subspace of $L^p(\mathcal{M}, \mu)$, where

$(\mathcal{M}, \mu)$ is an arbitrary measure space, and $1 \leq p < \infty$. We shall denote by R a strongly continuous representation of G in X such that

$$c \equiv \sup\{\|R_u\| : u \in G\} < \infty. \tag{2.1}$$

For $k \in L^1(G)$, let $H_k : X \to X$ be the corresponding *transferred convolution operator* defined by X-valued Bochner integration as follows:

$$H_k g = \int_G k(u) R_{-u} g \, d\lambda(u), \text{ for all } g \in X.$$

The operator H_k is clearly a bounded linear operator such that $\|H_k\| \leq c\|k\|_1$. The Coifman-Weiss General Transference Result improves the order of magnitude of this bound-specifically, $\|H_k\|$ does not exceed $c^2 N_p(k)$, where $N_p(k)$ denotes the norm of convolution by k on $L^p(G, \lambda)$ (see [**7**, §2], or, for the generality stated here, [**5**, Theorem (2.3)]). Each of the two theorems in the present section extends the Coifman-Weiss General Transference Result to square functions by suitably adapting its proof.

Theorem 2.2. *Suppose that $\{k_j\}_{j \geq 1} \subseteq L^1(G)$. For each j let T_{k_j} be the convolution operator on $L^p(G, \lambda)$ defined by k_j. If C is a constant such that*

$$\left\| \left\{ \sum_{j \geq 1} |T_{k_j} f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \leq C \left\| \left\{ \sum_{j \geq 1} |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)}, \tag{2.3}$$

for all sequences $\{f_j\}_{j \geq 1} \subseteq L^p(G)$,

then

$$\left\| \left\{ \sum_{j \geq 1} |H_{k_j} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c^2 C \left\| \left\{ \sum_{j \geq 1} |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}, \tag{2.4}$$

for all sequences $\{g_j\}_{j \geq 1} \subseteq X$.

Proof: For purposes of utilizing Fubini's Theorem, we observe at the outset that [**5**, Lemma (2.5)] permits us to assume without loss of generality that the measure space $(\mathcal{M}, \mu)$ is sigma-finite, and that for $g \in X$, expressions such as $(R_u g)(\omega)$ have a version jointly measurable in (u, ω). By Monotone Convergence we can assume that $\{k_j\}_{j\geq 1}$ is a finite sequence $\{k_j\}_{j=1}^N$. Suppose first that each k_j is a continuous function with compact support K_j. Put $K = \cup_{j=1}^N K_j$. Let ϵ be a positive real number, and choose a relatively compact open neighborhood V of the identity in G such that $\frac{\lambda(V-K)}{\lambda(V)} < 1+\epsilon$ [**11**, Lemma (18.12)]. Let χ denote the characteristic function of $V - K$, and fix $\{g_j\}_{j=1}^N \subseteq X$. By the Marcinkiewicz-Zygmund Inequality [**10**, p. 203] and (2.1), we have for each $s \in V$,

$$\left\| \left\{ \sum_{j=1}^N |H_{k_j} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}^p \leq c^p \left\| \left\{ \sum_{j=1}^N |R_s H_{k_j} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}^p .$$

Averaging this inequality over V with respect to $d\lambda(s)$, we see with the aid of Fubini's Theorem that:

$$\left\| \left\{ \sum_{j=1}^N |H_{k_j} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}^p \leq \frac{c^p}{\lambda(V)} \int_V \int_M$$

$$\left\{ \sum_{j=1}^N \left| \int_G k_j(u) \left(R_{s-u} g_j\right)(\omega) d\lambda(u) \right|^2 \right\}^{\frac{p}{2}} d\mu(\omega) d\lambda(s) = \frac{c^p}{\lambda(V)} \int_M \int_V \tag{2.5}$$

$$\left\{ \sum_{j=1}^N \left| \int_G k_j(u) \chi(s-u) \left(R_{s-u} g_j\right)(\omega) d\lambda(u) \right|^2 \right\}^{\frac{p}{2}} d\lambda(s) d\mu(\omega).$$

From (2.3) we obtain for μ-almost all $\omega \in \mathcal{M}$:

$$\int_V \left\{ \sum_{j=1}^N \left| \int_G k_j(u)\chi(s-u)\left(R_{s-u}g_j\right)(\omega)d\lambda(u) \right|^2 \right\}^{\frac{p}{2}} d\lambda(s)$$

$$= \int_V \left\{ \sum_{j=1}^N \left| k_j * \left[\left(R_{(\cdot)}g_j\right)(\omega)\chi \right] \right|^2 \right\}^{\frac{p}{2}} d\lambda$$

$$\leq C^p \int_{V-K} \left\{ \sum_{j=1}^N \left| \left(R_u g_j\right)(\omega) \right|^2 \right\}^{\frac{p}{2}} d\lambda(u).$$

Using this on the right of (2.5), we see with the aid of a further application of Fubini's Theorem that:

$$\left\| \left\{ \sum_{j=1}^N |H_{k_j} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}^p$$

$$\leq \frac{c^p C^p}{\lambda(V)} \int_{V-K} \int_{\mathcal{M}} \left\{ \sum_{j=1}^N \left| \left(R_u g_j\right)(\omega) \right|^2 \right\}^{\frac{p}{2}} d\mu(\omega) d\lambda(u)$$

$$= \frac{c^p C^p}{\lambda(V)} \int_{V-K} \left\| \left\{ \sum_{j=1}^N |R_u g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}^p d\lambda(u).$$

Consequently, from the Marcinkiewicz-Zygmund Inequality and (2.1) we have:

$$\begin{aligned} \left\| \left\{ \sum_{j=1}^N |H_{k_j} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}^p &\leq c^{2p} C^p \left\| \left\{ \sum_{j=1}^N |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}^p \frac{\lambda(V-K)}{\lambda(V)} \\ &\leq c^{2p} C^p \left\| \left\{ \sum_{j=1}^N |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}^p (1+\epsilon). \end{aligned} \tag{2.6}$$

Letting $\epsilon \to 0$ in (2.6) completes the proof of (2.4) for the case when the finite sequence $\{k_j\}$ consists of continuous functions with compact support. The general case of a finite sequence $\{k_j\} \subseteq L^1(G)$ follows from this by standard approximations in $L^1(G)$ in conjunction with the following elementary lemma. ■

Lemma 2.7. *If N is a positive integer, and $\{W_j\}_{j=1}^N \subseteq \mathfrak{B}(L^p(G))$, then*

$$\Gamma : \{f_j\}_{j=1}^N \to \{W_j f_j\}_{j=1}^N$$

is a linear mapping of $L^p(\lambda, \ell_N^2)$ into itself such that

$$\|\Gamma\| \leq \sum_{j=1}^N \|W_j\|.$$

Proof: Given $\{f_j\}_{j=1}^N \subseteq L^p(G)$, we have:

$$\left\| \left\{ \sum_{j=1}^N |W_j f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \leq \left\| \sum_{j=1}^N |W_j f_j| \right\|_{L^p(\lambda)}$$

$$\leq \sum_{j=1}^N \|W_j\| \, \|f_j\|_{L^p(\lambda)} \leq \left(\sum_{j=1}^N \|W_j\| \right) \left\| \left\{ \sum_{j=1}^N |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)}. \quad ■$$

The following companion to Theorem (2.2) is easily seen by entirely analogous reasoning.

Theorem 2.8. *Suppose that $\{k_j\}_{j \geq 1} \subseteq L^1(G)$. If C is a constant such that*

$$\left\| \left\{ \sum_{j \geq 1} |T_{k_j} f|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \leq C \|f\|_{L^p(\lambda)},$$

for all $f \in L^p(G)$,

then

$$\left\| \left\{ \sum_{j \geq 1} |H_{k_j} g|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c^2 C \|g\|_{L^p(\mu)},$$

for all $g \in X$.

3. Square functions defined by multiplier transforms. In this section we continue with the notation established in §2. We shall take $L^p(\mu)$ to be the space X of the representation R described at the beginning of §2, and shall further assume that:

(3.1) for each $u \in G$, the operator R_u extends from $L^p(\mu) \cap L^2(\mu)$ to a bounded linear mapping $R_u^{(2)}$ of $L^2(\mu)$ into $L^2(\mu)$;

(3.2) $\sup \left\{ \left\| R_u^{(2)} \right\| : u \in G \right\} < \infty$.

Under these circumstances, the map $u \to R_u^{(2)}$ is readily seen to be a weakly continuous representation of G in $L^2(\mu)$, and hence $R^{(2)}$ is strongly continuous [**11**, Theorem (22.8)]. Moreover, $R^{(2)}$ is similar to a unitary representation of G in $L^2(\mu)$ [**9**, Theorem 8.1]. Applying Stone's Theorem for unitary representations, we obtain a unique strongly countably additive regular spectral measure $\mathcal{E}(\cdot)$, defined on the Borel sets of $\hat{G}$ and acting in $L^2(\mu)$, such that

$$R_u^{(2)} = \int_{\hat{G}} \gamma(u) d\mathcal{E}(\gamma), \text{ for all } u \in G. \tag{3.3}$$

Suppose now that $\varphi \in M_p(\hat{G})$ is continuous as a complex-valued function on $\hat{G}$. Then the extension in [**6**, Theorem (2.1) and Remarks preceding (2.7)] of the Coifman-Weiss Multiplier Transference Theorem [**7**, Theorem (3.7)] asserts that $\int_{\hat{G}} \varphi d\mathcal{E}$ extends from $L^p(\mu) \cap L^2(\mu)$ to a bounded linear mapping $J_\varphi : L^p(\mu) \to L^p(\mu)$ whose norm does not exceed $c^2 \|\varphi\|_{M_p(\hat{G})}$, where c is the constant in (2.1). We remark that as an immediate consequence of Fubini's Theorem we have:

$$J_{\hat{k}} = H_k, \text{ for all } k \in L^1(G). \tag{3.4}$$

It will be convenient to establish one further item of notation: for $\varphi \in M_p(\hat{G})$, we shall denote by S_φ the corresponding multiplier transform on $L^p(G)$. Having attended to these preliminaries, we can now state the central result of this section.

Theorem 3.5. *Let $L^p(\mu)$ be the space X of the representation R described in §2, and suppose that (3.1) and (3.2) hold. Let $\{\varphi_j\}_{j\geq 1} \subseteq M_p(\hat{G})$ consist of functions which are continuous on $\hat{G}$. If C is a constant such that*

$$\left\| \left\{ \sum_{j\geq 1} |S_{\varphi_j} f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \leq C \left\| \left\{ \sum_{j\geq 1} |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)}, \tag{3.6}$$

for all sequences $\{f_j\}_{j\geq 1} \subseteq L^p(G)$,

then

$$\left\| \left\{ \sum_{j\geq 1} |J_{\varphi_j} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c^2 C \left\| \left\{ \sum_{j\geq 1} |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}, \tag{3.7}$$

for all sequences $\{g_j\}_{j\geq 1} \subseteq L^p(\mu)$.

The proof of Theorem (3.5) will be facilitated by the following linearization lemma for square functions and its corollary. We include a simple proof of the lemma for lack of a reference suitable for our setting. Notice that the lemma is somewhat reminiscent of the Kenig-Tomas linearization lemma for maximal operators [**13**, §1].

Lemma 3.8. *Let N be a positive integer, and let q be the index conjugate to $p (q = \infty$ if $p = 1)$. If $\{f_j\}_{j=1}^N \subseteq L^p(\mu)$, then*

$$\begin{aligned} \left\| \left\{ \sum_{j=1}^N |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} &= \max \\ \left\{ \left| \int_{\mathcal{M}} \sum_{j=1}^N f_j g_j \, d\mu \right| : \{g_j\}_{j=1}^N \subseteq L^q(\mu), \right.&\left. \left\| \left\{ \sum_{j=1}^N |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^q(\mu)} \leq 1 \right\}. \end{aligned} \tag{3.9}$$

Proof: Let $\{g_j\}_{j=1}^N$ be as in (3.9). Applying the Schwarz Inequality for ℓ_N^2 pointwise and then Hölder's Inequality for $L^p(\mu)$, we obviously have:

$$\left| \int_{\mathcal{M}} \sum_{j=1}^N f_j g_j \, d\mu \right| \leq \left\| \left\{ \sum_{j=1}^N |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}. \tag{3.10}$$

We complete the proof by showing that for a suitable choice of $\{g_j\}_{j=1}^N$ equality can be ahieved in (3.10). For $j = 1, \ldots, N$, define the μ-measurable function h_j on $\mathcal{M}$ by writing:

$$h_j(\omega) = \begin{cases} 0, & \text{if } \{\sum_{k=1}^N |f_k(\omega)|^2\}^{\frac{1}{2}} = 0; \\ \overline{f_j(\omega)}\{\sum_{k=1}^N |f_k(\omega)|^2\}^{-\frac{1}{2}}, & \text{otherwise.} \end{cases}$$

Obviously,

$$\left\{\sum_{j=1}^N |f_j|^2\right\}^{\frac{1}{2}} = \sum_{j=1}^N f_j h_j \text{ on } \mathcal{M},$$

and, whenever $\{\sum_{j=1}^N |f_j(\omega)|^2\}^{\frac{1}{2}} \neq 0$,

$$\left\{\sum_{j=1}^N |h_j(\omega)|^2\right\}^{\frac{1}{2}} = 1.$$

Choose $g \in L^q(\mu)$ so that $\|g\|_q = 1$, and

$$\left\|\left\{\sum_{j=1}^N |f_j|^2\right\}^{\frac{1}{2}}\right\|_{L^p(\mu)} = \int_{\mathcal{M}} \left\{\sum_{j=1}^N |f_j|^2\right\}^{\frac{1}{2}} g\,d\mu \tag{3.11}$$
$$= \int_{\mathcal{M}} \left(\sum_{j=1}^N f_j h_j\right) g\,d\mu.$$

Defining $g_j = h_j g$ for $j = 1, \ldots, N$, we have

$$\left\|\left\{\sum_{j=1}^N |g_j|^2\right\}^{\frac{1}{2}}\right\|_{L^q(\mu)} \leq \|g\|_{L^q(\mu)} \left\|\left\{\sum_{j=1}^N |h_j|^2\right\}^{\frac{1}{2}}\right\|_{L^\infty(\mu)} \leq 1.$$

In view of (3.11), this particular choice of $\{g_j\}_{j=1}^N$ achieves equality in (3.10). ■

Corollary 3.12. *Suppose that N is a positive integer and $\{\varphi_j\}_{j=1}^N \subseteq M_p(\hat{G})$. Let τ be the Haar measure in $\hat{G}$ normalized with respect to λ for Fourier inversion, and let $k \in L^1(\tau)$. If C is a constant such that*

$$\left\| \left\{ \sum_{j=1}^N |S_{\varphi_j} f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \leq C \left\| \left\{ \sum_{j=1}^N |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \tag{3.13}$$

for all sequences $\{f_j\}_{j=1}^N \subseteq L^p(G)$,

then

$$\left\| \left\{ \sum_{j=1}^N |S_{k*\varphi_j} f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \leq C\|k\|_{L^1(\tau)} \left\| \left\{ \sum_{j=1}^N |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)}, \tag{3.14}$$

for all sequences $\{f_j\}_{j=1}^N \subseteq L^p(G)$.

Proof: Suppose first that $\{f_j\}_{j=1}^N \subseteq L^2(G) \cap L^p(G)$ and $\{g_j\}_{j=1}^N \subseteq L^2(G) \cap L^q(G)$, where q is the index conjugate to p (if $p = 1$, then we take $L^q(G)$ to be the space of bounded measurable functions *modulo* equality a.e., equipped with the essential supremum norm). Using Plancherel's Theorem, Fubini's Theorem, and a change of variable, we see that:

$$\begin{aligned}
&\left| \sum_{j=1}^N \int_G (S_{k*\varphi_j} f_j)\, \overline{g_j} d\lambda \right| \\
&= \left| \sum_{j=1}^N \int_{\hat{G}} (k * \varphi_j)(\gamma) \hat{f}_j(\gamma) \overline{\hat{g}_j(\gamma)} d\tau(\gamma) \right| \\
&= \left| \int_{\hat{G}} k(\alpha) \left\{ \sum_{j=1}^N \int_{\hat{G}} \varphi_j(\gamma) (\overline{\alpha} f_j)^\wedge(\gamma) \overline{(\overline{\alpha} g_j)^\wedge(\gamma)} d\tau(\gamma) \right\} d\tau(\alpha) \right| \\
&\leq \int_{\hat{G}} |k(\alpha)| \left| \sum_{j=1}^N \int_G \{S_{\varphi_j}(\overline{\alpha} f_j)\} \overline{(\overline{\alpha} g_j)} d\lambda \right| d\tau(\alpha).
\end{aligned}$$

Applying Lemma (3.8) (for the measure space (G, λ)) to the sum in the last expression, we see from (3.13) that:

$$\left|\sum_{j=1}^{N}\int_G \left(S_{k*\varphi_j} f_j\right) \overline{g_j} d\lambda\right|$$

$$\leq C\|k\|_{L^1(\tau)} \left\|\left\{\sum_{j=1}^{N}|f_j|^2\right\}^{\frac{1}{2}}\right\|_{L^p(\lambda)} \left\|\left\{\sum_{j=1}^{N}|g_j|^2\right\}^{\frac{1}{2}}\right\|_{L^q(\lambda)}, \tag{3.15}$$

for $\{f_j\}_{j=1}^N \subseteq L^2(G) \cap L^p(G)$ and $\{g_j\}_{j=1}^N \subseteq L^2(G) \cap L^q(G)$.

If $1 < p < \infty$, then standard approximations show that the inequality in (3.15) continues to hold for arbitrary $\{f_j\}_{j=1}^N \subseteq L^p(G), \{g_j\}_{j=1}^N \subseteq L^q(G)$. Another application of Lemma (3.8) now gives (3.14) when $1 < p < \infty$.

To complete the proof of the corollary, suppose that $p = 1$, and that $\{f_j\}_{j=1}^N \subseteq L^2(G) \cap L^p(G)$, $\{g_j\}_{j=1}^N \subseteq L^q(G)$. For $1 \leq j \leq N$ there is an increasing sequence $\{E_{j,n}\}_{n=1}^\infty$ of subsets of G such that $\lambda(E_{j,n}) < \infty$ for all n, and $\left(S_{k*\varphi_j} f_j\right)$ vanishes on $G \backslash (\cup_{n=1}^\infty E_{j,n})$. Let $\chi_{j,n}$ denote the characteristic function, defined on G, of $E_{j,n}$. Notice that as $n \to \infty$, we have by Dominated Convergence:

$$\left(S_{k*\varphi_j} f_j\right) \chi_{j,n} \to \left(S_{k*\varphi_j} f_j\right), \text{ in the norm topology of } L^1(G). \tag{3.16}$$

For each positive integer n we apply (3.15) to $\{f_j\}_{j=1}^N$ and $\{\chi_{j,n} g_j\}_{j=1}^N$ to get:

$$\left|\sum_{j=1}^{N}\int_G \left(S_{k*\varphi_j} f_j\right) \chi_{j,n}\overline{g_j} d\lambda\right|$$

$$\leq C\|k\|_{L^1(\tau)} \left\|\left\{\sum_{j=1}^{N}|f_j|^2\right\}^{\frac{1}{2}}\right\|_{L^p(\lambda)} \left\|\left\{\sum_{j=1}^{N}|g_j|^2\right\}^{\frac{1}{2}}\right\|_{L^q(\lambda)}.$$

Letting $n \to \infty$, we see from this, (3.16), and Lemma (3.8) that for $p = 1$, (3.14) holds for any sequence $\{f_j\}_{j=1}^N \subseteq L^2(G) \cap L^p(G)$. Standard approximations now complete the proof for the case $p = 1$. ∎

Remark 3.17. With obvious modifications, the foregoing reasoning also shows that the statement of Corollary (3.12) remains valid if throughout we replace $\{f_j\}_{j=1}^N \subseteq L^p(G)$ by $f \in L^p(G)$ and $\left\|\left\{\sum_{j=1}^N |f_j|^2\right\}^{\frac{1}{2}}\right\|_{L^p(\lambda)}$ by $\|f\|_{L^p(\lambda)}$.

We now return to the context of Theorem (3.5).

Proof of Theorem (3.5): By Monotone Convergence we can assume without loss of generality that $\{\varphi_j\}_{j\geq 1}$ is a finite sequence $\{\varphi_j\}_{j=1}^N$. By Lemma (3.8) and simple approximation arguments similar to those used in the proof of Corollary (3.12), it suffices for (3.7) to show that

$$\left|\sum_{j=1}^N \int_{\mathcal{M}} (J_{\varphi_j} f_j)\, g_j d\mu\right| \leq c^2 C \left\|\left\{\sum_{j=1}^N |f_j|^2\right\}^{\frac{1}{2}}\right\|_{L^p(\mu)} \left\|\left\{\sum_{j=1}^N |g_j|^2\right\}^{\frac{1}{2}}\right\|_{L^q(\mu)}, \tag{3.18}$$

for all $\{f_j\}_{j=1}^N \subseteq L^2(\mu) \cap L^p(\mu)$ and all $\{g_j\}_{j=1}^N \subseteq L^2(\mu) \cap L^q(\mu)$,

where, as previously, q is the index conjugate to p. Fix $\{f_j\}_{j=1}^N \subseteq L^2(\mu) \cap L^p(\mu)$, $\{g_j\}_{j=1}^N \subseteq L^2(\mu) \cap L^q(\mu)$, and, for $1 \leq j \leq N$, define the regular Borel measure Θ_j, on $\hat{G}$ by putting $\Theta_j(\cdot) = (\mathcal{E}(\cdot) f_j, \overline{g_j})$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(\mu)$, and $\mathcal{E}$ is the regular spectral measure in (3.3).

We first establish (3.18) under the additional assumption that each $\varphi_j, j = 1,\ldots,N$, is a compactly supported function on $\hat{G}$. Let τ be the normalized Haar measure of $\hat{G}$ described in the statement of Corollary (3.12), and, in accordance with [**12**, Theorems (28.52) and (33.12)], let $\{h_\delta\}_{\delta\in\Delta}$ be an approximate identity for $L^1(\tau)$ such that: for each δ, we have $h_\delta \geq 0$, $\|h_\delta\|_{L^1(\tau)} = 1$, $\hat{h}_\delta$ is compactly supported; and, for each open neighborhood W of the identity in $\hat{G}$, $\int_{\hat{G}\setminus W} h_\delta d\tau \to 0$, as δ runs through Δ. For $1 \leq j \leq N$ and $\delta \in \Delta$, put $\varphi_{j,\delta} = h_\delta * \varphi_j$. Then $\varphi_{j,\delta} \in L^1(\tau), \varphi_{j,\delta}$ is bounded and continuous on $\hat{G}$, $\hat{\varphi}_{j,\delta}$ is compactly supported, and, for $1 \leq j \leq N$, $\varphi_{j,\delta} \to \varphi_j$ uniformly on $\hat{G}$ as δ runs through Δ. It follows that for each j and each δ, we have $\varphi_{j,\delta} = \hat{k}_{j,\delta}$ for some $k_{j,\delta} \in L^1(G)$, and that for each j,

$$\int_{\hat{G}} \varphi_{j,\delta} d\Theta_j \to \int_{\hat{G}} \varphi_j d\Theta_j, \quad \text{as } \delta \text{ runs through } \Delta. \tag{3.19}$$

Moreover, from (3.4),

$$\int_{\hat{G}} \varphi_{j,\delta} d\Theta_j = \int_{\mathcal{M}} \left(H_{k_{j,\delta}} f_j\right) g_j d\mu, \quad \text{for all } j \text{ and } \delta. \tag{3.20}$$

From Corollary (3.12) and (3.6), we see that for each δ,

$$\begin{aligned}\left\| \left\{ \sum_{j=1}^{N} |k_{j,\delta} * x_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} &= \left\| \left\{ \sum_{j=1}^{N} |S_{\varphi_{j,\delta}} x_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \\ &\leq C \left\| \left\{ \sum_{j=1}^{N} |x_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)}, \\ &\text{for all sequences } \{x_j\}_{j=1}^{N} \subseteq L^p(G).\end{aligned}$$

Applying Theorem (2.2) in order to transfer this inequality to $\mathcal{M}$, we find with the aid of Lemma (3.8) that for each $\delta \in \Delta$:

$$\begin{aligned}&\left| \sum_{j=1}^{N} \int_{\mathcal{M}} \left(H_{k_{j,\delta}} f_j\right) g_j d\mu \right| \\ &\leq c^2 C \left\| \left\{ \sum_{j=1}^{N} |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \left\| \left\{ \sum_{j=1}^{N} |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^q(\mu)}.\end{aligned} \tag{3.21}$$

Letting δ run through Δ in (3.21), and taking account of (3.19) and (3.20), we obtain (3.18) in the special case when each $\varphi_j, j = 1, \ldots, N$, is compactly supported.

It remains now to establish (3.18) for $\{\varphi_j\}_{j=1}^{N}$ in the general case; this will be accomplished by a suitable reduction to the foregoing case of compact supports. We continue with $\{f_j\}_{j=1}^N$, $\{g_j\}_{j=1}^N$, and $\{\Theta_j\}_{j=1}^N$ as previously described. Given $\epsilon > 0$, it follows from the regularity of the Borel measures Θ_j, $j = 1, \ldots, N$, that there is a compact subset K of $\hat{G}$ such that $|\Theta_j|(\hat{G}\backslash K) < \epsilon$ for $1 \le j \le N$. Calling on a standard fact [**12**, Theorem (31.37)], we get a function $f \in L^1(G)$ such that $\hat{f}$ has compact support, $\hat{f} = 1$ on K, and $\|f\|_{L^1(\lambda)} < 1 + \epsilon$. Obviously the following holds:

(3.22)
$$\left|\sum_{j=1}^N \int_{\mathcal{M}} \left(J_{\varphi_j} f_j\right) g_j d\mu\right| = \left|\sum_{j=1}^N \int_{\hat{G}} \varphi_j d\Theta_j\right| \le \left|\sum_{j=1}^N \int_{\hat{G}} \varphi_j \hat{f} d\Theta_j\right| + \left|\sum_{j=1}^N \int_{\hat{G}\backslash K} \varphi_j (1-\hat{f}) d\Theta_j\right|.$$

Using the Marcinkiewicz-Zygmund Inequality [**10**, p. 203] and (3.6), we see that for all $\{x_j\}_{j=1}^N \subseteq L^p(G)$:

$$\left\|\left\{\sum_{j=1}^N \left|S_{\varphi_j \hat{f}} x_j\right|^2\right\}^{\frac{1}{2}}\right\|_{L^p(\lambda)} \le C (1+\epsilon) \left\|\left\{\sum_{j=1}^N |x_j|^2\right\}^{\frac{1}{2}}\right\|_{L^p(\lambda)}.$$

From this, and the previous case applied to $\left\{\varphi_j \hat{f}\right\}_{j=1}^N$, we find that:

$$\left|\sum_{j=1}^N \int_{\hat{G}} \varphi_j \hat{f} d\Theta_j\right| \le c^2 C (1+\epsilon) \left\|\left\{\sum_{j=1}^N |f_j|^2\right\}^{\frac{1}{2}}\right\|_{L^p(\mu)} \left\|\left\{\sum_{j=1}^N |g_j|^2\right\}^{\frac{1}{2}}\right\|_{L^q(\mu)}. \tag{3.23}$$

In order to express an estimate for the second term in the majorant of (3.22), let $M_j = \sup \left\{ |\varphi_j(\gamma)| : \gamma \in \hat{G} \right\}$, for $j = 1, \ldots, N$, and put $M = \sum_{j=1}^{N} M_j$. Elementary reasoning using the choice of K and f shows that

$$\left| \sum_{j=1}^{N} \int_{\hat{G} \setminus K} \varphi_j (1 - \hat{f}) d\Theta_j \right| \leq \epsilon(2 + \epsilon) M. \tag{3.24}$$

Using (3.23) and (3.24) in (3.22), and letting $\epsilon \to 0$, we obtain (3.18), and thereby complete the proof of Theorem (3.5). ■

The method of proof of Theorem (3.5), with obvious modifications, also supplies the following companion result (note Remark (3.17) in this regard).

Theorem 3.25. *Assume the hypotheses of Theorem (3.5) except that (3.6) is to be replaced by*

$$\left\| \left\{ \sum_{j \geq 1} |S_{\varphi_j} f|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\lambda)} \leq C \|f\|_{L^p(\lambda)}, \qquad \text{for all } f \in L^p(G).$$

Then

$$\left\| \left\{ \sum_{j \geq 1} |J_{\varphi_j} g|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c^2 C \|g\|_{L^p(\mu)}, \qquad \text{for all } g \in L^p(\mu).$$

4. Special features when $\mathbf{G} = \mathbb{R}$. To open the discussion in this section, we go back to the set-up in §2, where we had a uniformly bounded, strongly continuous representation R of the general locally compact abelian group G in $X \subseteq L^p(\mu)$, and we seek counterparts of Theorems (3.5) and (3.25) in the case when R automatically has a spectral decomposition in X which will serve as a substitute for (3.3). One benefit of such circumstances is that they serve to eliminate the need for the extra assumptions of §3-specifically, that X should equal $L^p(\mu)$ and that R should have the auxiliary representation $R^{(2)}$ in $L^2(\mu)$ described in (3.1) and (3.2). As indicated in §1, the desired spectral decomposability of R in X itself, without such extra assumptions, automatically exists if $G = \mathbb{Z}$ or $G = \mathbb{R}$, and $1 < p < \infty$. We shall be concerned in the present section with the situation when $G = \mathbb{R}$ and $1 < p < \infty$. The results we shall obtain here, including the counterparts of Theorems (3.5) and (3.25) in Theorems (4.11) and (4.14) below, correspond to results already obtained for $G = \mathbb{Z}$ in [**2**, §§2,3]. The general framework afforded by Theorem (2.2), Lemma (3.8), and Corollary (3.12) makes it possible to obtain our results for $G = \mathbb{R}$ more easily than could be done by attempting to adapt directly the reasoning in [**2**]. We remark in passing that the methods of this section can be applied equally well to the case $G = \mathbb{Z}$, where they afford some simplification of the reasoning in [**2**, §§2,3].

Before passing to the specific setting for this section, we shall review briefly the relevant general notion of spectral decomposability and its associated spectral integration.

Definition. A *spectral family* in a Banach space Y is a projection-valued function $F(\cdot)$ mapping the real line $\mathbb{R}$ into $\mathfrak{B}(Y)$, and having the following properties:

(i) $\sup \{\|F(\lambda)\| : \lambda \in \mathbb{R}\} < \infty$;
(ii) $F(\lambda)F(\tau) = F(\tau)F(\lambda) = F(\lambda)$ whenever $\lambda \leq \tau$;
(iii) $F(\cdot)$ is right-continuous on $\mathbb{R}$ with respect to the strong operator topology of $\mathfrak{B}(Y)$;
(iv) at each $\lambda \in \mathbb{R}$, $F(\cdot)$ has a left-hand limit $F(\lambda^-)$ in the strong operator topology of $\mathfrak{B}(Y)$;
(v) with respect to the strong operator topology of $\mathfrak{B}(Y)$, $F(\lambda) \to I$ as $\lambda \to +\infty$, and $F(\lambda) \to 0$ as $\lambda \to -\infty$.

If there is a compact interval $[a, b]$ such that $F(\lambda) = 0$ for $\lambda < a$ and $F(\lambda) = I$ for $\lambda \geq b$, then we say that $F(\cdot)$ is *concentrated* on $[a, b]$.

Corresponding to any spectral family $F(\cdot)$ of projections in Y, a Riemann-Stieltjes notion of spectral integration with respect to $F(\cdot)$ can be defined as follows. For a compact interval $J = [\alpha, \beta]$ of $\mathbb{R}$, let $\mathbf{BV}(J)$ denote the algebra consisting of all complex-valued functions φ on J whose total variation var (φ, J) is finite, equipped with the Banach algebra norm $\|\cdot\|_J$ specified by

$$\|\varphi\|_J = \sup \{|\varphi(\lambda)| : \lambda \in J\} + \operatorname{var}(\varphi, J).$$

Given $\varphi \in \mathbf{BV}(J)$, for each partition $\mathcal{P} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ of J put

$$\mathcal{S}(\mathcal{P}; \varphi,\ F) = \sum_{k=1}^{n} \varphi(\lambda_k)\{F(\lambda_k) - F(\lambda_{k-1})\}. \tag{4.1}$$

Then (see [**9**, Chapter 17] or the abbreviated account of spectral integration in [**3**, §2]) the net $\{\mathcal{S}(\mathcal{P}; \varphi, F)\}$ converges in the strong operator topology of $\mathfrak{B}(Y)$ as $\mathcal{P}$ increases through the partitions of J directed by refinement, and we denote the strong limit by $\int_{[\alpha,\beta]} \varphi dF$. If $\varphi \in \mathbf{BV}(J)$ is a continuous function, then for $k = 1, \ldots, n, \varphi(\lambda_k)$ can be replaced on the right of (4.1) by $\varphi(\lambda'_k)$, where $\lambda'_k \in [\lambda_{k-1}, \lambda_k]$ is chosen arbitrarily. The corresponding assertions for a notion of spectral integration likewise obtain when the compact interval J is replaced by $\mathbb{R}$ (see [**15**, Proposition 2.1.11 and Theorem 2.1.14]). In particular, for $\varphi \in \mathbf{BV}(\mathbb{R})$, the Riemann-Stieltjes approximating sums corresponding to (4.1) are taken on partitions of the extended real number system $[-\infty, +\infty]$, and have a limit in the strong operator topology of $\mathfrak{B}(Y)$, which is denoted by $\int_{\mathbb{R}} \varphi dF$. In this situation the functions $F(\cdot)$ and φ are extended from $\mathbb{R}$ to $[-\infty, +\infty]$ by defining $F(-\infty) = 0, F(+\infty) = I$, and $\varphi(\pm\infty) = \lim_{\lambda\to\pm\infty} \varphi(\lambda)$. The relationship between spectral integration over compact intervals and over $\mathbb{R}$ is expressed by the fact that for $\varphi \in \mathbf{BV}(\mathbb{R})$, $\int_{[-a,a]} \varphi dF$ converges in the strong operator topology of $\mathfrak{B}(Y)$ to $\int_{\mathbb{R}} \varphi dF$, as $a \to +\infty$.

We now describe the setting for the results of this section. As previously, $(\mathcal{M}, \mu)$ will be an arbitrary measure space. We shall denote by X_0 a closed subspace of $L^p(\mu)$, where $1 < p < \infty$, and we shall consider a strongly continuous one-parameter group $\{U_t : t \in \mathbb{R}\}$ of operators on X_0 such that

$$c_0 \equiv \sup\ \{\|U_t\| : t \in \mathbb{R}\} < \infty. \tag{4.2}$$

By [**3**, Theorem (4.21)], $\{U_t : t \in \mathbb{R}\}$ automatically has a spectral decomposition acting in X_0-specifically, there is a unique spectral family $E(\cdot)$ of projections in X_0 such that:

$$U_t x = \lim_{a\to+\infty} \int_{[-a,a]} e^{it\lambda} dE(\lambda)x, \qquad \text{for } t \in \mathbb{R}, x \in X_0. \tag{4.3}$$

Moreover, it follows from [**4**, Theorems (5.12)-(ii) and (5.16)], in combination with the Coifman-Weiss General Transference Result [**5**, Theorem (2.3)] and the boundedness properties of the classical Hilbert transform, that

$$\sup \{\|E(\lambda)\| : \lambda \in \mathbb{R}\} \leq c_0^2 C_p, \tag{4.4}$$

where, here and henceforth, C_p denotes a positive real constant which depends only on p and may change in value from one occurrence to another. Given a function $\varphi \in \mathbf{BV}(\mathbb{R})$, we define $\Phi \in \mathbf{BV}(\mathbb{R})$ by writing

$$\Phi(\lambda) \equiv 2^{-1} \left\{ \lim_{s \to \lambda+} \varphi(s) + \lim_{s \to \lambda-} \varphi(s) \right\}. \tag{4.5}$$

Notice that by Stečkin's Theorem [**10**, Theorem 6.2.5], $\mathbf{BV}(\mathbb{R}) \subseteq M_p(\mathbb{R})$, and consequently

$$\mathcal{J}_\varphi \equiv \int_{\mathbb{R}} \Phi dE \tag{4.6}$$

can be viewed as a transferred Fourier multiplier-a fact underscored by the next theorem (compare the discrete case in [**3**, Theorem (4.14)]).

Theorem 4.7. *Suppose that $\{U_t : t \in \mathbb{R}\}$ is a strongly continuous, uniformly bounded one-parameter group of operators acting on a closed subspace X_0 of $L^p(\mu)$, where μ is an arbitrary measure and $1 < p < \infty$. Let $\{\kappa_n\}_{n=1}^\infty$ be the (sequential) Fejér kernel of $\mathbb{R}$:*

$$\kappa_n(t) \equiv \frac{1}{2\pi n} \frac{\sin^2 \frac{nt}{2}}{\left(\frac{t}{2}\right)^2}.$$

If $\varphi \in \mathbf{BV}(\mathbb{R}) \cap L^1(\mathbb{R})$, then:

(i) *in the notation of (4.6), we have for each $x \in X_0$,*

$$\mathcal{J}_\varphi x = \lim_{n \to +\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\kappa}_n(t) \hat{\varphi}(t) U_t x dt,$$

where Bochner integration is used on the right;

(ii) $\|\mathcal{J}_\varphi\| \leq c_0^2 \|\varphi\|_{M_p(\mathbb{R})}$ *where c_0 is the constant in (4.2).*

Proof: Straightforward calculations show that:

$$\text{(4.8)} \qquad \operatorname{var}(\kappa_n * \varphi, \mathbb{R}) \le \operatorname{var}(\varphi, \mathbb{R}), \qquad \text{for all } n \in \mathbb{N},$$

and that, in the notation of (4.5),

$$\text{(4.9)} \qquad \lim_{n\to\infty} (\kappa_n * \varphi)(\lambda) = \Phi(\lambda), \qquad \text{for all } \lambda \in \mathbb{R}.$$

Suppose that $n \in \mathbb{N}$, a is a positive real number, and $x \in \{E(a)-E(-a)\}X_0$. Using (4.3) together with successive integrations by parts on the left-hand member of the following equation, we find with the aid of Fourier inversion that:

$$\text{(4.10)} \qquad \frac{1}{2\pi}\int_{[-n,n]} \hat{\kappa}_n(t)\hat{\varphi}(t)U_t x\,dt = \int_{[-a,a]} (\kappa_n * \varphi)\,dEx.$$

Letting $n \to \infty$ on the right of (4.10), we can use (4.8) and (4.9) to invoke a standard limit theorem for spectral integrals [**3**, Proposition (2.10)], thereby inferring that, in the norm topology of X_0:

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\kappa}_n(t)\hat{\varphi}(t)U_t x\,dt = \mathcal{J}_\varphi x,$$
$$\text{for all } x \in \cup_{a>0}\{E(a) - E(-a)\}X_0.$$

Since $\cup_{a>0}\{E(a)-E(-a)\}X_0$ is dense in X_0, the desired conclusions (4.7)-(i), (ii) are now immediate from the following uniform boundedness estimate, obtained for all $n \in \mathbb{N}$ with the aid of the Coifman-Weiss General Transference Result (stated just prior to Theorem (2.2)):

$$\begin{aligned}\left\|\frac{1}{2\pi}\int_{\mathbb{R}} \hat{\kappa}_n(t)\hat{\varphi}(t)U_t\,dt\right\| &\le c_0^2 N_p\left(\frac{\hat{\kappa}_n\hat{\varphi}}{2\pi}\right) \\ &= c_0^2\|\kappa_n * \varphi\|_{M_p(\mathbb{R})} \le c_0^2\|\varphi\|_{M_p(\mathbb{R})}. \qquad \blacksquare\end{aligned}$$

Theorem 4.7 bis. *Let $\{U_t : t \in \mathbb{R}\}$, X_0, μ, and p be as in the hypotheses of Theorem (4.7). Then the inequality in (4.7)-(ii) remains valid for each $\varphi \in \mathbf{BV}(\mathbb{R})$.*

Proof: For $n \in \mathbb{N}$, let $\varphi_n \in \mathbf{BV}(\mathbb{R}) \cap L^1(\mathbb{R})$ be given by $\varphi_n \equiv \hat{\kappa}_n\varphi$. Let Φ_n and Φ correspond to φ_n and φ as in (4.5). Thus, $\Phi_n = \hat{\kappa}_n\Phi$, and consequently $\Phi_n \to \Phi$ pointwise on $\mathbb{R}$, while

$$\sup_{n\in\mathbb{N}} \operatorname{var}(\Phi_n, \mathbb{R}) \le 2\left[\sup\{|\Phi(t)| : t \in \mathbb{R}\} + \operatorname{var}(\Phi, \mathbb{R})\right].$$

The analogue for $\mathbb{R}$ of the limit theorem for spectral integrals ([**3**, Proposition (2.10)]) now shows that $\mathcal{J}_{\varphi_n} \to \mathcal{J}_{\varphi}$ in the strong operator topology. The proof of Theorem (4.7) **bis** is completed by applying (4.7)-(ii) to φ_n to get

$$\|\mathcal{J}_{\varphi_n}\| \le c_0^2 \|\hat{\kappa}_n \varphi\|_{M_p(\mathbb{R})} \le c_0^2 \|\varphi\|_{M_p(\mathbb{R})}. \qquad \blacksquare$$

The next result generalizes Theorem (4.7) **bis** to square functions and provides the counterpart for Theorem (3.5) in the setting of the one-parameter group $\{U_t : t \in \mathbb{R}\}$, which is not required to have an $L^2(\mu)$-version. Recall the notation of Theorem (3.5) for multiplier transforms.

Theorem 4.11. *Let $\{U_t : t \in \mathbb{R}\}$, X_0, μ, and p be as in the hypotheses of Theorem (4.7), and suppose that $\{\varphi_j\}_{j \ge 1} \subseteq \mathbf{BV}(\mathbb{R})$. If C is a constant such that*

$$\left\| \left\{ \sum_{j \ge 1} |S_{\varphi_j} f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \le C \left\| \left\{ \sum_{j \ge 1} |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})},$$

for all sequences $\{f_j\}_{j \ge 1} \subseteq L^p(\mathbb{R})$,

then

$$\left\| \left\{ \sum_{j \ge 1} |\mathcal{J}_{\varphi_j} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \le c_0^2 C \left\| \left\{ \sum_{j \ge 1} |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)},$$

for all sequences $\{g_j\}_{j \ge 1} \subseteq X_0$,

where c_0 is the constant in (4.2) and the operators $\mathcal{J}_{\varphi_j}$ are as defined in (4.6).

Proof: It suffices to consider the case of a finite sequence $\{\varphi_j\}_{j=1}^N \subseteq \mathbf{BV}(\mathbb{R})$. Moreover, the Marcinkiewicz-Zygmund Inequality [**10**, p. 203] together with the approximation reasoning used to prove Theorem (4.7) **bis** allows us to assume without loss of generality that $\varphi_j \in \mathbf{BV}(\mathbb{R}) \cap L^1(\mathbb{R})$, for $1 \le j \le N$. For each $n \in \mathbb{N}$, we infer from Corollary (3.12) that

$$\left\| \left\{ \sum_{j=1}^{N} |S_{\kappa_n * \varphi_j} f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \le C \left\| \left\{ \sum_{j=1}^{N} |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})}, \tag{4.12}$$

for all sequences $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{R})$.

Notice that for each j and n, we have $\kappa_n * \varphi_j = \hat{\psi}_{n,j}$, where $\psi_{n,j} \in L^1(\mathbb{R})$ is given by:

$$\psi_{n,j}(t) \equiv \frac{\hat{\kappa}_n(t)\hat{\varphi}_j(-t)}{2\pi}.$$

Denoting by $T_{n,j}$ the convolution operator on $L^p(\mathbb{R})$ defined by $\psi_{n,j}$, we can rewrite (4.12) in the form:

$$\left\| \left\{ \sum_{j=1}^{N} |T_{n,j} f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \leq C \left\| \left\{ \sum_{j=1}^{N} |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})},$$

for all sequences $\{f_j\}_{j=1}^N \subseteq L^p(\mathbb{R})$.

An application of Theorem (2.2) to this now gives:

$$\left\| \left\{ \sum_{j=1}^{N} |H_{\psi_{n,j}} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c_0^2 C \left\| \left\{ \sum_{j=1}^{N} |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}, \tag{4.13}$$

for all sequences $\{g_j\}_{j=1}^N \subseteq X_0$.

The proof of Theorem (4.11) is now easily completed by letting $n \to \infty$ in (4.13) and invoking (4.7)-(i). $\square$

Similar reasoning which takes account of Remark (3.17) and Theorem (2.8) provides the following counterpart to Theorem (3.25).

Theorem 4.14. *Let $\{U_t : t \in \mathbb{R}\}$, X_0, μ, and p be as in the hypotheses of Theorem (4.7), and suppose that $\{\varphi_j\}_{j\geq 1} \subseteq \mathbf{BV}(\mathbb{R})$. If C is a constant such that*

$$\left\| \left\{ \sum_{j\geq 1} |S_{\varphi_j} f|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}, \text{ for all } f \in L^p(\mathbb{R}),$$

then

$$\left\| \left\{ \sum_{j\geq 1} |\mathcal{J}_{\varphi_j} g|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c_0^2 C\|g\|_{L^p(\mu)}, \quad \text{for all } g \in X_0.$$

Theorem (4.11) enables us to transfer to X_0 the classical M. Riesz Property of $\mathbb{R}$ [**10**, Theorem 6.5.2]. This development, which is described in the next result, parallels the transference in [**2**, Theorem (3.15)] of the classical M. Riesz Property for $\mathbb{Z}$.

Theorem 4.15. *Let $\{U_t : t \in \mathbb{R}\}$, X_0, μ, and p be as in the hypotheses of Theorem (4.7), and let $E(\cdot)$ be the unique spectral family associated with $\{U_t : t \in \mathbb{R}\}$ by (4.3). Then there is a positive real constant C_p depending only on p such that:*

$$\left\| \left\{ \sum_{j=1}^{\infty} |E(a_j)g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c_0^2 C_p \left\| \left\{ \sum_{j=1}^{\infty} |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)},$$

for all sequences $\{g_j\}_{j=1}^{\infty} \subseteq X_0$, and all sequences $\{a_j\}_{j=1}^{\infty} \subseteq \mathbb{R}$.

Proof: For $j \in \mathbb{N}, n \in \mathbb{N}$, let $x_{j,n} = a_j - n$ and $y_{j,n} = a_j + \frac{1}{n}$. Denote the characteristic function of the interval $(x_{j,n}, y_{j,n}]$ by $\varphi_{j,n}$. Temporarily fix $N \in \mathbb{N}$. According to the classical M. Riesz Property for $\mathbb{R}$, we have for each $n \in \mathbb{N}$:

$$\left\| \left\{ \sum_{j=1}^{N} |S_{\varphi_{j,n}} f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})} \leq C_p \left\| \left\{ \sum_{j=1}^{N} |f_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R})},$$

for all sequences $\{f_j\}_{j=1}^{N} \subseteq L^p(\mathbb{R})$.

It follows from this by direct application of Theorem (4.11) that:

$$\left\| \left\{ \sum_{j=1}^{N} |\mathcal{J}_{\varphi_{j,n}} g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c_0^2 C_p \left\| \left\{ \sum_{j=1}^{N} |g_j|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}. \tag{4.16}$$

Since simple direct calculations show that

$$\mathcal{J}_{\varphi_{j,n}} = \frac{1}{2}\{E(y_{j,n}) + E(y_{j,n}^-)\} - \frac{1}{2}\{E(x_{j,n}) + E(x_{j,n}^-)\},$$

it is easy to see that for each $j \in \mathbb{N}$,

$$\mathcal{J}_{\varphi_{j,n}} g_j \to E(a_j) g_j, \text{ as } n \to \infty.$$

Hence the proof can be completed by letting $n \to \infty$ in (4.16), and then letting $N \to \infty$. ■

We close with a brief account describing how Theorem (4.7)-(ii) permits the spectral family $E(\cdot)$ associated with $\{U_t : t \in \mathbb{R}\}$ to transfer to X_0 the Littlewood-Paley Theorem for the dyadic decomposition of $\mathbb{R}$ [**10**, Theorem 7.2.1]. Let $\{t_j\}_{j=-\infty}^{\infty}$ be the usual sequence of dyadic points in $\mathbb{R}$ [**10**, §7.1.2]:

$$t_j = \begin{cases} 2^{j-1}, & \text{if } j > 0; \\ -1, & \text{if } j = 0; \\ -2^{|j|}, & \text{if } j < 0. \end{cases}$$

For each $j \in \mathbb{Z}$, we let Γ_j be the open interval (t_j, t_{j+1}), and we define the *dyadic* sigma-algebra $\sum_d$ to be the sigma-algebra of subsets of $\mathbb{R}$ generated by the collection

$$\mathfrak{G} \equiv \{\Gamma_j : j \in \mathbb{Z}\} \cup \{\{t_j\} : j \in \mathbb{Z}\}.$$

Clearly each σ belonging to $\sum_d$ can be expressed as the union of a uniquely determined subcollection $\mathfrak{K}_\sigma$ of $\mathfrak{G}$.

We define a projection-valued function $\mathcal{F}_0$ on $\mathfrak{G}$ by writing for each $j \in \mathbb{Z}$:

$$\mathcal{F}_0(\Gamma_j) = E(t_{j+1}^-) - E(t_j);$$
$$\mathcal{F}_0(\{t_j\}) = E(t_j) - E(t_j^-).$$

We next extend $\mathcal{F}_0$ in the following way to a projection-valued function defined on the algebra $\mathfrak{U}_d$ of subsets of $\mathbb{R}$ generated by $\mathfrak{G}$. If $\sigma \in \mathfrak{U}_d$ and $\mathfrak{K}_\sigma$ is finite, we put

$$\mathcal{F}_0(\sigma) = \sum \{\mathcal{F}_0(\alpha) : \alpha \in \mathfrak{K}_\sigma\};$$
$$\mathcal{F}_0(\mathbb{R} \backslash \sigma) = I - \sum \{\mathcal{F}_0(\alpha) : \alpha \in \mathfrak{K}_\sigma\}.$$

We claim that

$$\sup\{\|\mathcal{F}_0(\sigma)\| : \sigma \in \mathfrak{U}_d\} \le c_0^2 C_p. \tag{4.17}$$

The essence of the proof of (4.17) occurs in the case when $\sigma \in \mathfrak{U}_d$ has the form $\sigma = \cup_{k=1}^{n} \Gamma_{j_k}$, with $j_1, \ldots, j_n$ distinct integers, and we shall establish (4.17) by treating this case. For $k = 1, \ldots, n$, let a_{j_k} and b_{j_k} satisfy $t_{j_k} < a_{j_k} < b_{j_k} < t_{j_k+1}$. Let φ be the characteristic function of the union of the closed intervals $[a_{j_k}, b_{j_k}]$, $k = 1, \ldots, n$. It follows from the Strong Marcinkiewicz Multiplier Theorem [**10**, Theorem 8.3.1] that $\|\varphi\|_{M_p(\mathbb{R})} \le C_p$. Hence applying Theorem (4.7)-(ii) to φ, we get:

$$\|\mathcal{J}_\varphi\| \le c_0^2 C_p. \tag{4.18}$$

It is easy to see by direct calculations that:

$$\mathcal{J}_\varphi = \sum_{k=1}^{n} \frac{1}{2}\left\{E(b_{j_k} + E(b_{j_k}^-)\right\} - \sum_{k=1}^{n} \frac{1}{2}\left\{E(a_{j_k}) + E(a_{j_k}^-)\right\}.$$

Substituting this in (4.18), we let $a_{j_k} \to t_{j_k}$ and $b_{j_k} \to t_{j_k+1}$. This gives

$$\|\mathcal{F}_0(\sigma)\| \le c_0^2 C_p,$$

and so we can regard the claim in (4.17) as established. By using [**1**, Corollary 2] in conjunction with (4.17), we can now extend $\mathcal{F}_0$ from $\mathfrak{U}_d$ to a projection-valued function $\mathcal{F}$ defined on the dyadic sigma-algebra $\sum_d$ by writing for each $\sigma \in \sum_d$,

$$\mathcal{F}(\sigma) = \sum\{\mathcal{F}_0(\alpha) : \alpha \in \mathfrak{K}_\sigma\}, \tag{4.19}$$

where the convergence of the sum on the right is in the sense of unordered summation with respect to the strong operator topology of $\mathfrak{B}(X_0)$.

It can now be shown that $\mathcal{F}$ is a strongly countably additive spectral measure, whereupon recourse to Khintchine's Inequality [**14**, Theorem 2. b. 3] provides a transferred Littlewood-Paley Theorem for X_0. We specify these end-results in the following theorem. The remaining details of its demonstration parallel the treatment in [**2**, Theorem (2.12) and Corollary (2.14)], and will be omitted for expository reasons.

Theorem 4.20. *The projection-valued function $\mathcal{F}$ defined in (4.19) is a strongly countably additive spectral measure on $\sum_d$ such that*

$$\sup\left\{\|\mathcal{F}(\sigma)\| : \sigma \in \sum\nolimits_d\right\} \leq c_0^2 C_p,$$

where C_p is a positive real constant depending only on p. Moreover, there is a positive real constant B_p depending only on p such that whenever $g \in X_0$ and $\{\sigma_j\}_{j\geq 1}$ is a sequence of mutually disjoint elements of $\sum_d$ whose union is $\mathbf{R}$, then

$$c_0^{-2} B_p^{-1} \|g\|_{L^p(\mu)} \leq \left\| \left\{ \sum_{j\geq 1} |\mathcal{F}(\sigma_j) g|^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leq c_0^2 B_p \|g\|_{L^p(\mu)}.$$

REFERENCES

1. J. Y. Barry, *On the convergence of ordered sets of projections*, Proc. Amer. Math. Soc. **5** (1954), 313-314.
2. E. Berkson, J. Bourgain, and T. A. Gillespie, *On the almost everywhere convergence of ergodic averages for power-bounded operators on L^p-subspaces*, Integral Equations and Operator Theory (to appear).
3. E. Berkson and T. A. Gillespie, *Stečkin's theorem, transference, and spectral decompositions*, J. Functional Analysis **70** (1987), 140-170.
4. E. Berkson, T. A. Gillespie, and P. S. Muhly, *Abstract spectral decompositions guaranteed by the Hilbert transform*, Proc. London Math. Soc. (3) **53** (1986), 489-517.
5. E. Berkson, T. A. Gillespie, and P. S. Muhly, *Generalized analyticity in UMD spaces*, Arkiv för Mat. **27** (1989), 1-14.
6. E. Berkson, T. A. Gillespie, and P. S. Muhly, *L^p-multiplier transference induced by representations in Hilbert space*, Studia Math. **94** (1989), 51-61.
7. R. R. Coifman and G. Weiss, *Operators associated with representations of amenable groups, singular integrals induced by ergodic flows, the rotation method and multipliers*, Studia Math. **47** (1973), 285-303.
8. R. R. Coifman and G. Weiss, *Transference Methods in Analysis*, C. B. M. S. Regional Conference Series in Math, No. 31, American Math. Soc., Providence, R. I. (1977).
9. H. R. Dowson, *Spectral Theory of Linear Operators*, London Math. Soc. Monographs, No. 12, Academic Press, New York (1978).
10. R. E. Edwards and G. I. Gaudry, *Littlewood-Paley and Multiplier Theory*, Ergebnisse der Math. und ihrer Grenzgebiete 90, Springer-Verlag, Berlin (1977).
11. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Grundlehren der Math. Wissenschaften, Band 115, Academic Press, New York (1963).

12. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis II*, Grundlehren der math. Wissenschaften, Band 152, Springer-Verlag, Berlin (1970).
13. C. Kenig and P. Tomas, *Maximal operators defined by Fourier multipliers*, Studia Math. **68** (1980), 79-83.
14. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I (Sequence Spaces)*, Ergebnisse der Math. und ihrer Grenzgebiete 92, Springer-Verlag, Berlin (1977).
15. D. J. Ralph, *Semigroups of Well-Bounded Operators and Multipliers*, Thesis, Univ. of Edinburgh (1977).

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